QCD at finite isospin density

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with

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QCD at finite isospin density

- Special case of the physically relevant situation $\mu_u \neq \mu_d \neq \mu_s$,
- Platform to assess limitations of various numerical approaches to finite baryon density,
- Provides a rich range of physical phenomena:
  - (almost free) pion gas at low temperature and density,
  - (almost free) quark gas at high temperature,
  - Bose condensation of charged pions at large density.

QCD at finite isospin density

- Isospin chemical potential: $\mu_d = -\mu_u = \mu_I/2$

$$m_u = m_d \;\Rightarrow \det(m_u, -\mu_I/2) = \det(m_d, +\mu_I/2)^*$$
$$\det_u \times \det_d = |\det|^2$$

  - No sign problem (phase quenched)
  - System does not carry net baryon number: $Q_u = -Q_d$

- Chemical potential favors creation of $\bar{u}d$ mesons, especially the lightest one, $\pi^- \sim \bar{u}\gamma_5 d \quad \Rightarrow$ Bose condensation

- QCD inequalities (Son & Stephanov) $\longrightarrow$ symmetry breaking driven by $\langle \bar{\psi} i \gamma_5 \tau_{1,2} \psi \rangle$

Small isospin density

- At small isospin densities one can use chiral perturbation theory

$$\mathcal{L} = \frac{1}{4} f_\pi^2 \mathrm{Tr}[D_\mu \Sigma D_\mu \Sigma^\dagger - 2 m_\pi^2 \mathrm{Re}\Sigma]$$

where $\Sigma \in SU(2)$ is the matrix pion field:

  - μ_I breaks $SU(2)_{L+R} \to U(1)_{L+R}$,
  - no additional low energy constant needed (to leading order),
  - valid for $\mu_I \lesssim m_\rho$.

- Effective potential can be minimized as a function of μ_I using

$$\overline{\Sigma} = \cos\alpha + i(\tau_1 \cos\phi + \tau_2 \sin\phi) \sin\alpha,$$

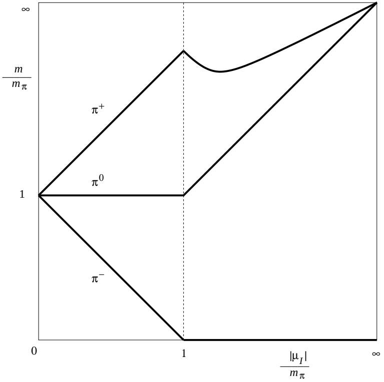
  - flavour rotation angle ϕ irrelevant.

Small isospin density

- Two distinct regimes can be identified:
- $|\mu_I| < m_\pi$:
  - no pion can be excited,
  - $\overline{\Sigma} = 1$, i.e. $\langle \bar{u}u + \bar{d}d \rangle = 2\langle \bar{\psi}\psi \rangle_0$
  - normal QCD vacuum.
- $|\mu_I| \geq m_\pi$:
  - π^- particles can be excited,
  - a Bose condensate of π^- may form where $\langle \bar{u}\gamma_5 d \rangle \neq 0$,
  - chiral condensate rotates into pion condensate as a function of μ_I,
  - $U(1)_{L+R}$ spontaneously broken $\longrightarrow$ 3*d* XY universality class,
  - π^- becomes massless, π^+, π^0 remain massive.
- When $|\mu_I| \gtrsim m_\rho$ chiral perturbation theory breaks down.

Free energy at low temperature

- Energies m to excite a pion from the vacuum at low temperature:



$\Rightarrow$ at $\mu_I = m_\pi$ the π^- Bose condense.

'Equation of state' (EoS)

- 'Equation of state' (EoS) : density as a function of isospin chemical potential:

$$\rho_I = \frac{Q}{V} = \rho_I(\hat{\mu}_I)$$

where $\hat{\mu}_I = \mu_I / T$.

- Canonical simulations give the free energy $F(Q) = -\ln Z_C(Q)$ and its derivative

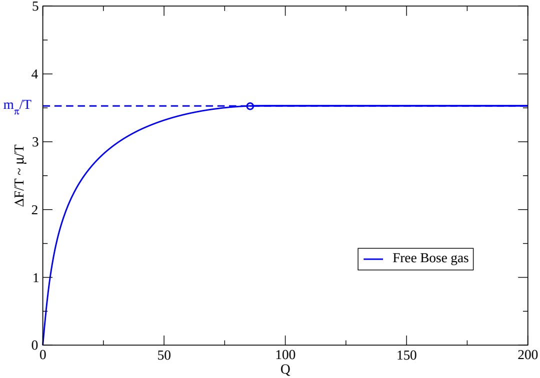
$$F(Q) - F(Q-1) \overset{V\to\infty}{\Longrightarrow} \frac{dF}{d\rho_I} = \mu_I .$$

EoS at low temperature

- EoS for free bosons, i.e. non-interacting pions at low density ($am_\pi \approx 0.89\ \forall\beta$):

$$\rho(\hat{\mu},\hat{m}) = \frac{T^3}{2\pi^2}\int_0^{+\infty} d\hat{p}\,\hat{p}^2\left(\frac{1}{e^{(\omega-\hat{\mu})}-1} - \frac{1}{e^{(\omega+\hat{\mu})}-1}\right)$$

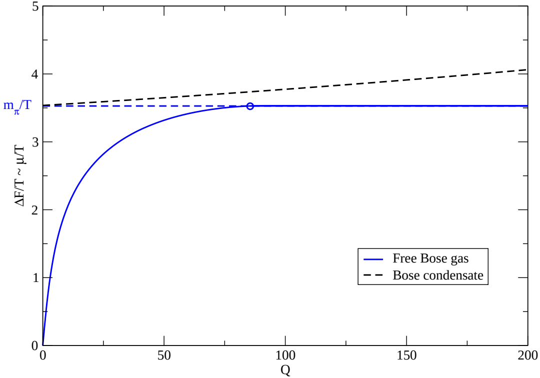
where $\omega = \sqrt{\hat{p}^2+\hat{m}^2}$.



EoS at low temperature

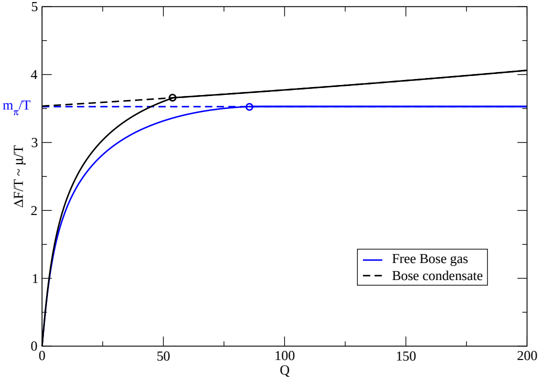
- Weak pion repulsion ($\sim \frac{1}{f_\pi^2}$) at $T = 0 \longrightarrow$ Bose condensation:

$$\rho_I = f_\pi^2 \mu_I \left(1 - \left(\frac{m_\pi}{\mu_I}\right)^4\right)$$



EoS at low temperature

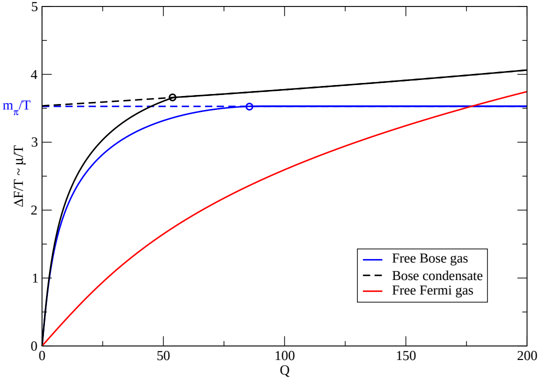
- Weak pion repulsion ($\sim \frac{1}{f_\pi^2}$) at low T: $T_c(\mu = 0)$ increases $\mu_{\text{crit}} > m_\pi$, but interaction pushes critical density down



EoS at high temperature

- EoS for a massless, free Fermi gas via the pressure:

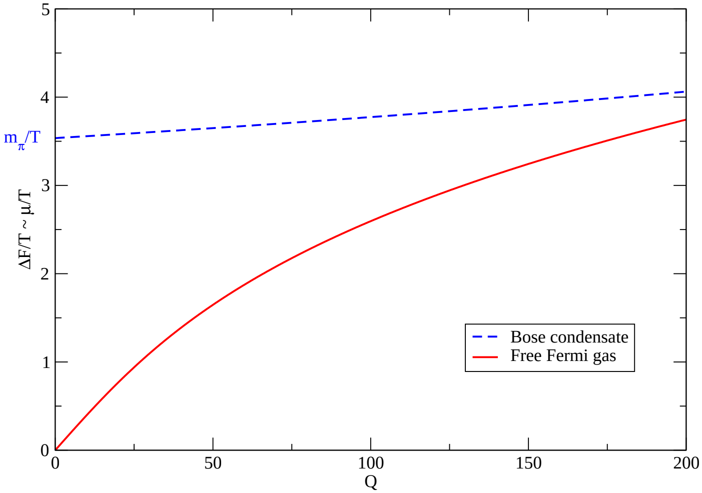
$$\frac{P(\mu_I) - P(\mu_I = 0)}{T^4} = \frac{1}{2}\left(\frac{\mu_I}{T}\right)^2 + \frac{1}{4\pi^2}\left(\frac{\mu_I}{T}\right)^4 .$$



Lattice simulation details

- $N_f = 4+4$, i.e. 2 staggered fermions on $8^3 \times 4$ at $am = 0.14$:
  $\Rightarrow$ deconfinement transition at $\mu = 0$ is 1^{st} order
- Temperature range $\frac{T}{T_c} \sim [\frac{1}{2}, 1]$
- Pion mass am_π changes only by few percent:
  $\Rightarrow m_\pi/T \sim$ constant
- Combine 69 ensembles at 6 values of μ up to $\mu/T = 4$ with Ferrenberg-Swendsen reweighting.
- No $U(1)$ breaking term (à la Kogut-Sinclair):
  - maintain importance sampling
  - order parameter $\frac{1}{V}\chi_{\pi^-}$, $\chi_{\pi^-} \equiv \langle \sum_x \pi^-(0)\pi^-(x) \rangle$

Free energy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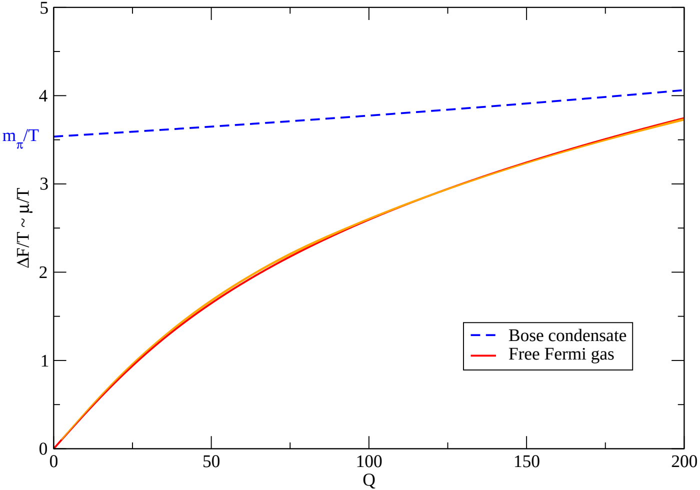
ΔF/T ~ μ/T
m_π/T
Bose condensate
Free Fermi gas

Free energy



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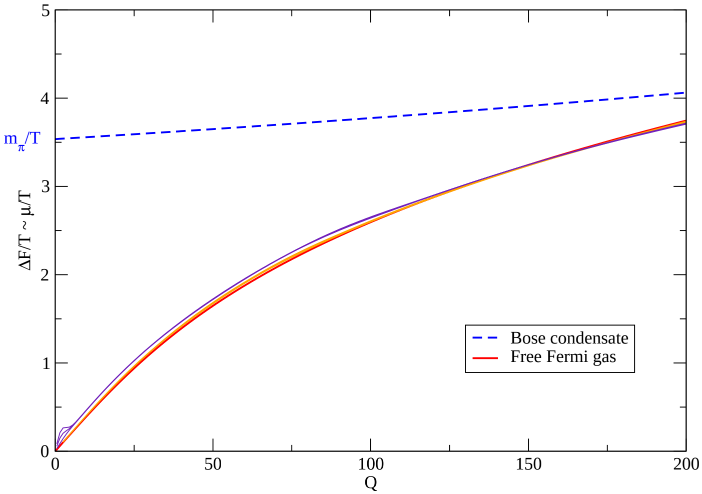
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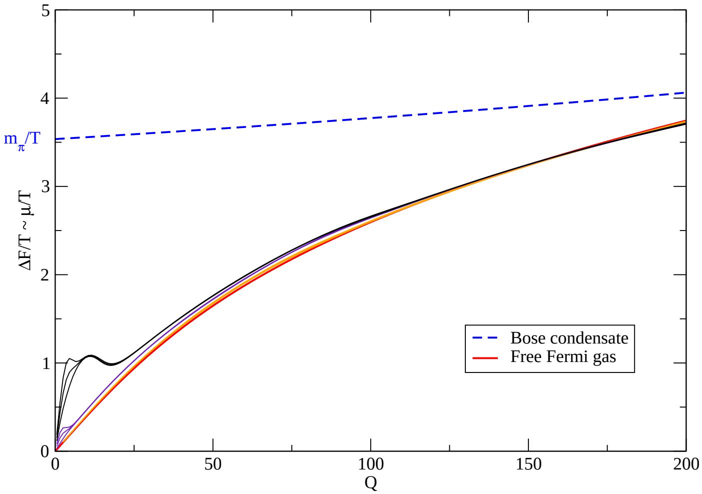
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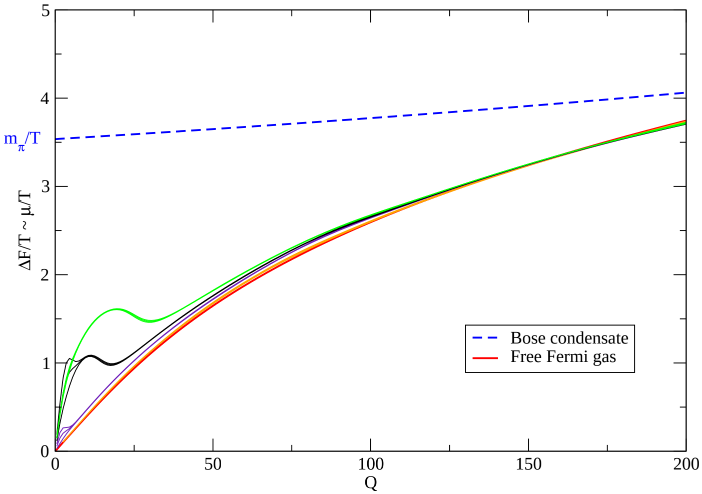

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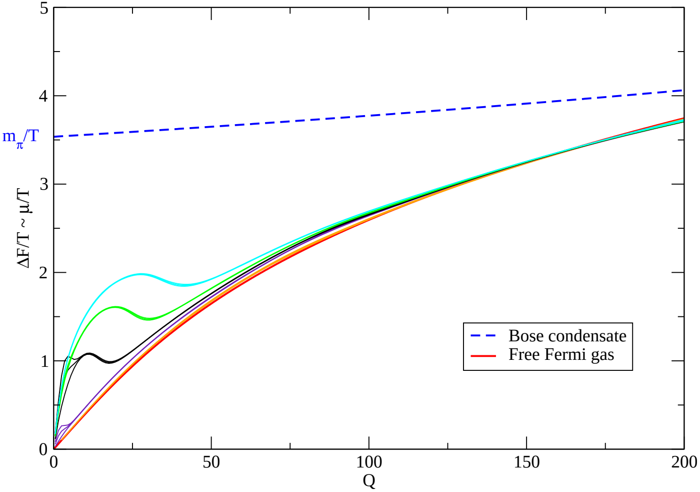
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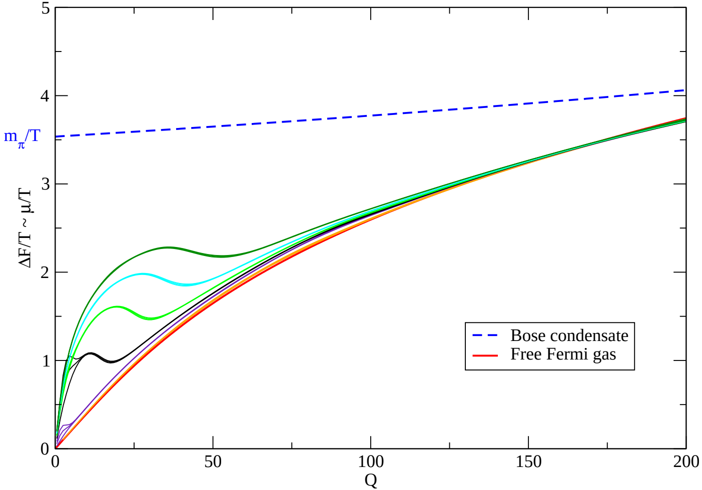
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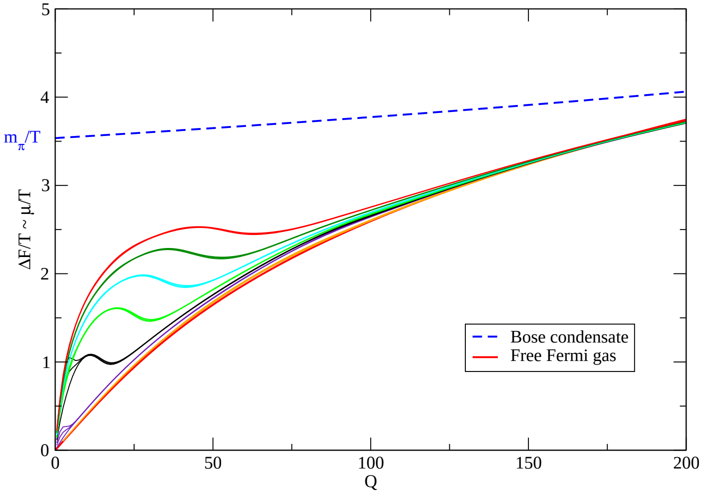
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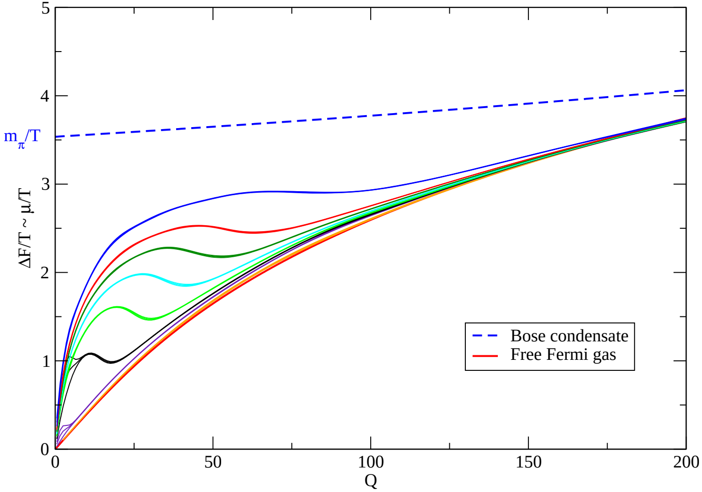
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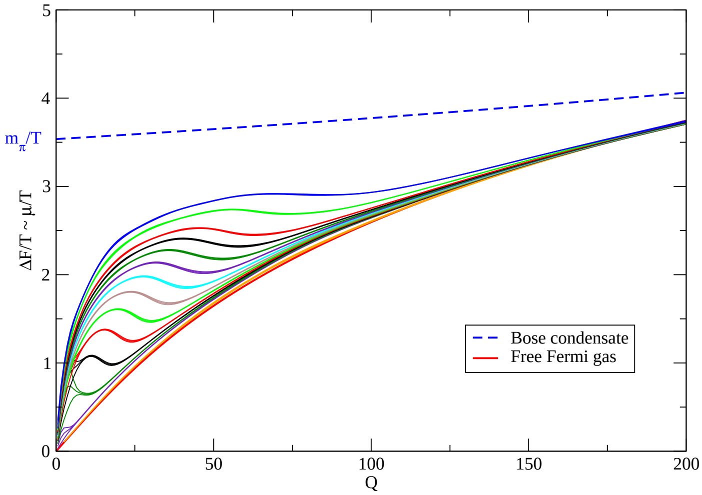

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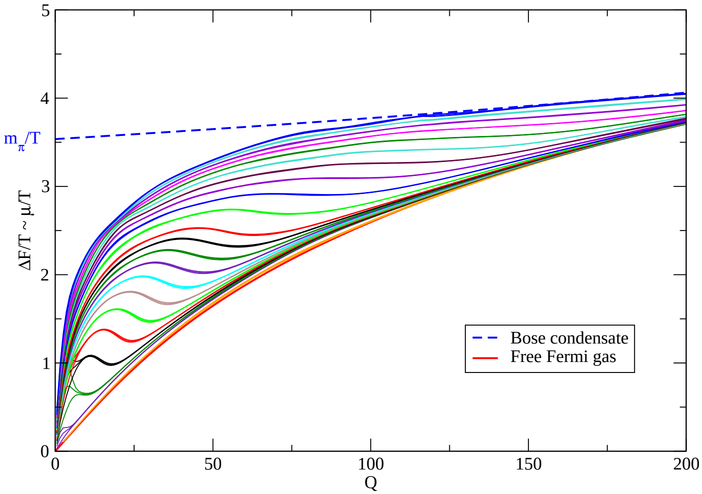

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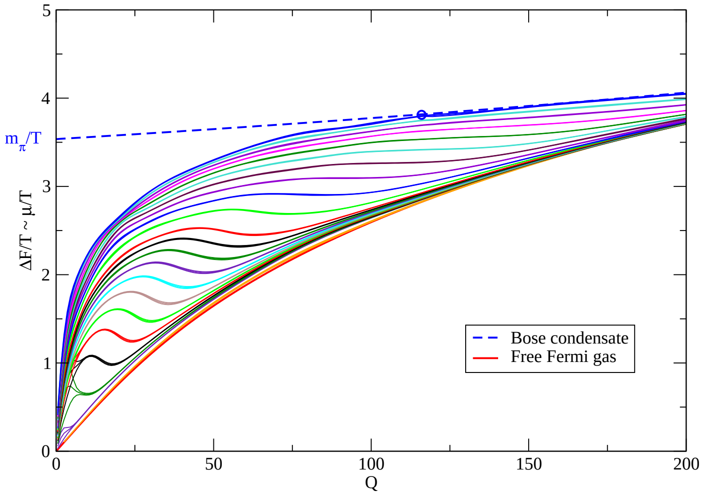

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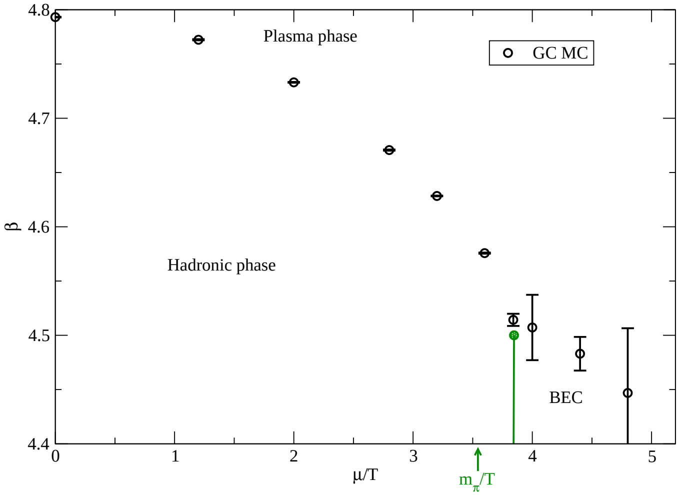
Phase diagram



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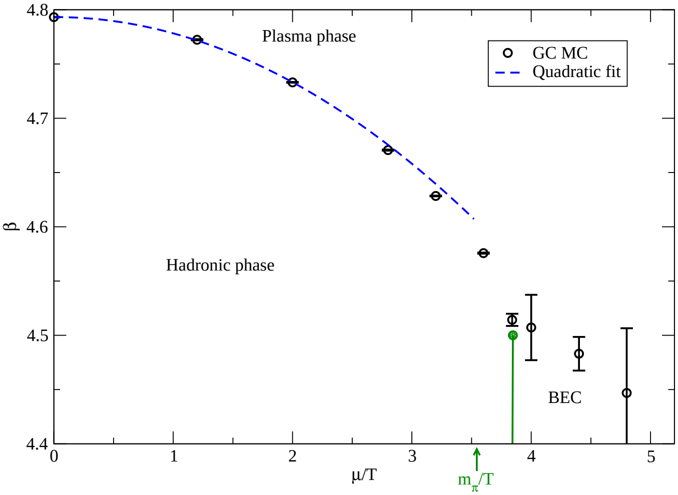

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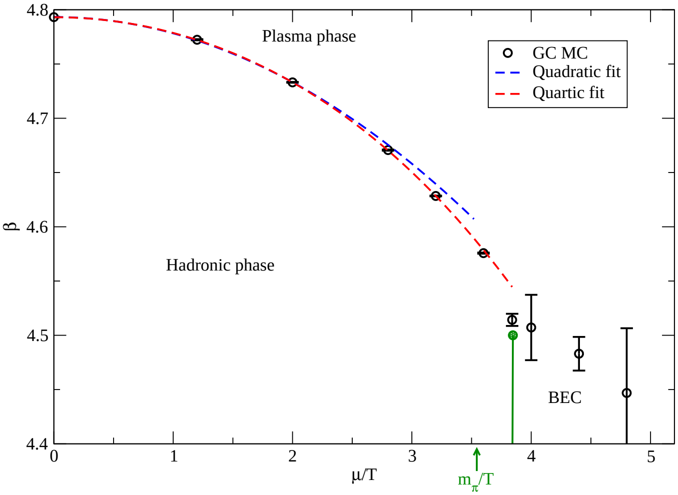

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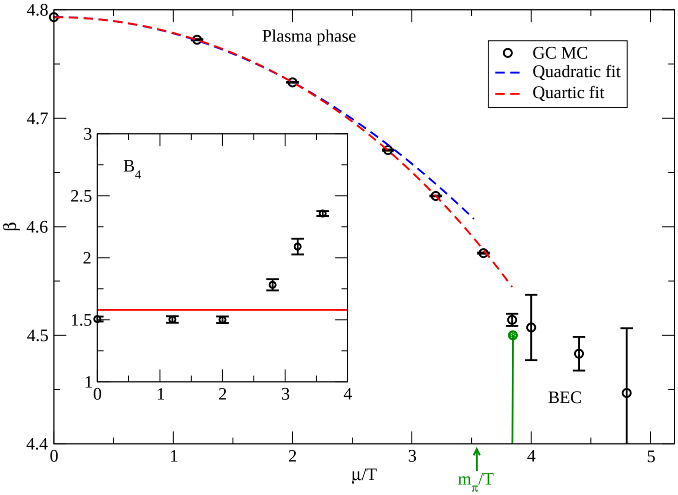

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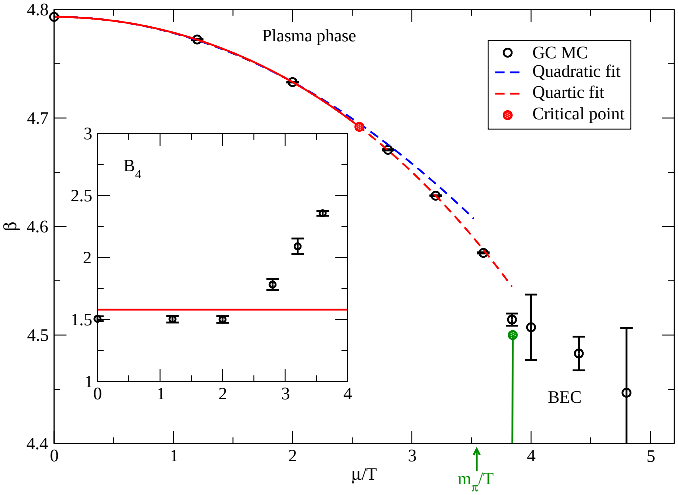
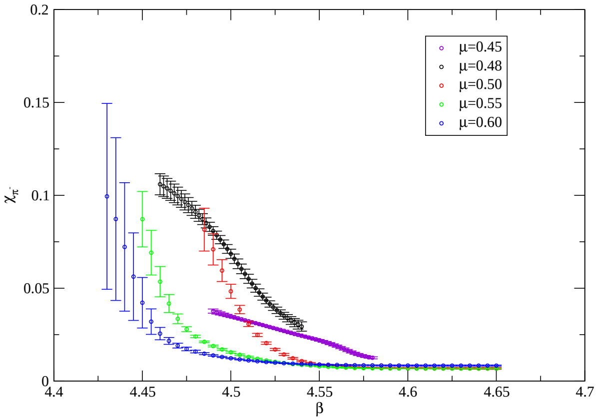


Bose condensation

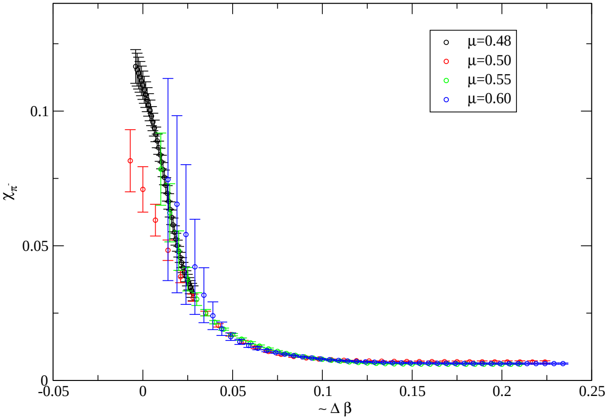
- Transition BEC $\leftrightarrow$ Fermi gas:

  $\Rightarrow$ measure order parameter: pion susceptibility χ_{π^-}



Bose condensation

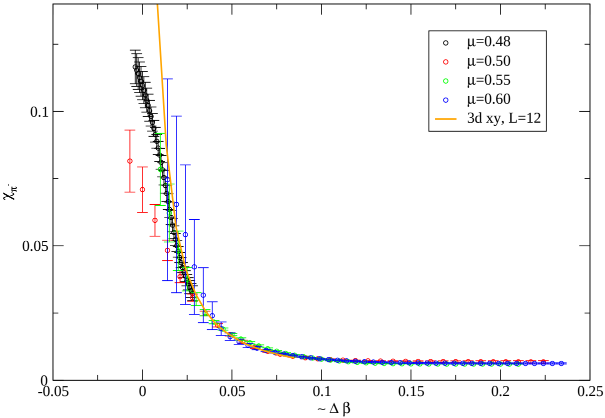
- Rescale to recover universal behaviour:



⇒ good agreement

Bose condensation

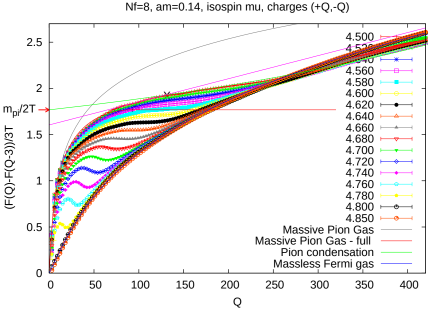
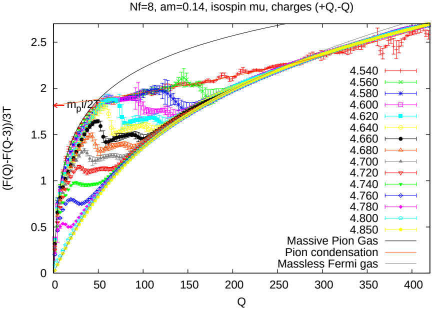
- Universality class of the 3*d* xy-model:



$\Rightarrow$ good agreement

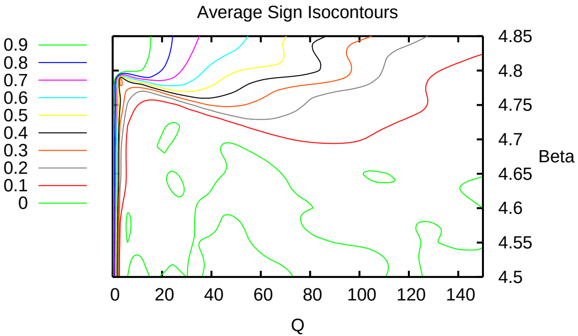
Lessons for finite baryon density

- Reweighting from $\mu = 0$ ensembles alone gives unreliable results

full	from $\mu = 0$ only
	

Lessons for finite baryon density

- Average sign of the determinant smaller than commonly believed



Reweighting from isospin to baryonic μ over very limited range
(Onset of BEC phase at $(Q,\beta) \sim (120, 4.52)$)
To be expected: $Z_{\text{baryon}}(\beta, \frac{1}{3}\mu_B) \ll Z_{\text{isospin}}(\beta, \frac{1}{2}\mu_I)$ at low T

Summary

- We determined the EoS and the phase diagram of $N_f = 4+4$ QCD at finite isospin density and finite temperature.
- We exposed the two mechanisms at work:
  - Bose condensation at high density,
  - deconfinement at high temperature (first-order $\longrightarrow$ crossover).
- Implications for the baryonic density case.