Random matrices and the QCD sign problem

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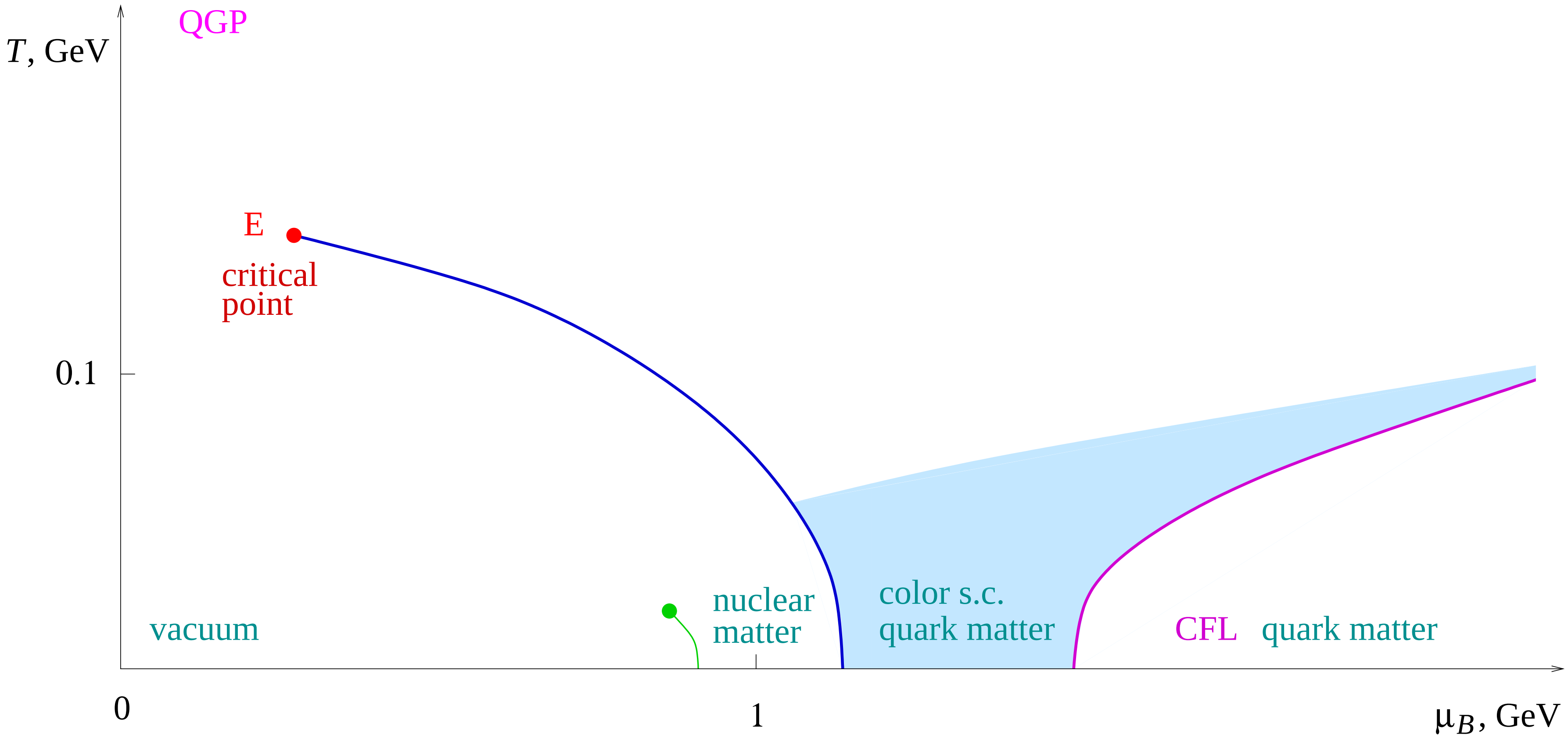
.

with J. Han

QCD Phase Diagram



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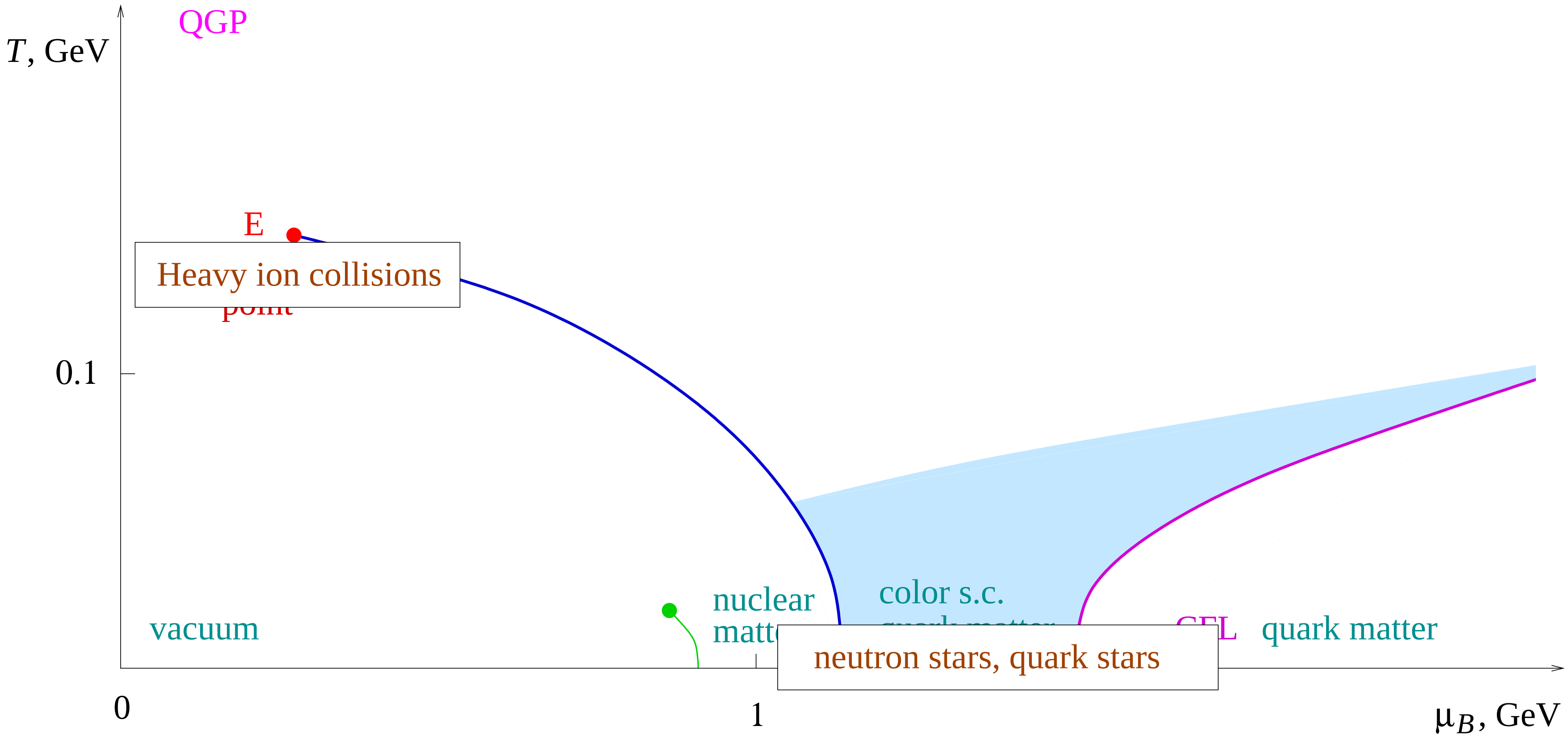
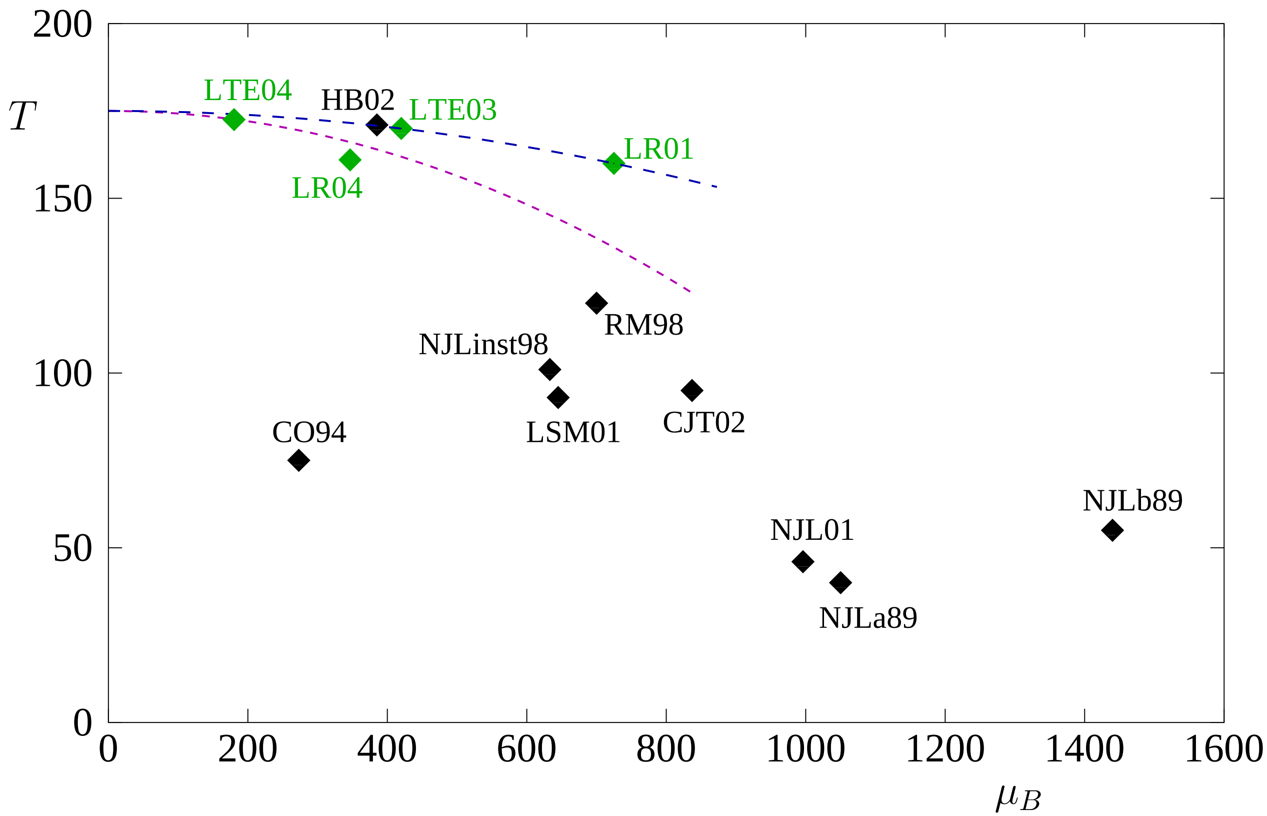


Locating the QCD critical point



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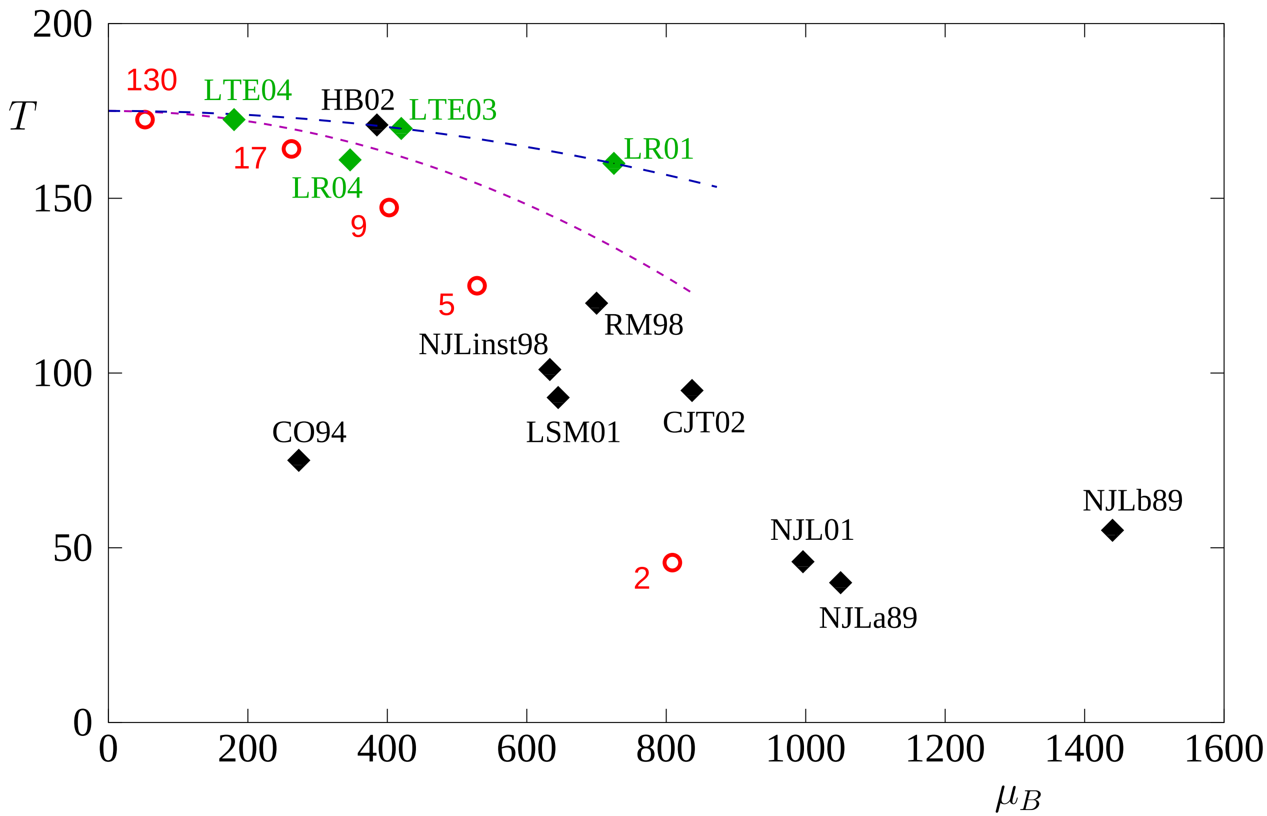

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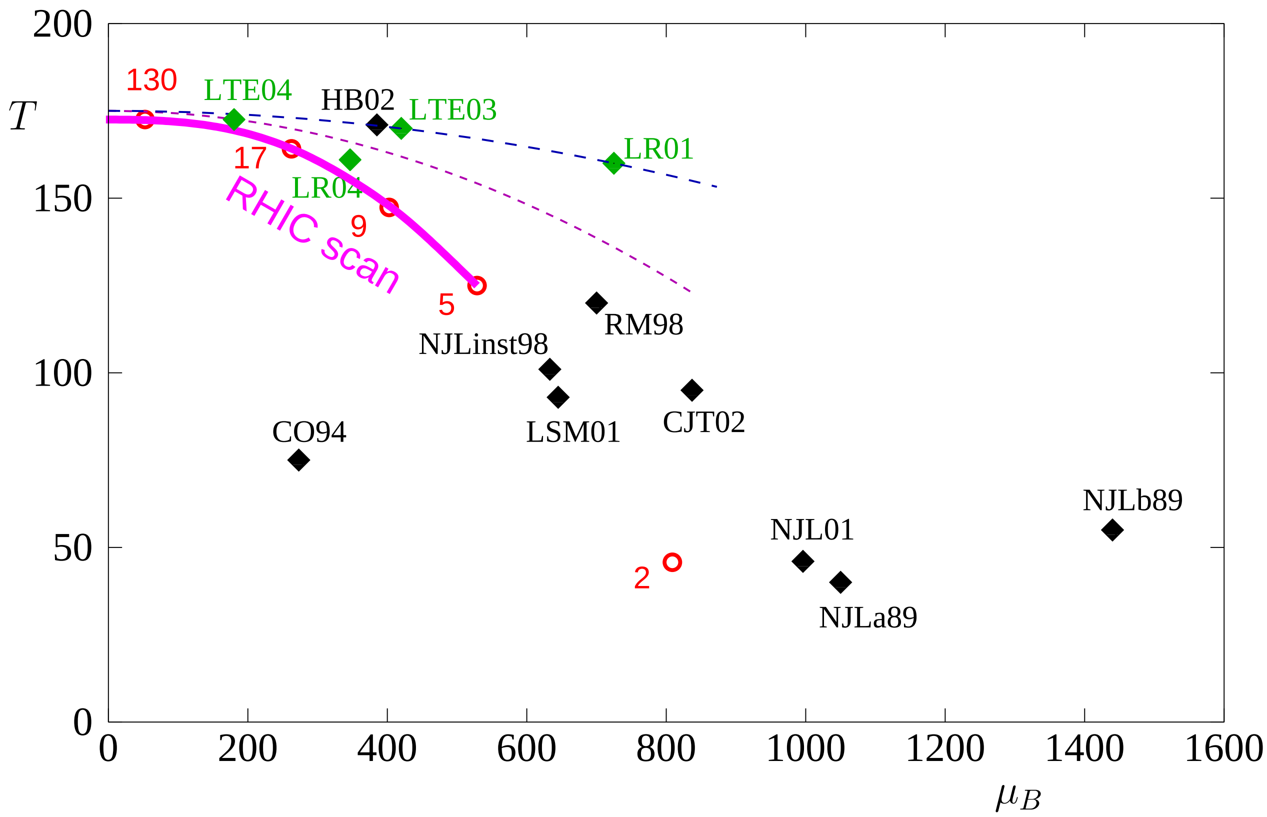
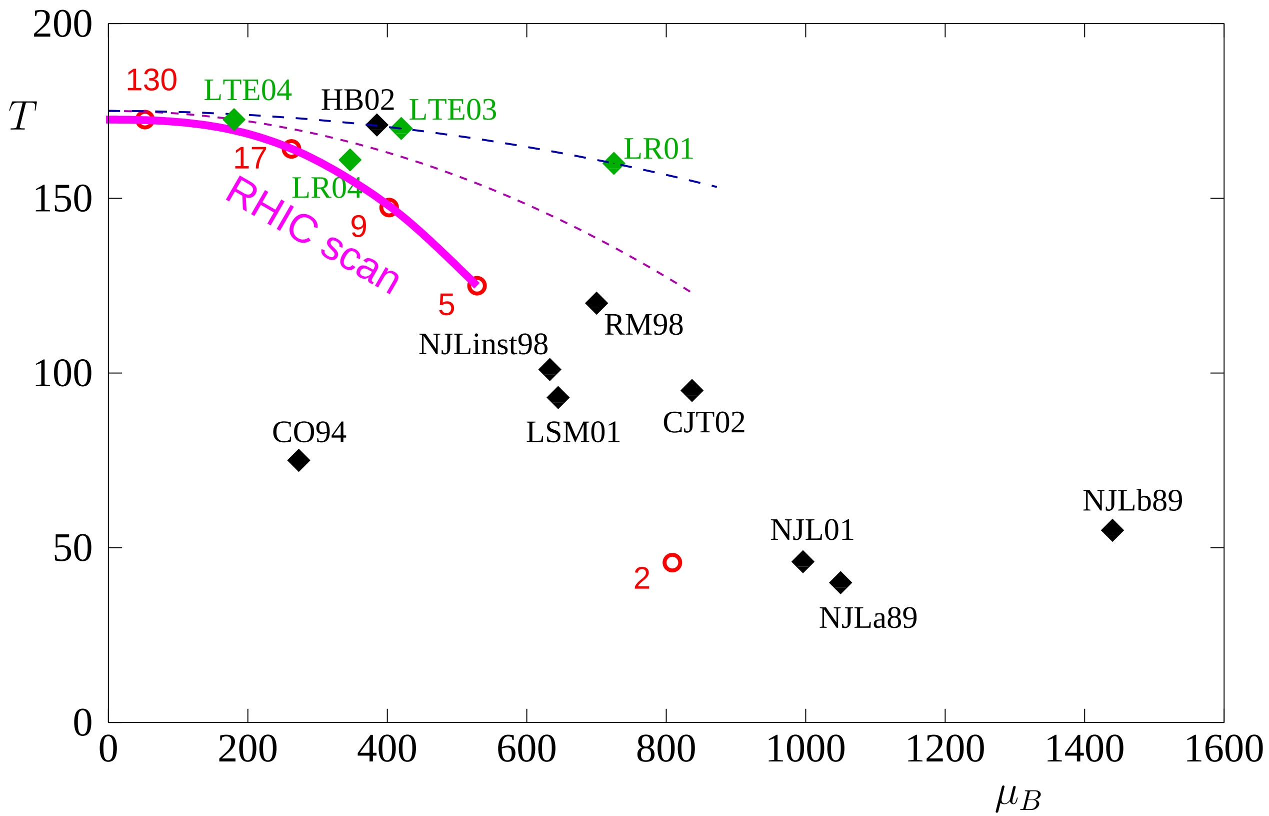


Experiments can scan the phase diagram by changing $\sqrt{s}$: RHIC, SPS, FAIR.

Signatures: event-by-event fluctuations.

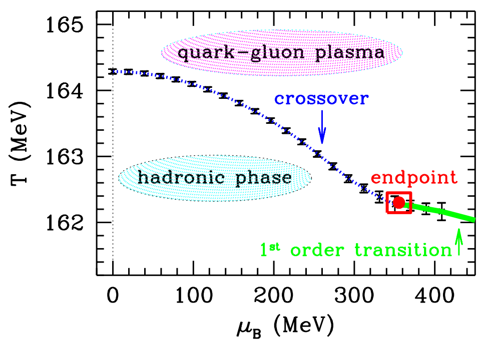
Susceptibilities diverge $\Rightarrow$ fluctuations grow towards the critical point.

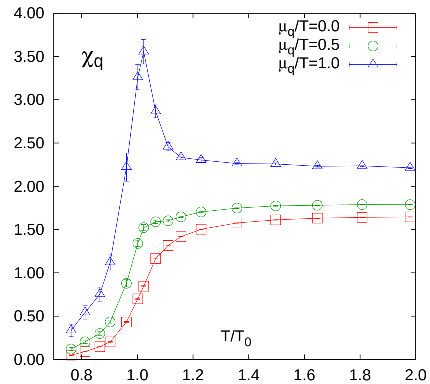
Locating the QCD critical point



Sign problem. Lattice:

- Reweighting $\Leftarrow$
- Taylor expansion
- Imaginary μ
- Canonical





The phase of the Dirac determinant

Let $\det \mathbb{M} = |\det \mathbb{M}|\, e^{i\theta}$. The *average phase factor*:

$$\langle e^{i2\theta} \rangle_{1+1^*} = \frac{\langle e^{i2\theta} |\det \mathbb{M}|^2 \rangle_0}{\langle |\det \mathbb{M}|^2 \rangle_0} = \frac{Z_{1+1}}{Z_{1+1^*}} \equiv R$$

- R measures the severity of the sign problem.

In QCD:

$$R(T,\mu) = \frac{Z_{1+1}}{Z_{1+1^*}} = \frac{e^{VP_{1+1}/T}}{e^{VP_{1+1^*}/T}}$$

For example, when $T \ll m_\pi$,

$$P_{1+1^*} \sim \mu^2 e^{-m_\pi/T}$$

while $$P_{1+1} \sim \mu^2 e^{-m_N/T} \ll P_{1+1^*}$$ (⇒ Cohen: $m_N \geq 3/2\, m_\pi$)

Thus

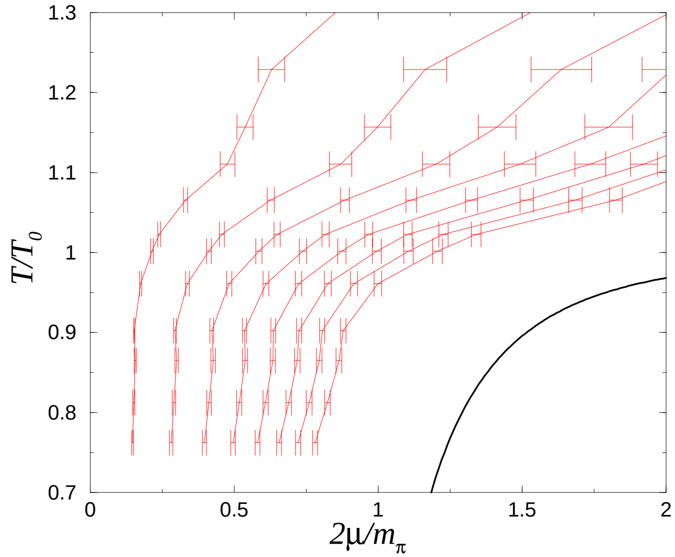
$$R \sim \exp\left(-V\mu^2 e^{-m_\pi/T}\right) \to 0 \quad \text{as} \quad V \to \infty$$ (Splittorff)

R and the severity of the sign problem

- In a finite volume V (as in lattice simulations) R is also finite.

- In a MC calculation, when R becomes small, noise may cause spurious zeros in $Z_{1+1} \sim R$, which might be misidentified as Lee-Yang zeros. (Ejiri)

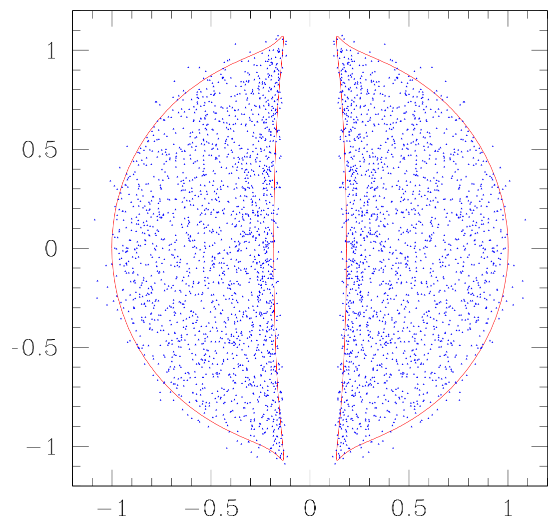
- These fluctuations are large when $1+1^*$ approaches phase transition to pion condensation. (Splittorff)



- This happens because μ enters the domain of eigenvalues of Dirac operator in μ-plane (right):

$$\det \mathbb{M} = \prod_i (\mu - \mu_i) = 0.$$

Small fluctuation in the position of an eigenvalue μ_i translates into a large change in phase of $\det \mathbb{M}$.



R and pion condensation boundary in RMM

To guard against possible misidentification of the critical point it is important to know where the boundary of pion condensation occurs at $T \neq 0$.

An approach: use RMM to study the behavior of $R(T, \mu)$.

$$Z_{1+1} = \langle \det{}^2 \mathbb{M} \rangle_0 = \int \mathcal{D}X \, e^{-N \operatorname{Tr} XX^\dagger} \det{}^2 \mathbb{M}$$

where $\mathbb{M}$ is the $2N \times 2N$ matrix approximating the Dirac operator:

$$\mathbb{M} = \begin{pmatrix} 0 & iX + C \\ iX^\dagger + C & 0 \end{pmatrix} + m + \mu\gamma_0; \qquad C = \underbrace{iT \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}_{\text{“Matsubara”}},$$

X is $N \times N$ complex random matrix. $N \to \infty$ corresponds to thermodynamic limit.

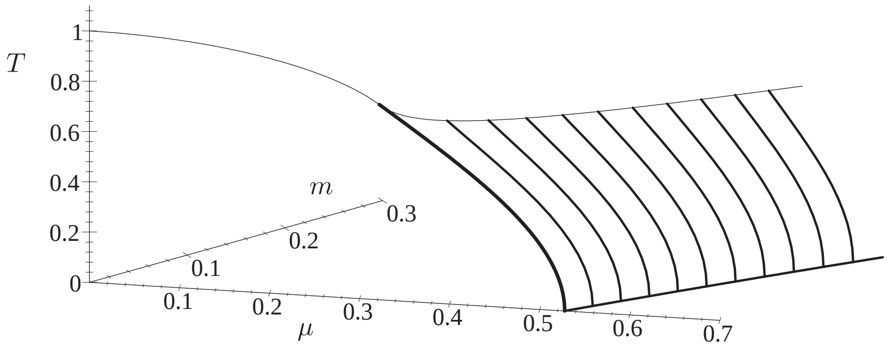
$$Z_{1+1^*} = \langle \det \mathbb{M} \det \mathbb{M}^* \rangle_0$$

Properties of the Random Matrix Model

- For $\mu \neq 0 \quad \det \mathbb{M}$ is complex $\Rightarrow$ sign problem.
- Solvable *analytically*.

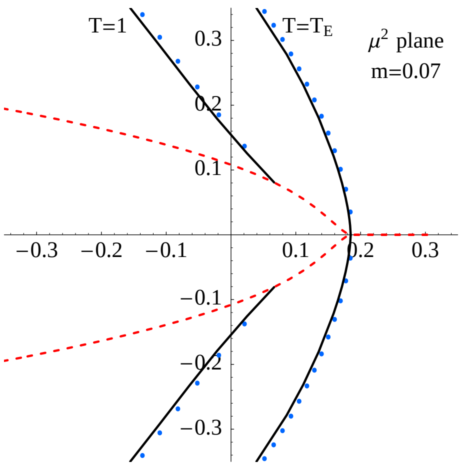
Examples:

- Phase diagram (1+1):



(Halasz *et al*)

- Complex μ singularities
(Taylor exp. convergence radius)



Analytical solution of RMM

- After Hubbard-Stratonovich:

$$Z_{1+1} = \int \mathcal{D}A\, e^{-N\,\mathrm{tr}\,AA^\dagger} \det{}^{\frac{N}{2}} \begin{pmatrix} A+m & \mu + i\pi T \\ \mu + i\pi T & A^\dagger + m \end{pmatrix} \times \text{(same with } T \to -T\text{)}$$

where A is complex 2×2 (i.e., $N_\mathrm{f} \times N_\mathrm{f}$) matrix.

$$Z_{1+1^*} = Z_{1+1}\Big|_{\mu \to \mu\tau_3}; \qquad\qquad (\mathbb{M}^* = \mathbb{M}\Big|_{\mu \to -\mu})$$

- Define $\Omega(A)$: $Z = \int \mathcal{D}A\, e^{-N\Omega(A)}$, $N \to \infty$ dominated by saddle point of $\Omega(A)$:

$$A - \frac{(A+m)[(A+m)^2 - \mu^2 + T^2]}{[(A+m)^2 - \mu^2 + T^2]^2 + 4\mu^2 T^2} = 0$$

- This saddle point is the same for $1+1$ and $1+1^*$ RMM (outside the pion condensation domain) and also $\min \Omega_{1+1} = \min \Omega_{1+1^*}$. I.e.

$$R \sim \frac{e^{-N\Omega_{1+1}}}{e^{-N\Omega_{1+1^*}}} \to e^{0\cdot N} \sim 1 \qquad \text{as } N \to \infty\text{, not 0.}$$

Analytical solution of RMM (contd.)

Need second derivative matrix $\frac{\partial^2 \Omega}{\partial A_{..} \partial A_{..}} \equiv \Omega''_{....}$:

$$R = \frac{Z_{1+1}}{Z_{1+1^*}} = \left(\frac{\det \Omega''_{1+1}}{\det \Omega''_{1+1^*}} \right)^{-1/2} = \left| \frac{b_3^2 - b_4^2}{b_1^2 - b_2^2} \right|$$

where

$$b_1 = \frac{(A+m)^2}{W} \left(1 - \frac{8T^2\mu^2}{W} \right)$$

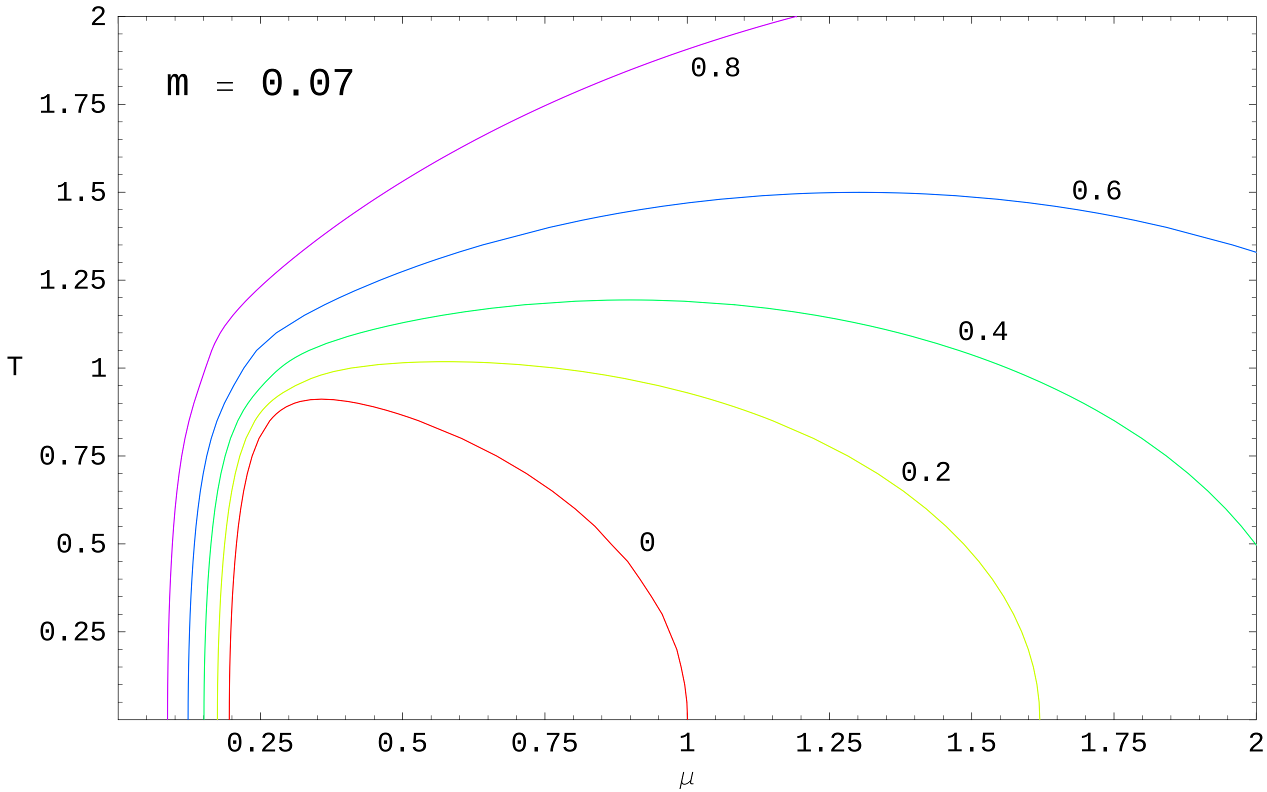
$$b_2 = 1 - \frac{T^2 - \mu^2}{W} - \frac{8T^2\mu^2(A+m)^2}{W^2}$$

$$b_3 = 1 - \frac{T^2 + \mu^2}{W}$$

$$b_4 = \frac{(A+m)^2}{W}$$

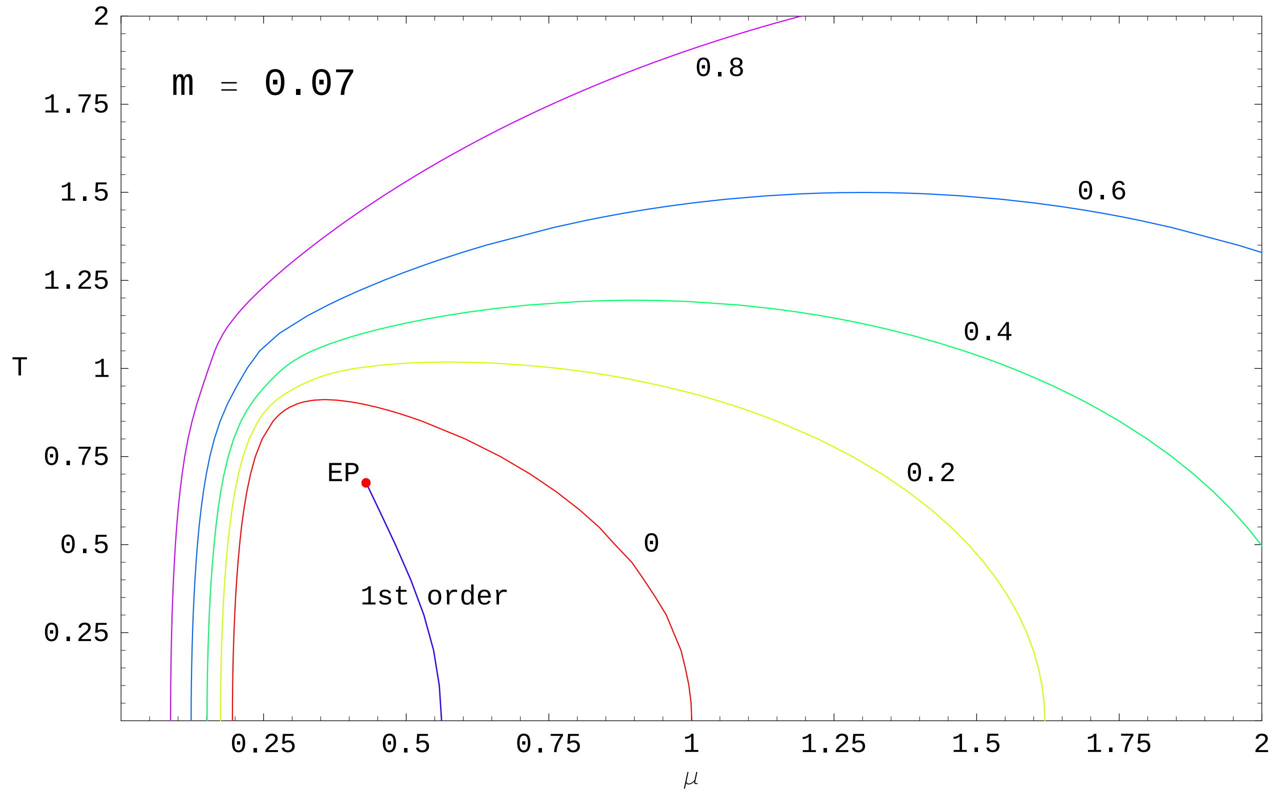
$$W = (A+m)^4 + 2(A+m)^2(T^2 - \mu^2) + (T^2 + \mu^2)^2$$

$R(T,\mu)$ contour plot



- Sign problem is less severe at higher temperature :)

$R(T, \mu)$ contour plot



- Sign problem is less severe at higher temperature :)
- First order transition of 1+1 is inside the $R = 0$ boundary :(

Interesting limits

- T=0, small m and $\mu \sim \sqrt{m}$ (Splittorff, Verbaarschot)

$$R \approx 1 - \frac{2\mu^2}{m} = 1 - \left(\frac{2\mu}{m_\pi}\right)^2 .$$

- $T < 1$, small m and $\mu \sim \sqrt{m}$

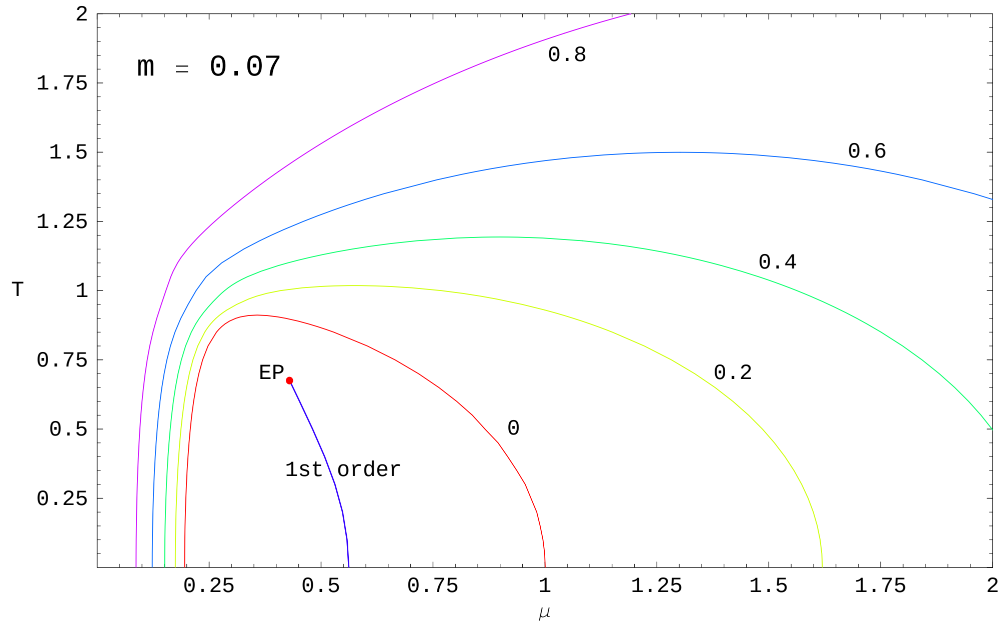
$$R \approx 1 - \sqrt{1-T^2}\left(\frac{2\mu}{m_\pi}\right)^2$$

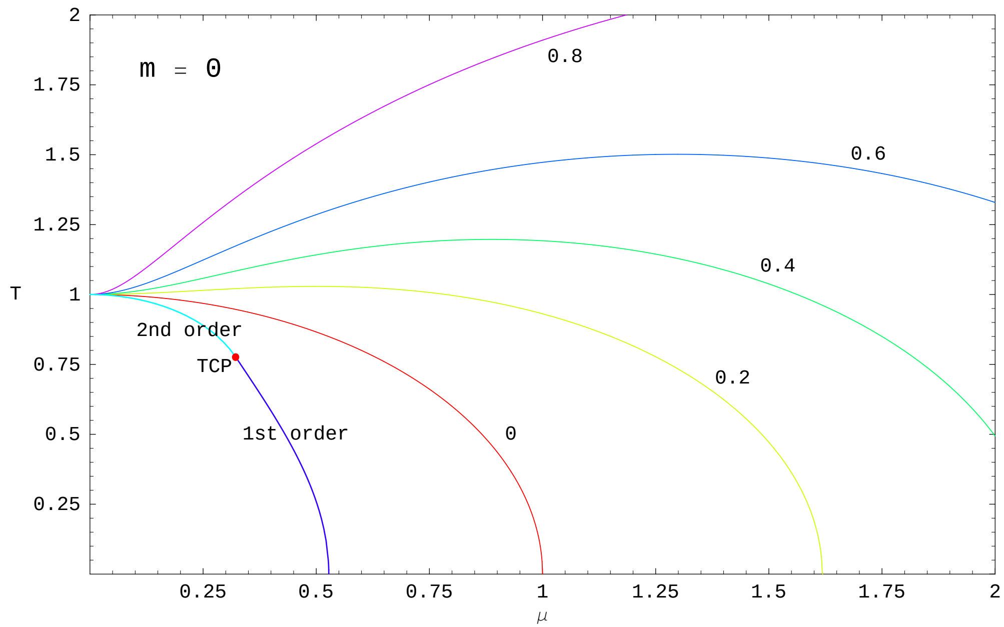
sign problem weakens with T.

- Chiral limit ($m = 0$), any μ, T

$$R = \frac{[(T^2+\mu^2)^2 - (T^2+\mu^2)]^2}{[(T^2+\mu^2)^2 - (T^2-\mu^2)]^2}$$

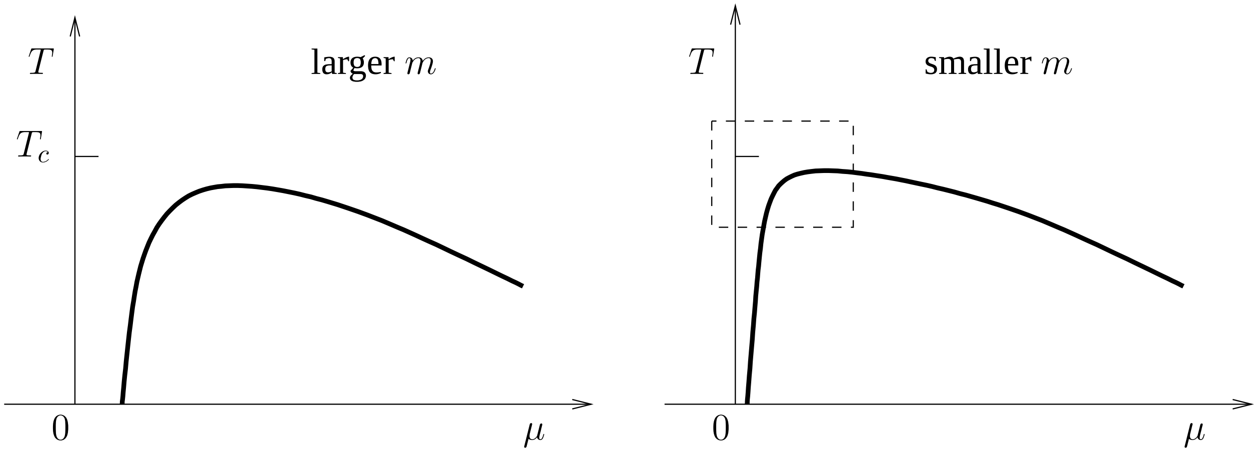
$R = 0$ in a 90° pie: $T^2 + \mu^2 < 1$.





$R = 0$ boundary near $T = 1, \mu = 0$

How does the $R = 0$ (pion condensation) boundary approach $T = 1$, $\mu \to 0$ as $m_\pi \to 0$?



- In RMM, expanding the analytic solution (notation: $t \equiv T^2 - 1$)

$$A^3 + A(t + 3\mu^2) - m = 0$$

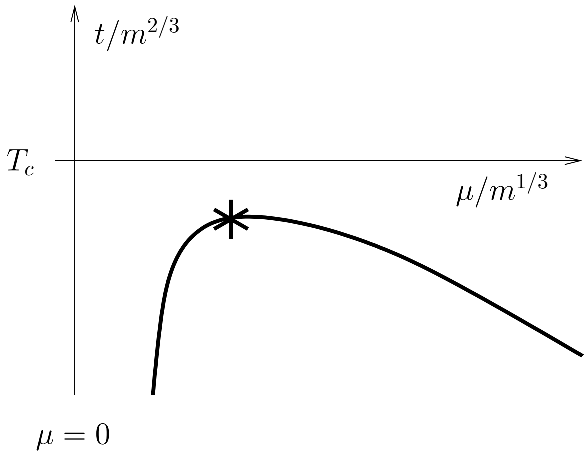
i.e. at $m = 0$ $A \sim (-t - 3\mu^2)^{1/2} = (t_c - t)^{1/2}$ or $A \sim m^{1/3}$ at $t = t_c$.

- The $R = 0$ curve is

$$T^2 = 1 - \mu^2 - \frac{m^2}{4\mu^4} = 1 - m^{2/3} F\left(\frac{\mu}{m^{1/3}}\right),$$

with $F(x) = x^2 + 1/(4x^2)$ – a scaling function.

Pion condensation boundary as $m_\pi \to 0$ – scaling



In RMM: $T_c - T_* \sim m^{2/3}$ and $\mu_* = m^{1/3}$ (slower than $m_\pi \sim m^{1/2}$).

In QCD: $T_c - T_* \sim m^{1/(\beta\delta)}$ and $\mu_* = m^{1/(2\beta\delta)}$?

Summary

- Using analytical solution of RMM we found $R = \langle e^{2i\theta} \rangle$ at finite T and μ.
- Sign problem is less severe at higher T.
- The $1+1$ phase transition is hidden inside the $R = 0$ (pion condensation) domain.
- As $m \to 0$ the domain $R = 0$ approaches $T = T_c$, $\mu = 0$ point in a self-similar way, with $\mu_* \sim m^{1/3}$ (in RMM) – slower than m_π.