

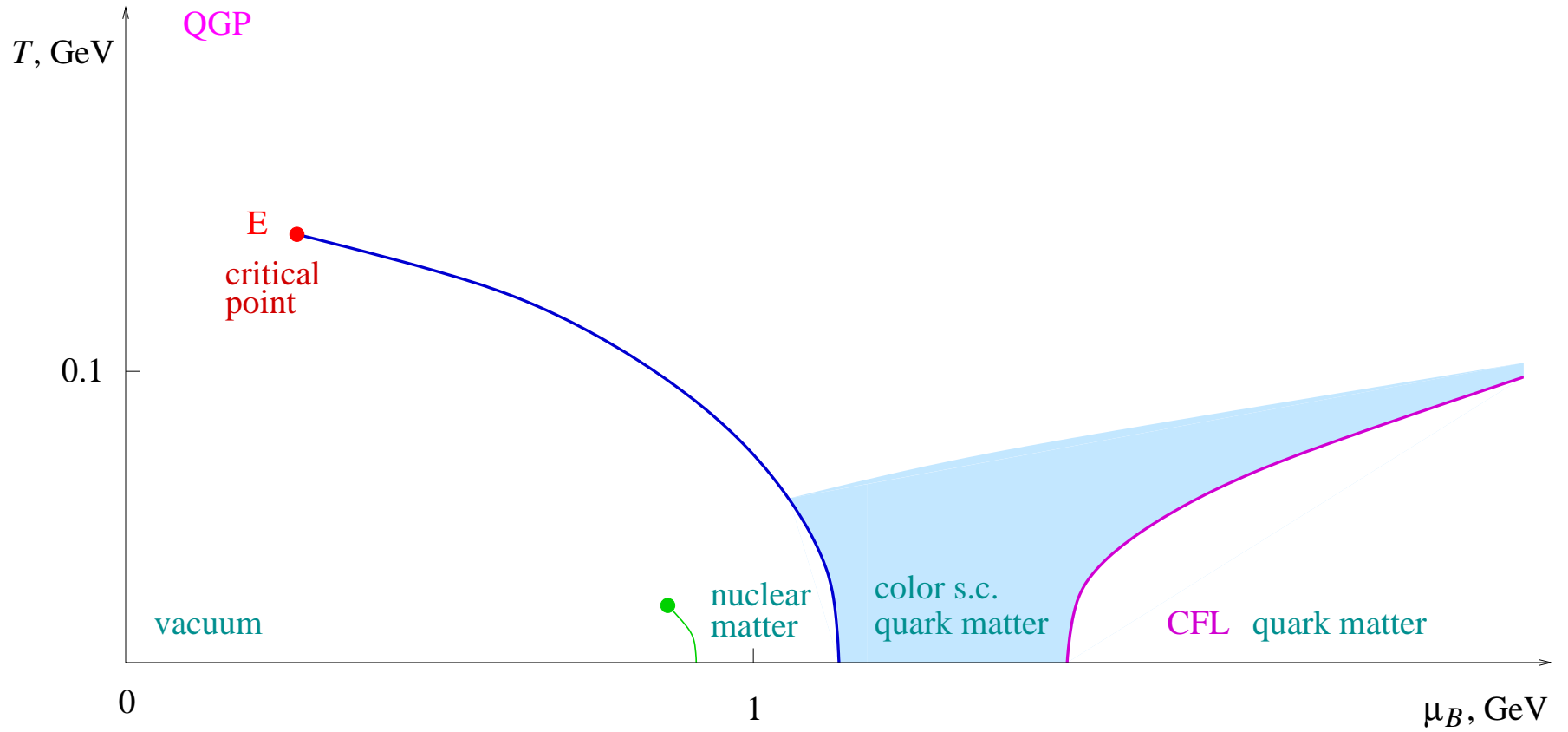
# Random matrices and the QCD sign problem

M. Stephanov

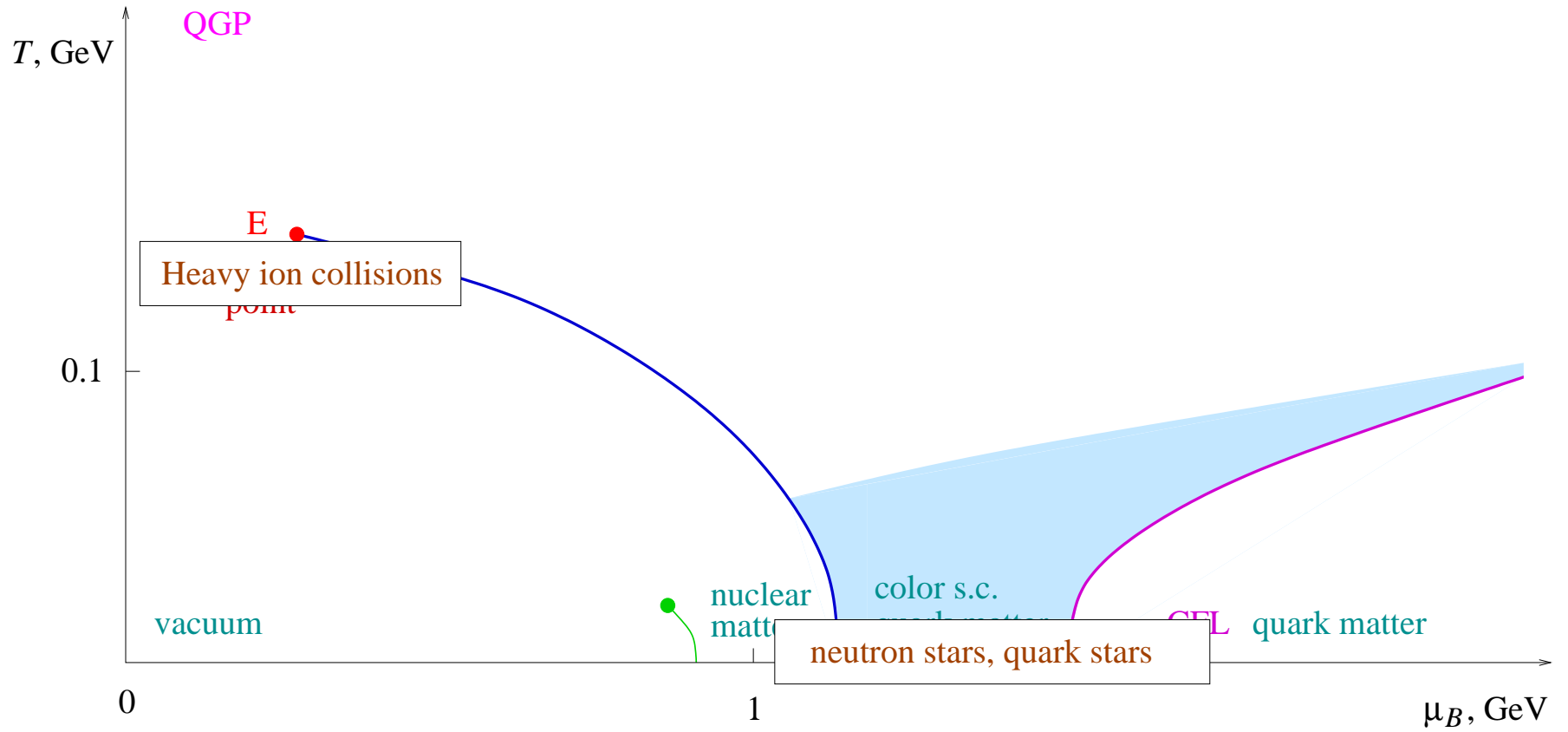
*U. of Illinois at Chicago*

with J. Han

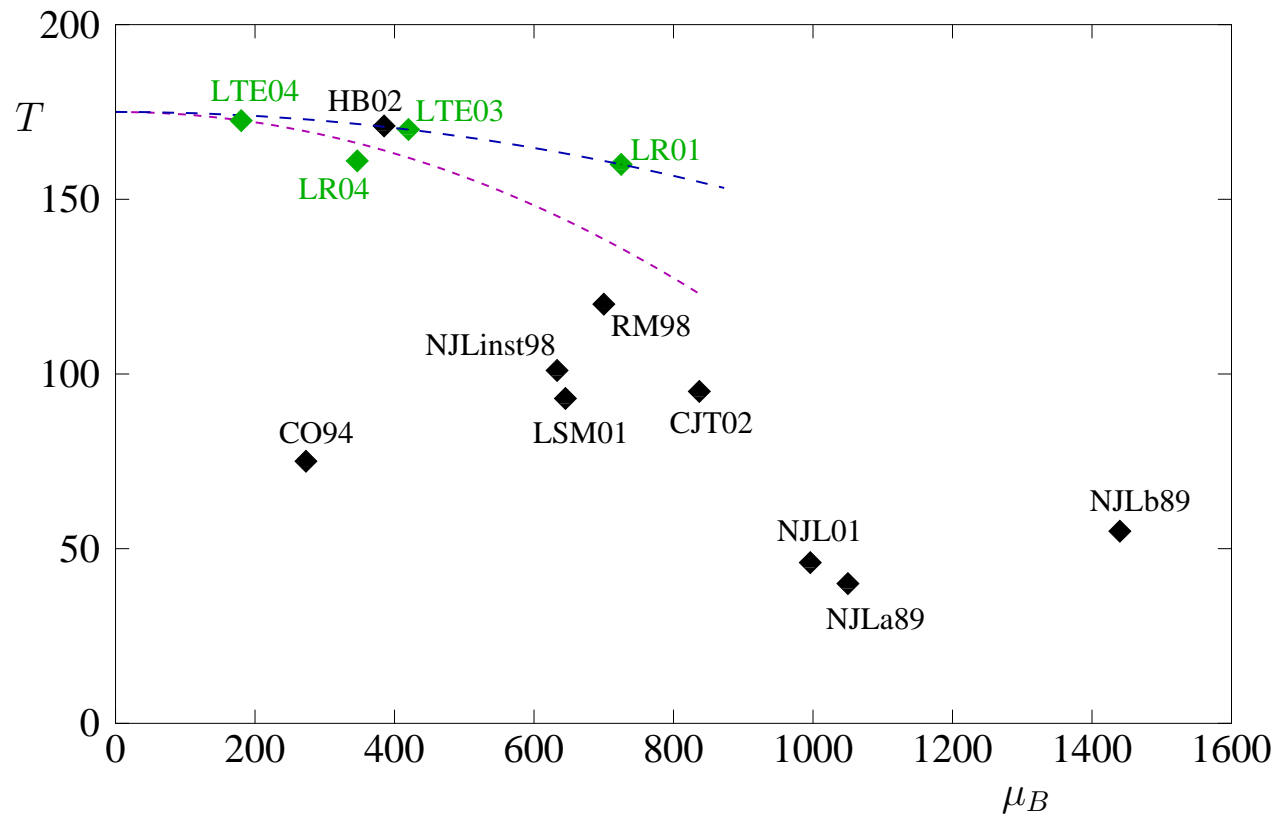
# QCD Phase Diagram



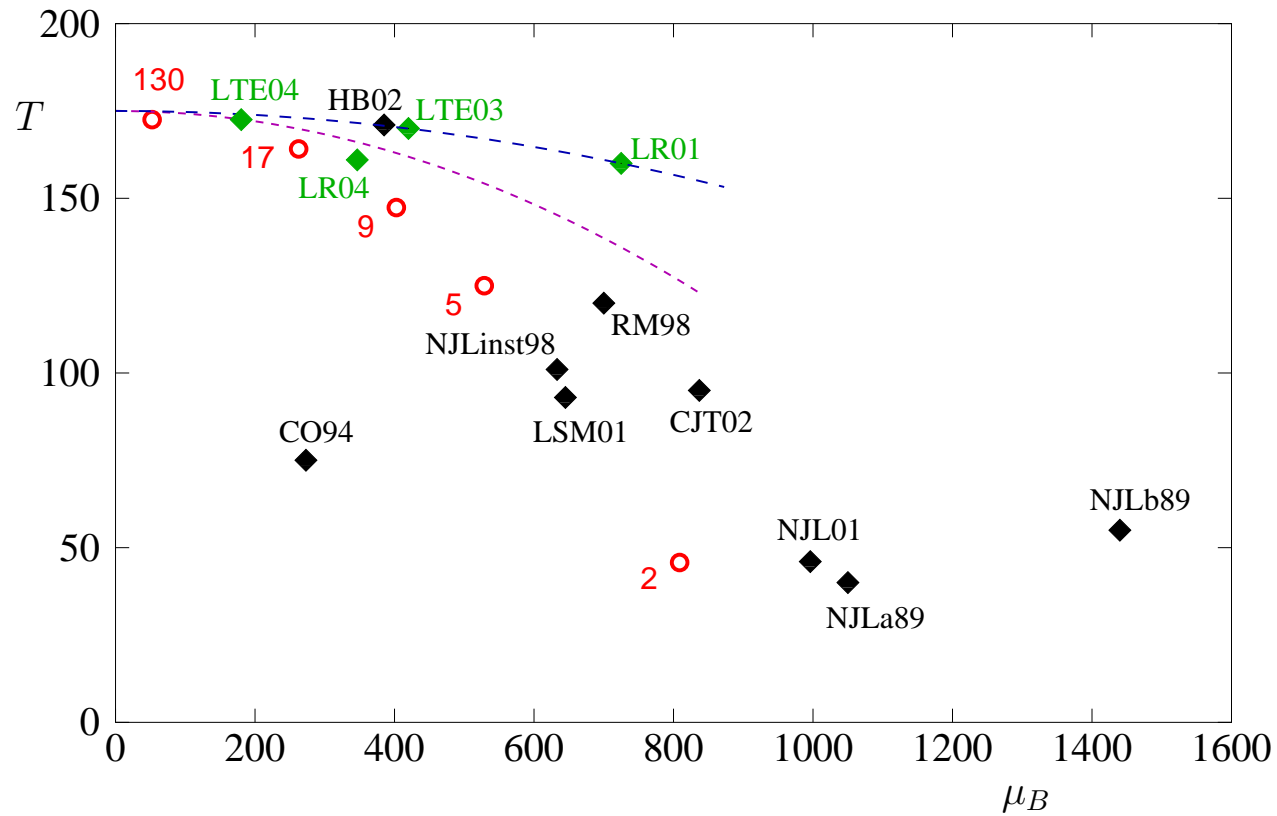
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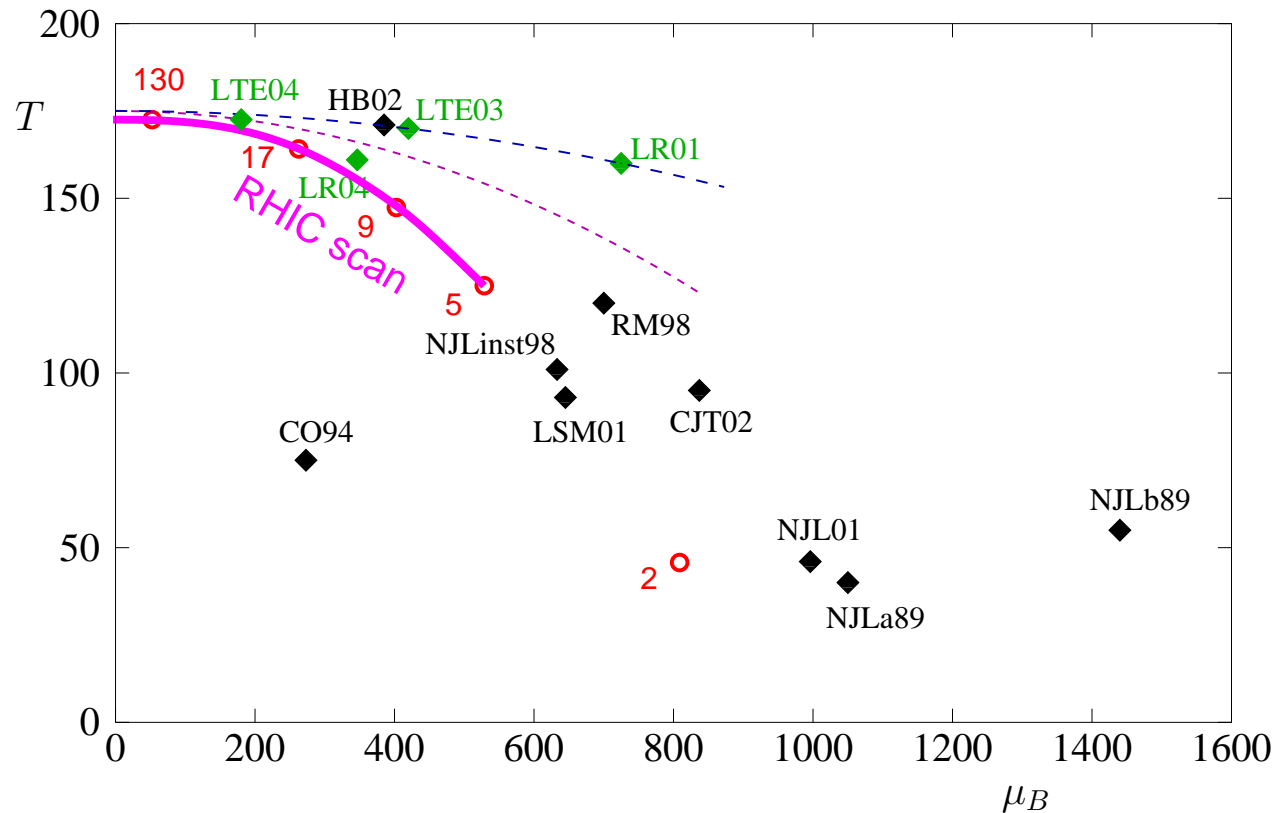
# Locating the QCD critical point



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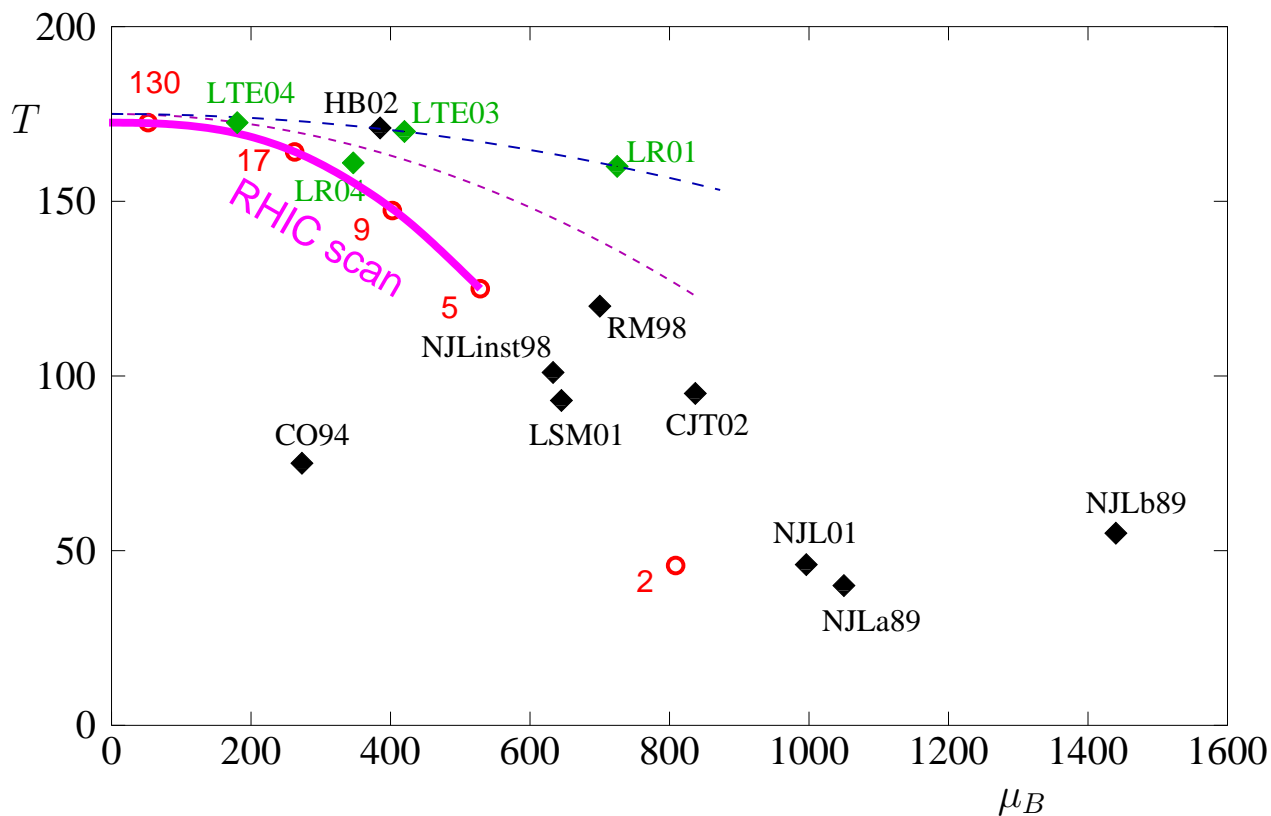


Experiments can scan the phase diagram by changing  $\sqrt{s}$ : RHIC, SPS, FAIR.

Signatures: event-by-event fluctuations.

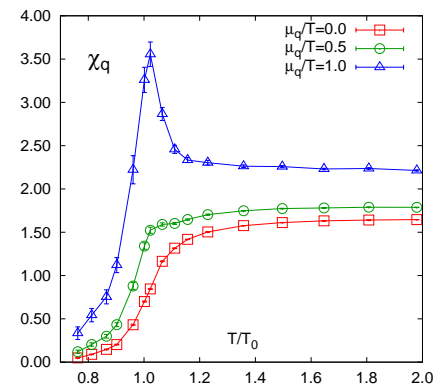
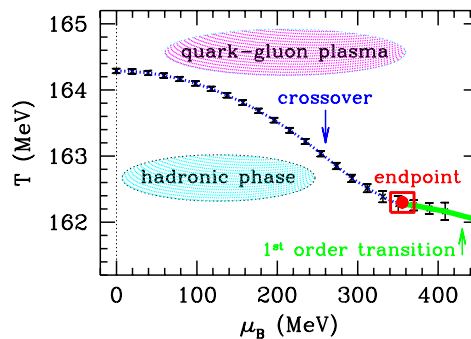
Susceptibilities diverge  $\Rightarrow$  fluctuations grow towards the critical point.

# Locating the QCD critical point



Sign problem. Lattice:

- Reweighting ⇐
- Taylor expansion
- Imaginary  $\mu$
- Canonical



# The phase of the Dirac determinant

Let  $\det \mathbb{M} = |\det \mathbb{M}| e^{i\theta}$ . The *average phase factor*:

$$\langle e^{i2\theta} \rangle_{1+1^*} = \frac{\langle e^{i2\theta} |\det \mathbb{M}|^2 \rangle_0}{\langle |\det \mathbb{M}|^2 \rangle_0} = \frac{Z_{1+1}}{Z_{1+1^*}} \equiv R$$

●  $R$  measures the severity of the sign problem.

In QCD:

$$R(T, \mu) = \frac{Z_{1+1}}{Z_{1+1^*}} = \frac{e^{VP_{1+1}/T}}{e^{VP_{1+1^*}/T}}$$

For example, when  $T \ll m_\pi$ ,

$$P_{1+1^*} \sim \mu^2 e^{-m_\pi/T}$$

while  $P_{1+1} \sim \mu^2 e^{-m_N/T} \ll P_{1+1^*}$  ( $\Rightarrow$  Cohen:  $m_N \geq 3/2 m_\pi$ )

Thus

$$R \sim \exp\left(-V\mu^2 e^{-m_\pi/T}\right) \rightarrow 0 \quad \text{as} \quad V \rightarrow \infty \quad (\text{Splittorff})$$

# $R$ and the severity of the sign problem

● In a finite volume  $V$  (as in lattice simulations)  $R$  is also finite.

● In a MC calculation, when  $R$  becomes small, noise may cause spurious zeros in  $Z_{1+1} \sim R$ , which might be misidentified as Lee-Yang zeros.

(Ejiri)

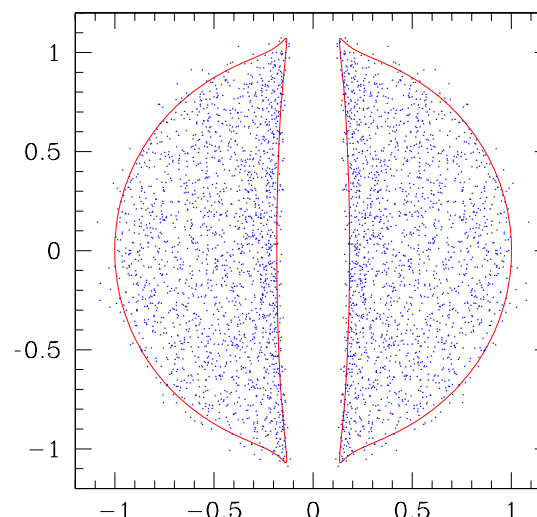
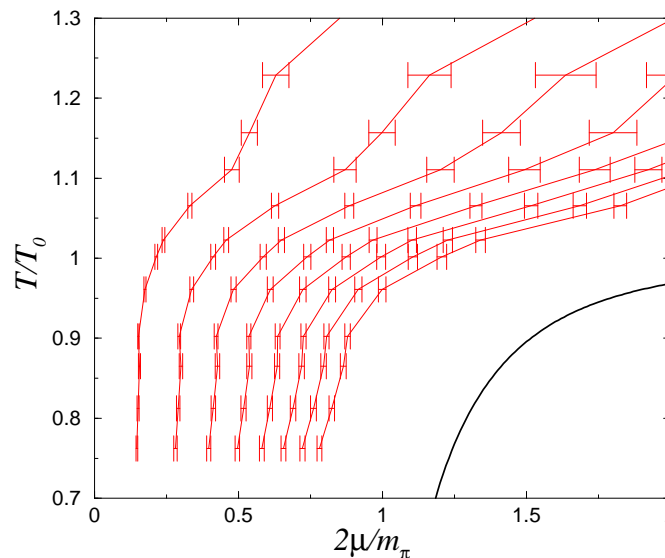
● These fluctuations are large when  $1+1^*$  approaches phase transition to pion condensation.

(Splittorff)

● This happens because  $\mu$  enters the domain of eigenvalues of Dirac operator in  $\mu$ -plane (right):

$$\det \mathbb{M} = \prod_i (\mu - \mu_i) = 0.$$

Small fluctuation in the position of an eigenvalue  $\mu_i$  translates into a large change in phase of  $\det \mathbb{M}$ .



# $R$ and pion condensation boundary in RMM

To guard against possible misidentification of the critical point it is important to know where the boundary of pion condensation occurs at  $T \neq 0$ .

An approach: use RMM to study the behavior of  $R(T, \mu)$ .

$$Z_{1+1} = \langle \det^2 \mathbb{M} \rangle_0 = \int \mathcal{D}X e^{-N \text{Tr} X X^\dagger} \det^2 \mathbb{M}$$

where  $\mathbb{M}$  is the  $2N \times 2N$  matrix approximating the Dirac operator:

$$\mathbb{M} = \begin{pmatrix} 0 & iX + C \\ iX^\dagger + C & 0 \end{pmatrix} + m + \mu\gamma_0; \quad C = \underbrace{iT \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}_{\text{“Matsubara”}},$$

$X$  is  $N \times N$  complex random matrix.  $N \rightarrow \infty$  corresponds to thermodynamic limit.

$$Z_{1+1}^* = \langle \det \mathbb{M} \det \mathbb{M}^* \rangle_0$$

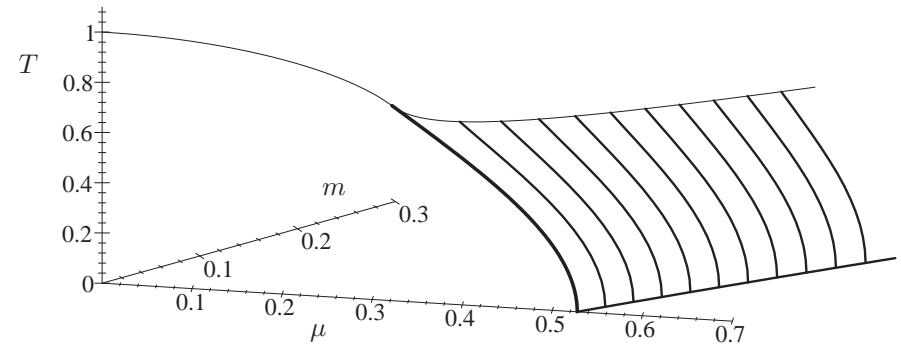
# Properties of the Random Matrix Model

● For  $\mu \neq 0$   $\det \mathbb{M}$  is complex  $\Rightarrow$  sign problem.

● Solvable *analytically*.

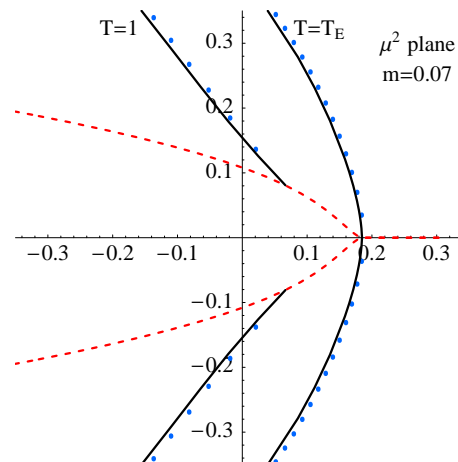
Examples:

● Phase diagram (1+1):



(Halasz *et al*)

● Complex  $\mu$  singularities  
(Taylor exp. convergence radius)



# Analytical solution of RMM

- After Hubbard-Stratonovich:

$$Z_{1+1} = \int \mathcal{D}A e^{-N \text{tr} AA^\dagger} \det^{\frac{N}{2}} \begin{pmatrix} A + m & \mu + i\pi T \\ \mu + i\pi T & A^\dagger + m \end{pmatrix} \times (\text{same with } T \rightarrow -T)$$

where  $A$  is complex  $2 \times 2$  (i.e.,  $N_f \times N_f$ ) matrix.

$$Z_{1+1}^* = Z_{1+1} \Big|_{\mu \rightarrow \mu \tau_3}; \quad (\mathbb{M}^* = \mathbb{M} \Big|_{\mu \rightarrow -\mu})$$

- Define  $\Omega(A)$ :  $Z = \int \mathcal{D}A e^{-N\Omega(A)}$ ,  $N \rightarrow \infty$  dominated by saddle point of  $\Omega(A)$ :

$$A - \frac{(A + m)[(A + m)^2 - \mu^2 + T^2]}{[(A + m)^2 - \mu^2 + T^2]^2 + 4\mu^2 T^2} = 0$$

- This saddle point is the same for  $1 + 1$  and  $1 + 1^*$  RMM (outside the pion condensation domain) and also  $\min \Omega_{1+1} = \min \Omega_{1+1^*}$ . I.e.

$$R \sim \frac{e^{-N\Omega_{1+1}}}{e^{-N\Omega_{1+1^*}}} \rightarrow e^{0 \cdot N} \sim 1 \quad \text{as } N \rightarrow \infty, \text{ not } 0.$$

# Analytical solution of RMM (contd.)

Need second derivative matrix  $\frac{\partial^2 \Omega}{\partial A_{..} \partial A_{..}} \equiv \Omega''_{...}$ :

$$R = \frac{Z_{1+1}}{Z_{1+1}^*} = \left( \frac{\det \Omega''_{1+1}}{\det \Omega''_{1+1}^*} \right)^{-1/2} = \left| \frac{b_3^2 - b_4^2}{b_1^2 - b_2^2} \right|$$

where

$$b_1 = \frac{(A + m)^2}{W} \left( 1 - \frac{8T^2 \mu^2}{W} \right)$$

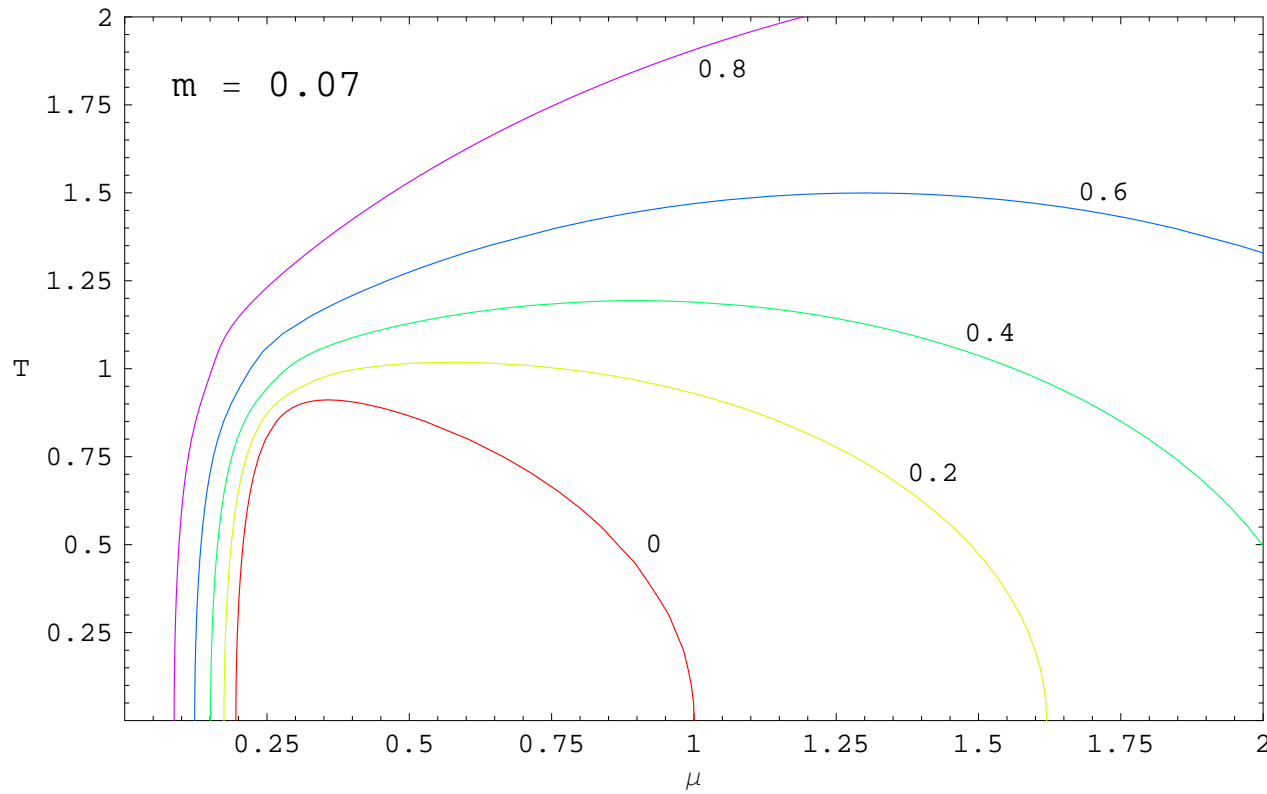
$$b_2 = 1 - \frac{T^2 - \mu^2}{W} - \frac{8T^2 \mu^2 (A + m)^2}{W^2}$$

$$b_3 = 1 - \frac{T^2 + \mu^2}{W}$$

$$b_4 = \frac{(A + m)^2}{W}$$

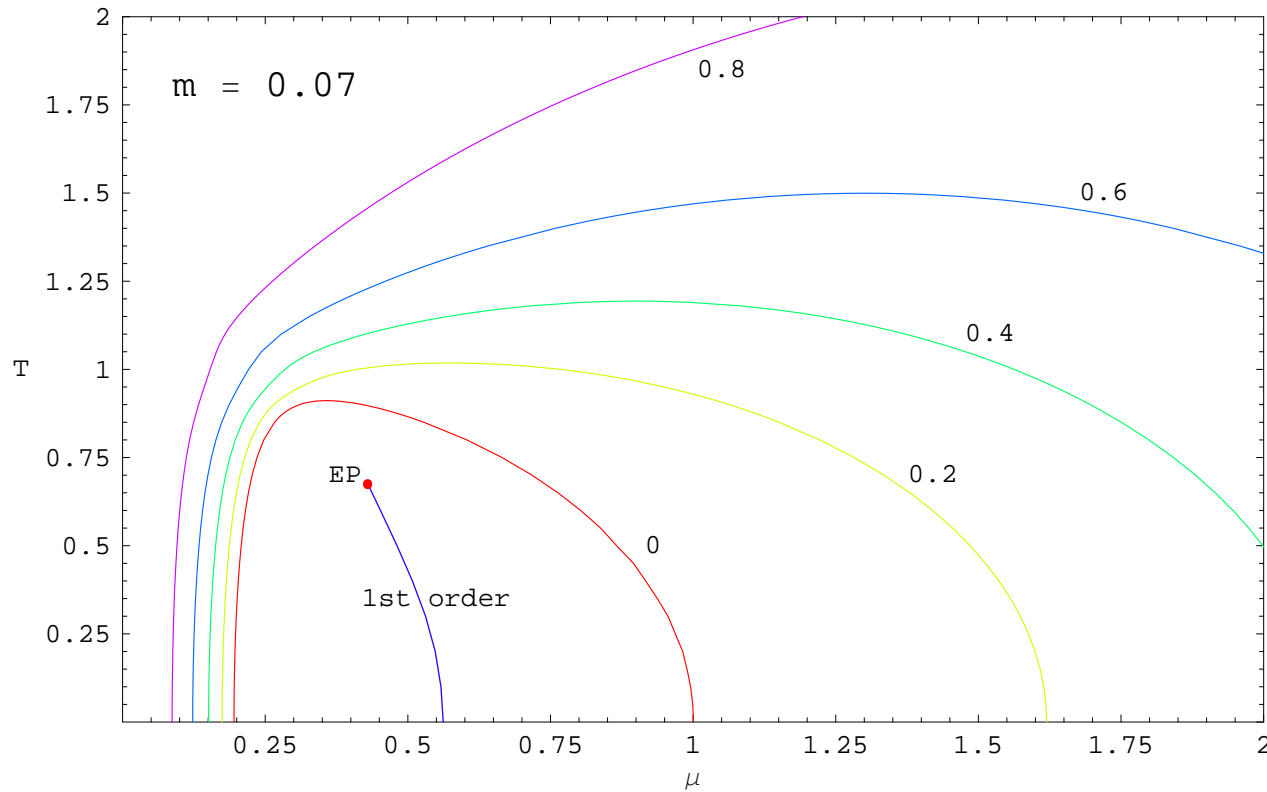
$$W = (A + m)^4 + 2(A + m)^2(T^2 - \mu^2) + (T^2 + \mu^2)^2$$

# $R(T, \mu)$ contour plot



● Sign problem is less severe at higher temperature :)

# $R(T, \mu)$ contour plot



- Sign problem is less severe at higher temperature :)
- First order transition of 1+1 is inside the  $R = 0$  boundary :(

# Interesting limits

- $T=0$ , small  $m$  and  $\mu \sim \sqrt{m}$   
(Splittorff, Verbaarschot)

$$R \approx 1 - \frac{2\mu^2}{m} = 1 - \left( \frac{2\mu}{m_\pi} \right)^2.$$

- $T < 1$ , small  $m$  and  $\mu \sim \sqrt{m}$

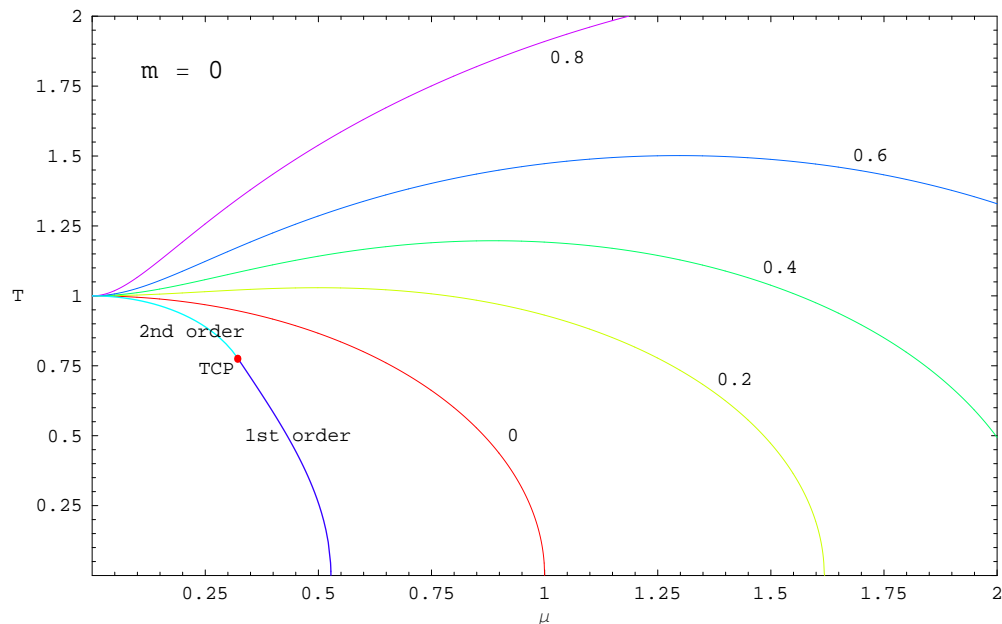
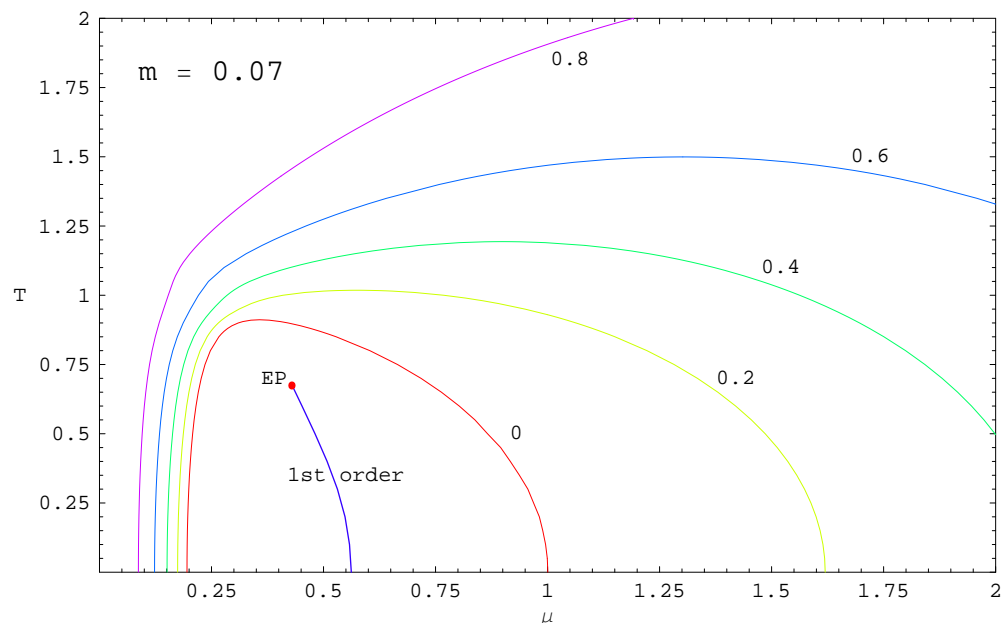
$$R \approx 1 - \sqrt{1 - T^2} \left( \frac{2\mu}{m_\pi} \right)^2$$

sign problem weakens with  $T$ .

- Chiral limit ( $m = 0$ ), any  $\mu$ ,  $T$

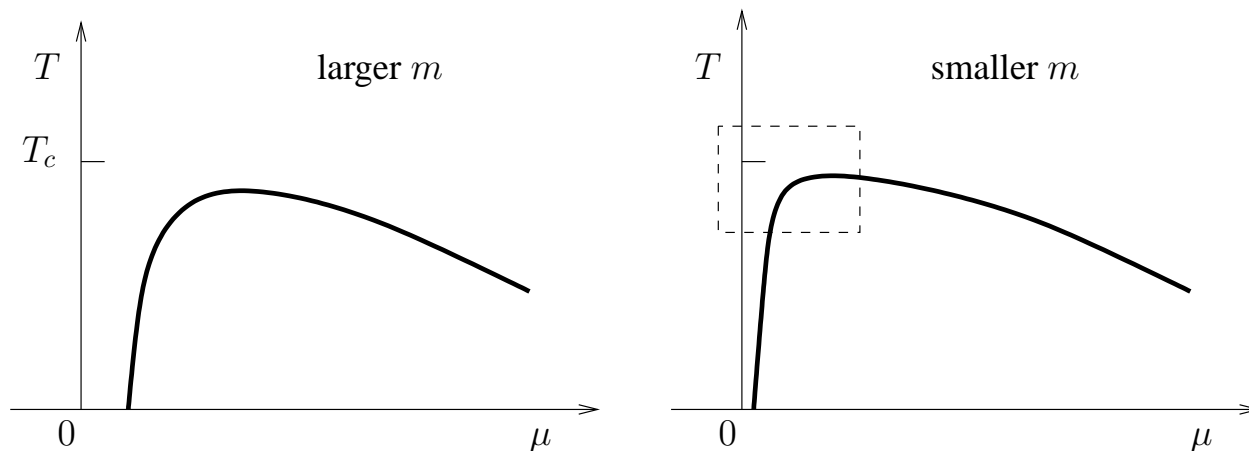
$$R = \frac{[(T^2 + \mu^2)^2 - (T^2 + \mu^2)]^2}{[(T^2 + \mu^2)^2 - (T^2 - \mu^2)]^2}$$

$R = 0$  in a  $90^\circ$  pie:  $T^2 + \mu^2 < 1$ .



# $R = 0$ boundary near $T = 1, \mu = 0$

How does the  $R = 0$  (pion condensation) boundary approach  $T = 1, \mu \rightarrow 0$  as  $m_\pi \rightarrow 0$ ?



● In RMM, expanding the analytic solution (notation:  $t \equiv T^2 - 1$ )

$$A^3 + A(t + 3\mu^2) - m = 0$$

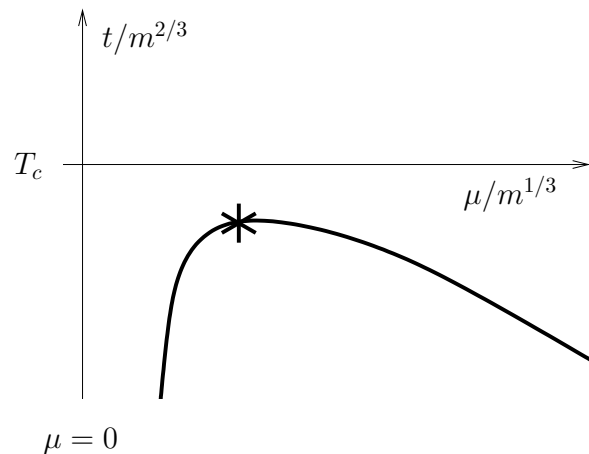
i.e. at  $m = 0$   $A \sim (-t - 3\mu^2)^{1/2} = (t_c - t)^{1/2}$  or  $A \sim m^{1/3}$  at  $t = t_c$ .

● The  $R = 0$  curve is

$$T^2 = 1 - \mu^2 - \frac{m^2}{4\mu^4} = 1 - m^{2/3} F\left(\frac{\mu}{m^{1/3}}\right),$$

with  $F(x) = x^2 + 1/(4x^2)$  – a scaling function.

# Pion condensation boundary as $m_\pi \rightarrow 0$ – scaling



In RMM:  $T_c - T_* \sim m^{2/3}$  and  $\mu_* = m^{1/3}$  (slower than  $m_\pi \sim m^{1/2}$ ).

In QCD:  $T_c - T_* \sim m^{1/(\beta\delta)}$  and  $\mu_* = m^{1/(2\beta\delta)}$  ?

# Summary

- Using analytical solution of RMM we found  $R = \langle e^{2i\theta} \rangle$  at finite  $T$  and  $\mu$ .
- Sign problem is less severe at higher  $T$ .
- The  $1 + 1$  phase transition is hidden inside the  $R = 0$  (pion condensation) domain.
- As  $m \rightarrow 0$  the domain  $R = 0$  approaches  $T = T_c, \mu = 0$  point in a self-similar way, with  $\mu_* \sim m^{1/3}$  (in RMM) – slower than  $m_\pi$ .