Fisher's Zeros at Zero and Finite Temperature

Yannick Meurice
The University of Iowa
yannick-meurice@uiowa.edu

XQCD, Frascati, August 7, 2007
Work done with A. Denbleyker, D. Du and A. Velytsky

Overview:

- Perturbative expansion of the average plaquette in pure gauge $SU(3)$ and complex singularities in the $\beta = 2N/g^2$ plane. Two models.

- Zeros of the partition function in the complex β plane (Fisher's zeros).

- New methods for complex values of $\Delta\beta$ where the MC reweighting calculation $< e^{-\Delta\beta S} >$ is not reliable. New definition of the region of confidence. Fits based on the assumption that

  ln(density of state (S)) $\simeq$ polynomial in S. (arXiv.0708.0438)

- Application of the new methods for pure gauge $SU(2)$ and $SU(3)$ on L^4 and $4 \times L^3$ lattices and comparison with existing results.

Lattice Perturbation Theory ($SU(3)$)

$P(1/\beta) = \sum_{m=0}^{10} b_m \beta^{-m} + \ldots.$

(F. Di Renzo et al. JHEP 10 038, P. Rakow Lat. 05)

Series analysis suggests a singularity: $P \propto (1/5.74 - 1/\beta)^{1.08}$

(Horsley et al, Rakow, Li and YM)

Not expected: zero radius of convergence (the plaquette changes discontinuously at $\beta \rightarrow \pm\infty$ (Li, YM PRD 71))

Not seen in 2d derivative of P (would requires massless glueballs!)

A Small Window for Complex Singularities

A simple alternative: the critical point in the fundamental-adjoint plane has mean field exponents and in particular $\alpha = 0$. On the $\beta_{adj.} = 0$ line, we assume an approximate logarithmic behavior (mean field)

$$-\partial P/\partial\beta \propto \ln((1/\beta_m - 1/\beta)^2 + \Gamma^2) \ , \tag{1}$$

This implies the approximate form (with params. to be fitted from the pert. series)

$$\partial^2 P/\partial\beta^2 \simeq -C\frac{(1/\beta_m - 1/\beta)}{\beta^3((1/\beta_m - 1/\beta)^2 + \Gamma^2)} \tag{2}$$

Typical Fits: $\beta_m \simeq 5.78$, $\Gamma \simeq 0.006$ (i.e $Im\ \beta \simeq 0.2$), and $C \simeq 0.15$

Bounds on Imaginary part (Li and YM PRD 73)

The stability of C and β_m can be used to set a lower bound on Γ. Given that the approximate form of $\partial^2 P/\partial\beta^2$ in Eq. (2) has extrema at $1/\beta = 1/\beta_m \pm \Gamma$. As we do not observe values larger than 0.3 near $\beta = 5.75$ we get the approximate bound $\frac{C}{2\beta_m^3\Gamma} < 0.3$. Large values of Γ would affect the low order coefficients. We never found fitted values of Γ close to 0.01.

$$0.001 < \Gamma < 0.01 \ . \tag{3}$$

This suggests zeroes of the partition function in the complex β plane with

$$0.03 \simeq 0.001\beta_m^2 < Im\beta < 0.01\beta_m^2 \simeq 0.33 \tag{4}$$

Large order extrapolations (YM PRD 74)

Model 1:

$\sum_{k=0} b_k \beta^{-k} \simeq C(\mathrm{Li}_2(\beta^{-1}/(\beta_m^{-1} + i\Gamma)) + \mathrm{h.c},$

$\mathrm{Li}_2(x) = \sum_{k=0} x^k/k^2$.

We fixed $\Gamma = 0.003$ and obtained $C = 0.0654$ and β_m=5.787 using of a_9 and a_{10}. The low order coefficients depend very little on Γ (when $\Gamma < 0.01$), larger series are needed!

Very good predictions of the values of $a_8, a_7, \ldots$!

order	predicted	numerical	rel.error
1	0.7567	2	-0.62
2	1.094	1.2208	-0.10
3	2.811	2.961	-0.05
4	9.138	9.417	-0.03
5	33.79	34.39	-0.017
6	135.5	136.8	-0.009
7	575.1	577.4	-0.004
8	2541	2545	-0.0016
9	*exact*	11590	
10	*exact*	54160	

Also $a_{16} = 7.7\ 10^8$ while from Fig. 1 of P. Rakow Lattice 2006 $a_{16} = 0.00027 \times 6^{16} = 7.6\ 10^8$;

Feynman diagram interpretation ???

Model 2 (Mueller 93, di Renzo 95):

$$\sum_{k=0} b_k \bar{\beta}^{-k} \simeq K \int_{t_1}^{t_2} dt \mathrm{e}^{-\bar{\beta}t} \left(1 - t\ 33/16\pi^2\right)^{-1-204/121} \tag{5}$$

$$\bar{\beta} = \beta(1 + d_1/\beta + \dots) \tag{6}$$

$t_1 = 0$ corresponds to the UV cutoff

$t_2 = 16\pi^2/33$: Landau pole; $t_2 = \infty$: usual perturbative series

If we want to study complex zeros, we need to regularize the Borel singularity; connection with the other model or density of states are not well understood.

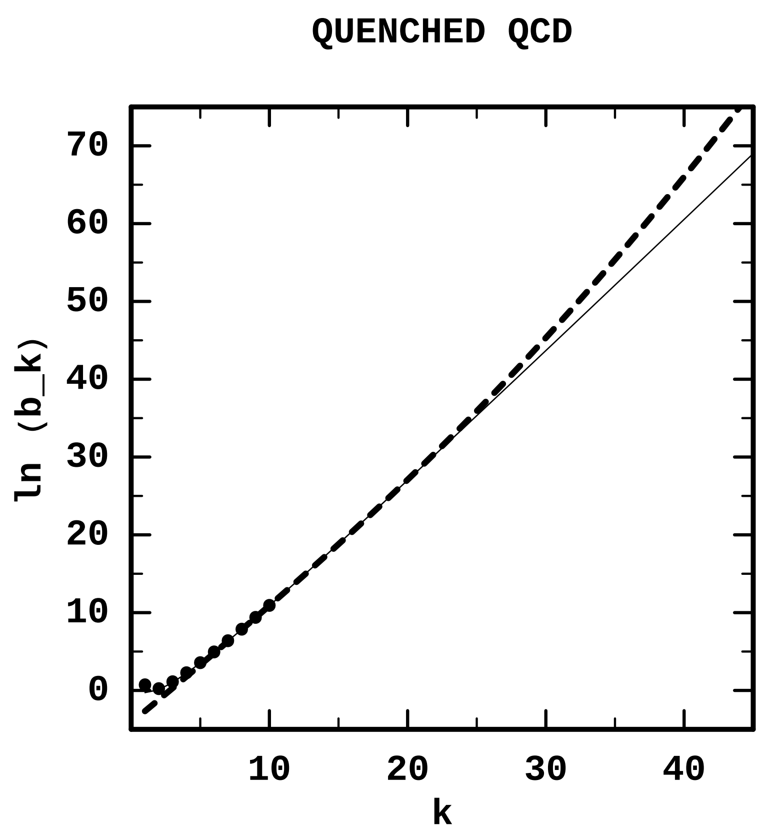


Figure 1: ln(b_k) for the dilogarithm model (solid line) and the integral model (dashes). The dots up to order 10 are the known values. The two models yields similar coefficients up to order 20. After that, the integral model has the logarithm of its coefficients growing faster than linear.

Gluon Condensate ????

The gluon condensate is not an order parameter, there is no absolute way to define this quantity. (G. Rossi)

$$P(\beta) - P_{pert.}(\beta) \simeq C(a/r_0)^4$$

with $a(\beta)$ defined with Sommer's scale, and P_{pert} appropriately truncated.

C is sensitive to resummation. $C \simeq 0.6$ with the bare series (YM PRD D74 096005) and 0.4 with the tadpole improved series (P. Rakow, Lattice 05). This gives values 2-3 times larger than the official value used in SVZ.

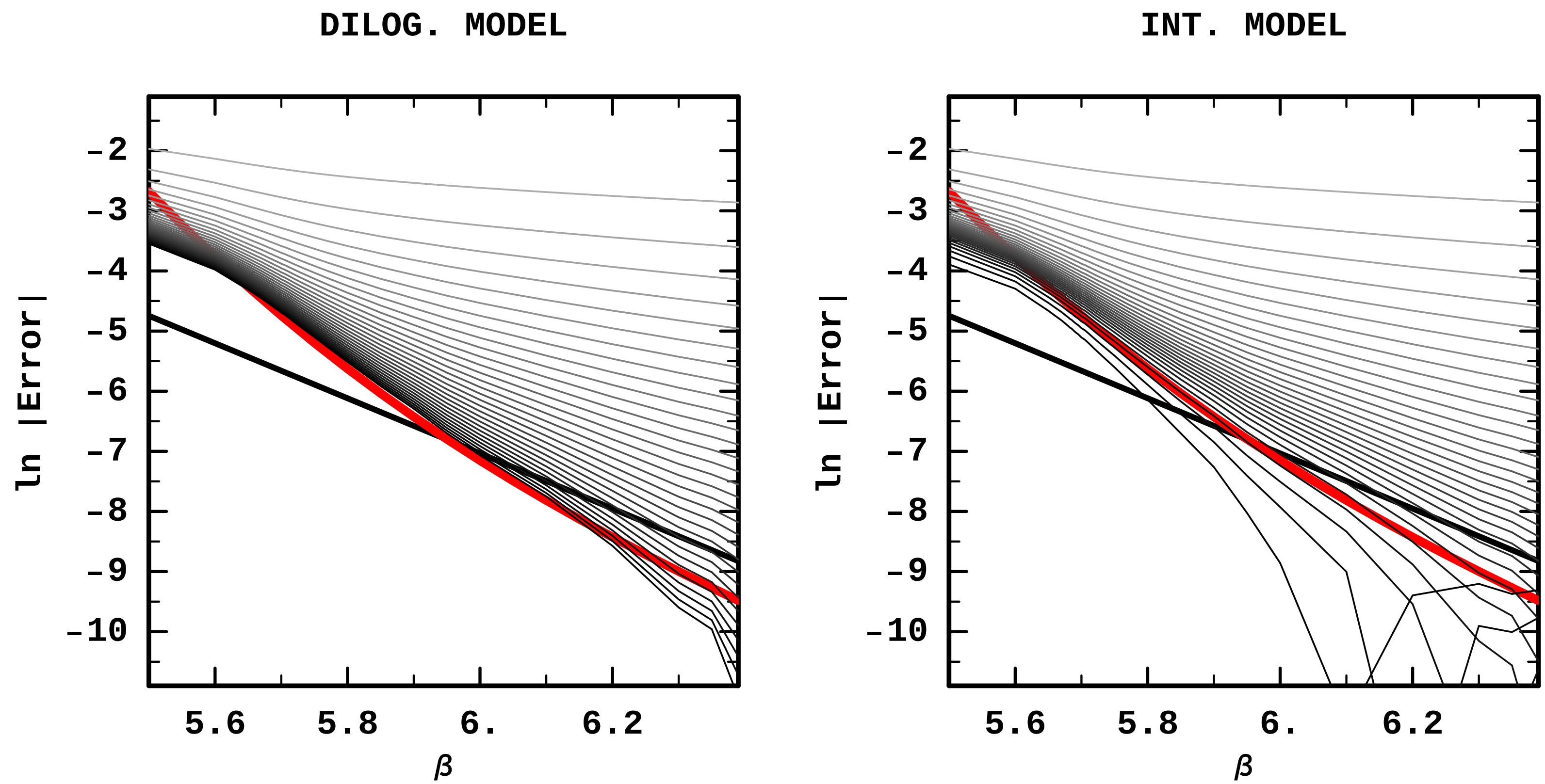


Figure 2: Accuracy curves for the dilogarithm model (left) and the integral model (right) at successive orders. As The red curve is ln$(0.65\ (a/r_0)^4)$. The solid curve is ln $\left(3.1 \times 10^8 \times (\beta)^{204/121-1/2} \mathrm{e}^{-(16\pi^2/33)\beta}\right)$

Zeros of the partition function

Reweighting (Falcioni et al. 82):

$$Z(\beta_0 + \Delta\beta) = Z(\beta_0) < \exp(-\Delta\beta S) >_{\beta_0} \quad . \tag{7}$$

$$\begin{aligned} < \quad \exp \quad & [-\Delta\beta(S - < S >_{\beta_0})] >_{\beta_0} \\ = \quad \exp \quad & [\Delta\beta < S >_{\beta_0}] Z(\beta_0 + \Delta\beta)/Z(\beta_0) \ , \end{aligned} \tag{8}$$

has the same complex zeros as $Z(\beta_0 + \Delta\beta)$.

$Z(\beta)$ is the Laplace transform of density of states $n(S)$:

$$Z(\beta) = \int_0^\infty dS \ n(S) \exp(-\beta S) \tag{9}$$

Few facts about the density of state $n(S)$

Depends on $L_1, L_2, \ldots$ only.

Can be obtained from $< \mathrm{e}^{-(\beta_1 + iu)S} >_{\beta_0}$ (inverse Laplace transform)

$S \sim 0$ probed at weak coupling

$S \sim \mathcal{N}_p$ (number of plaquettes) probed at strong coupling

$n(S) \propto \mathrm{e}^{-(a_1 S + a_2 S^2 + a_3 S^4 + a_4 S^4)}$ in the crossover ?

For $SU(2)$ with L_i even $Z(-\beta) = \mathrm{e}^{2\beta \mathcal{N}_p} Z(\beta)$ and $n(S) = n(2\mathcal{N}_p - S)$

Circle of confidence

Gaussian distributions (of S) have no complex zeros.

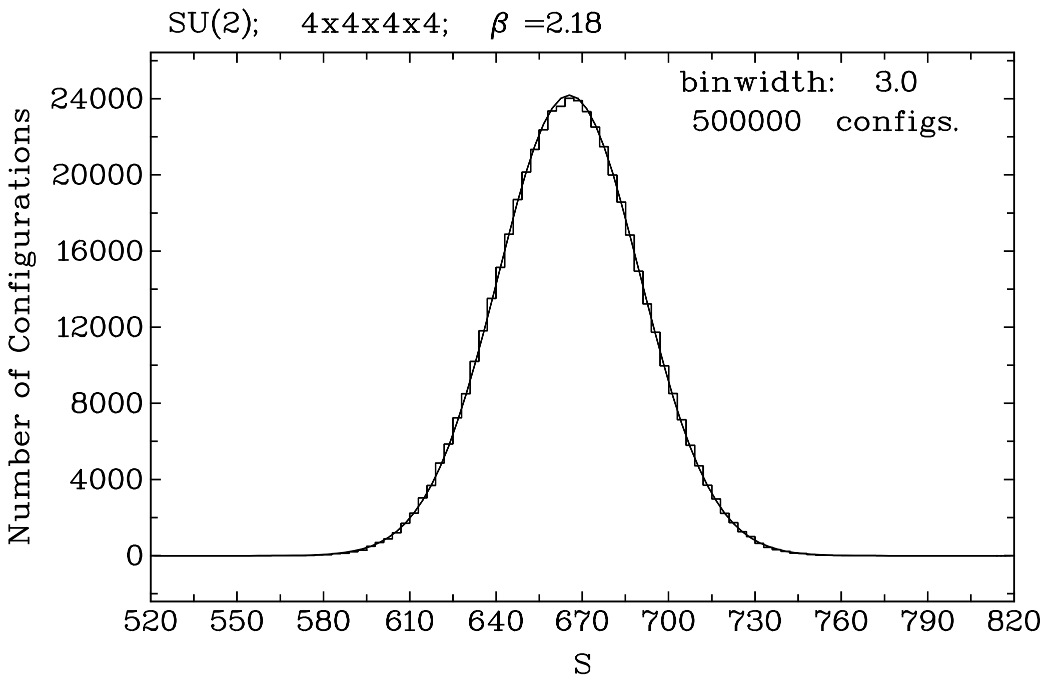
Criterion to determine a region of confidence for MC zeros (Alves and Berg 91, based on the Gaussian approximation):

$\sigma_S^2 =< S^2 > - < S >^2$ is the approximate width.

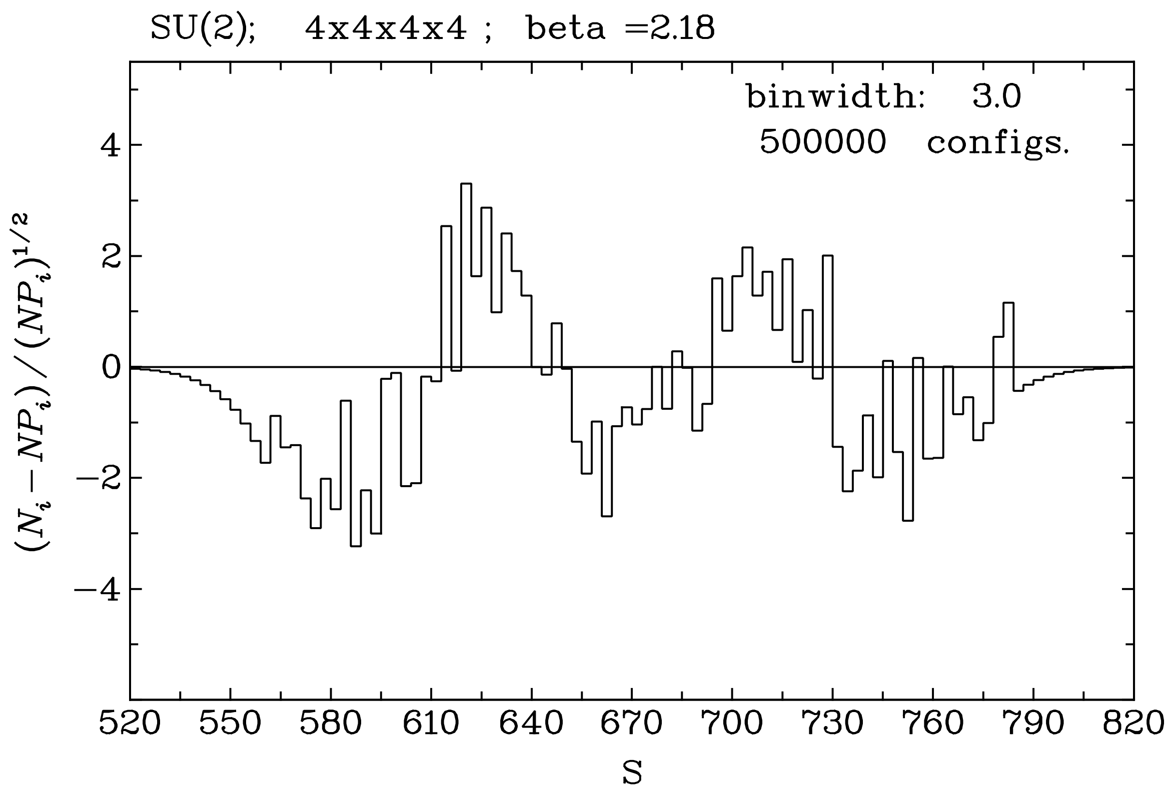
The fluctuation in $\exp(-\Delta\beta(S- < S >))$ become of the same size as the average for $|\Delta\beta|^2 < \ln(N_{conf.})/\sigma_S^2$

This defines a radius of confidence $\sqrt{\ln(N_{conf.})}/\sigma_S$ in the complex β plane. The radius shrinks like $V^{-1/2}$.

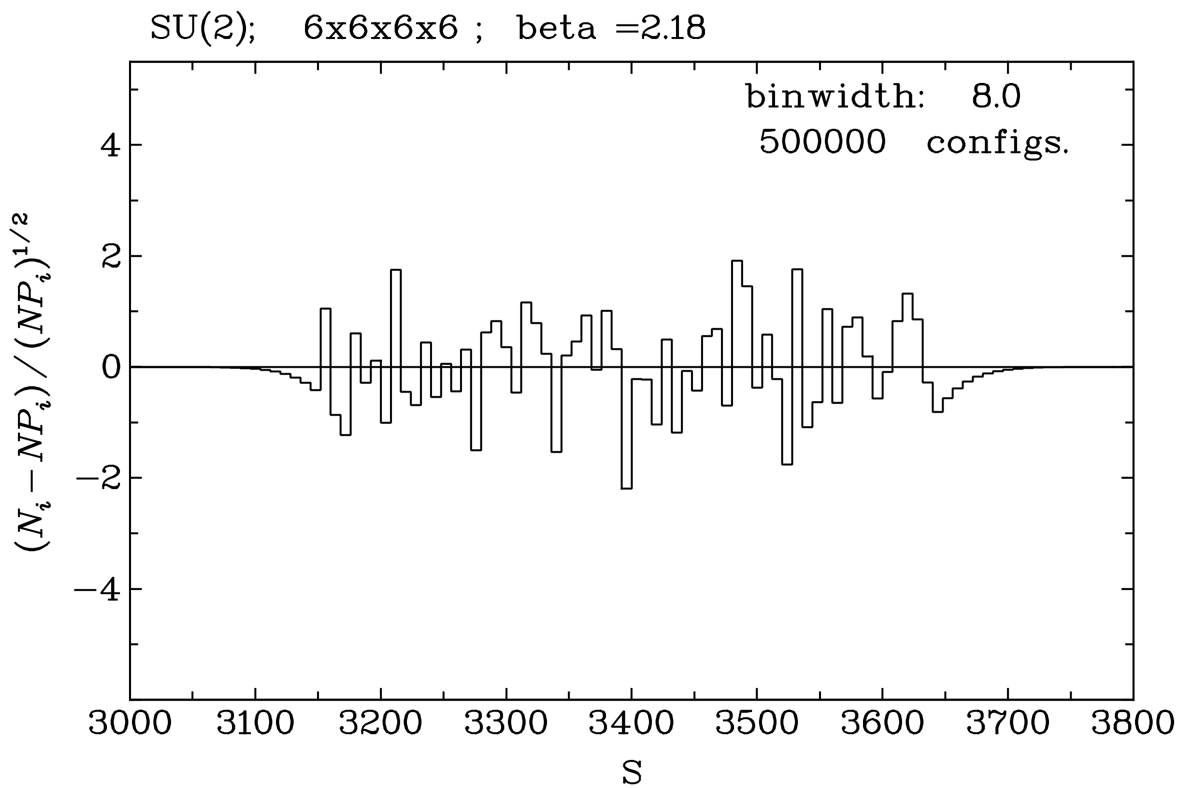
Quasi-Gaussian Histograms for S

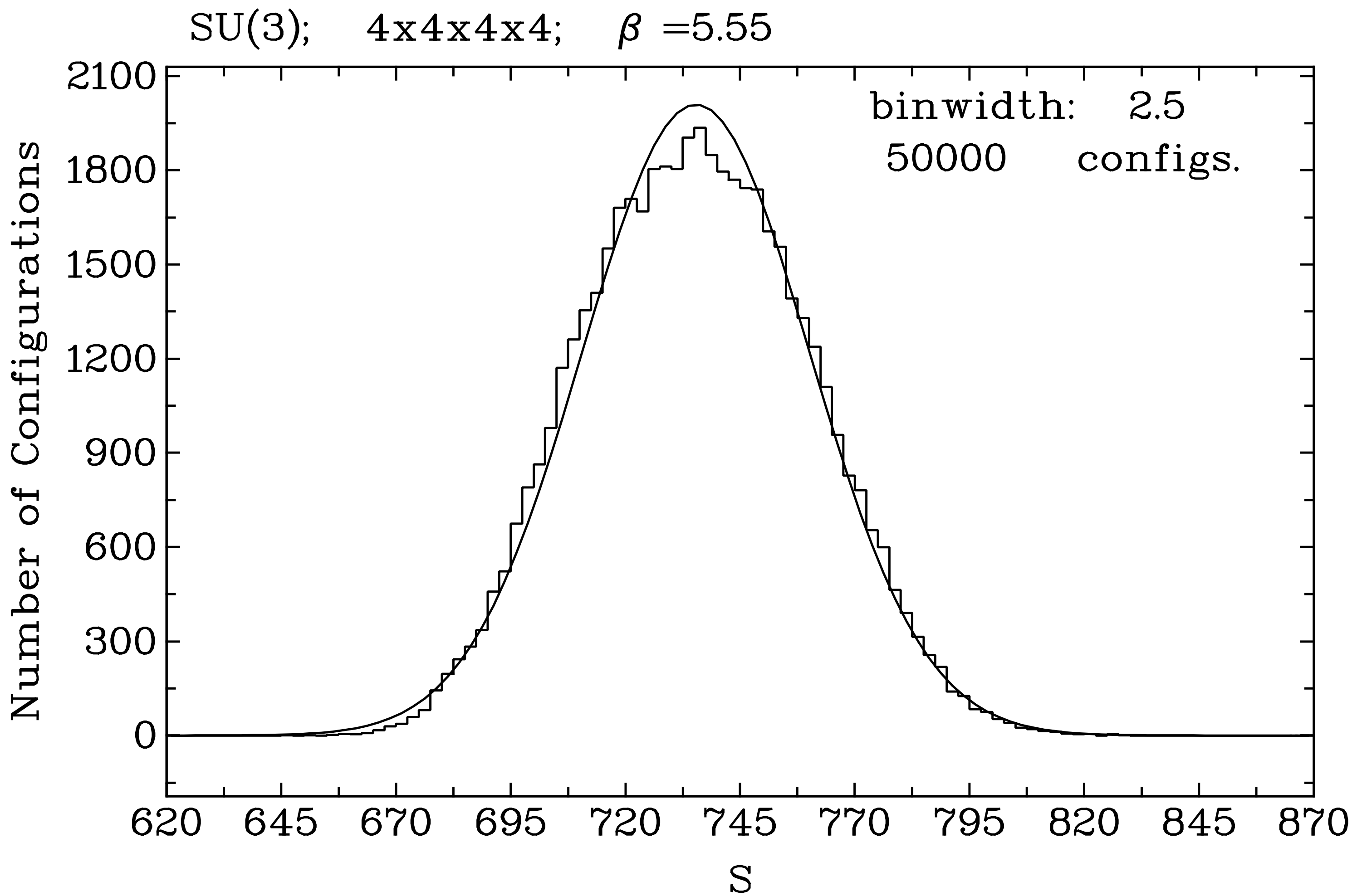


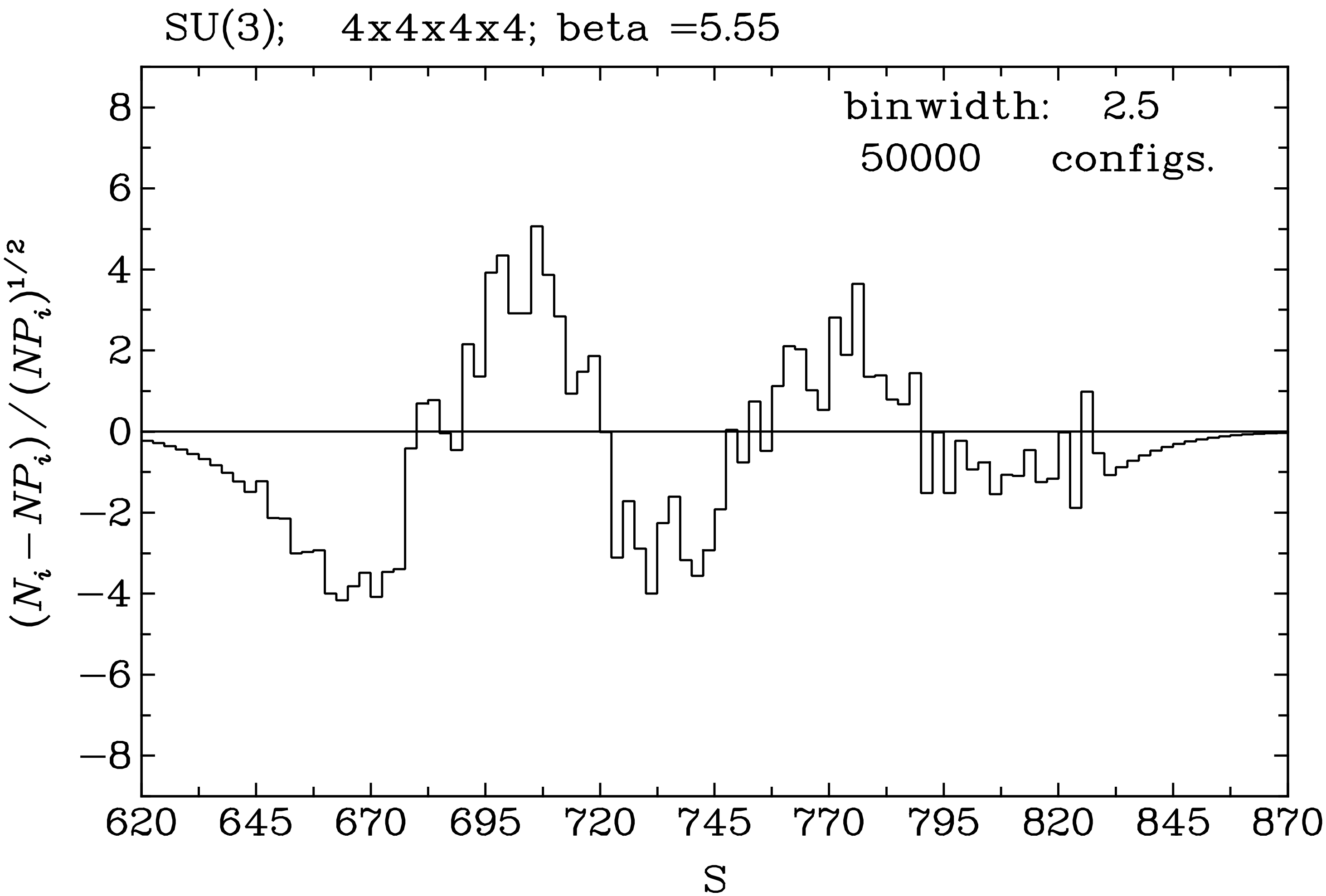
Discrepancies in unit of the expected fluctuations are coherent for 4^4.



As the volume increases, the signal gets lost in the noise




SU(3); 4x4x4x4; β =5.55
binwidth: 2.5
50000 configs.
Number of Configurations
2100
1800
1500
1200
900
600
300
0
620
645
670
695
720
745
770
795
820
845
870
S


SU(3); 4x4x4x4; beta =5.55
binwidth: 2.5
50000 configs.
$(N_i - NP_i)/(NP_i)^{1/2}$
8
6
4
2
0
−2
−4
−6
−8
620
645
670
695
720
745
770
795
820
845
870
S

Natural units, notations

$$S_{red.} = (S - < S >)/\sigma_S \; ; \; \beta_{red.} = \Delta\beta\sigma_S \; . \tag{10}$$

$$\begin{aligned} f &\equiv < \exp(-\beta_{red.} S_{red.}) > \\ R &\equiv Ref \\ I &\equiv Imf \\ \sigma^2_{Re} &\equiv < (Re\exp(-\beta_{red.} S_{red.}) - R)^2 > \\ \sigma^2_{Im} &\equiv < (Im\exp(-\beta_{red.} S_{red.}) - I)^2 > \\ \sigma^2_f &\equiv \sigma^2_{Re} + \sigma^2_{Im} \end{aligned}$$

New definition of the region of confidence

The Gaussian circle of confidence in the complex β plane is defined by the condition

$$\sigma_f/\sqrt{N_{conf.}} < |f| \tag{11}$$

We propose to consider the alternative region of confidence defined by a condition that controls the error on the level curves:

$$\frac{\sigma_f}{\sqrt{N_{conf.}}\ |f'|} < d\ . \tag{12}$$

In order to be useful d should be a fraction of the typical distance between zero level curves of the real and imaginary part.

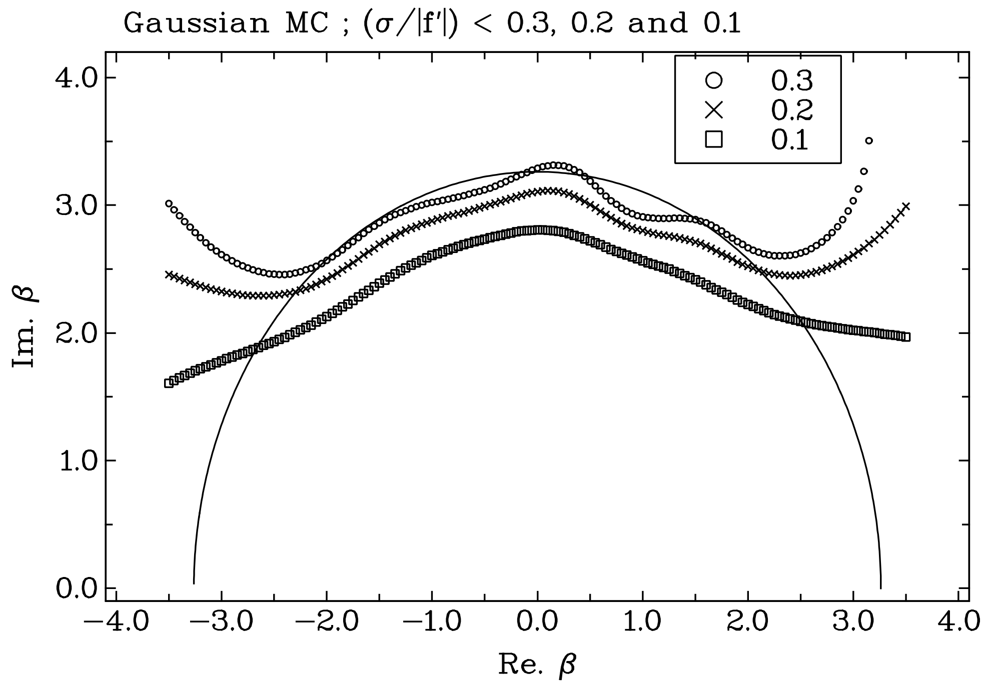


Figure 3: Boundary of the confidence region for $d = 0.3$ (circles), 0.2 (crosses) and 0.1 (boxes), compared to the Gaussian circle of confidence, all for 40,000 configurations.

Approximate models

The nice regularities of the difference with the Gaussian approximation (for small lattices) suggest

$$P(S) \propto \exp(-\lambda_1 S - \lambda_2 S^2 - \lambda_3 S^3 - \lambda_4 S^4) \tag{13}$$

The unknown parameters were determined from the fist four moments using Newton's methods and also by χ^2 minimization. Very good agreement between the two methods was found on 4^4 lattices.

Testing the new ideas with examples

Preprint arXiv 0708.0438.

Example 1: $\lambda_3 = 0.1,\ \lambda_4 = 0.01$. It has been chosen in such a way that we have zeros inside and outside the Gaussian region of confidence.

Example 2: $\lambda_3 = 0.01,\ \lambda_4 = 0.002$ The perturbation is much smaller and the first accurate zero is far away from the Gaussian circle of confidence.

When linear term is varied ($\sim$ changing β), the real part of the zero coincides with a maximum of the second moment.

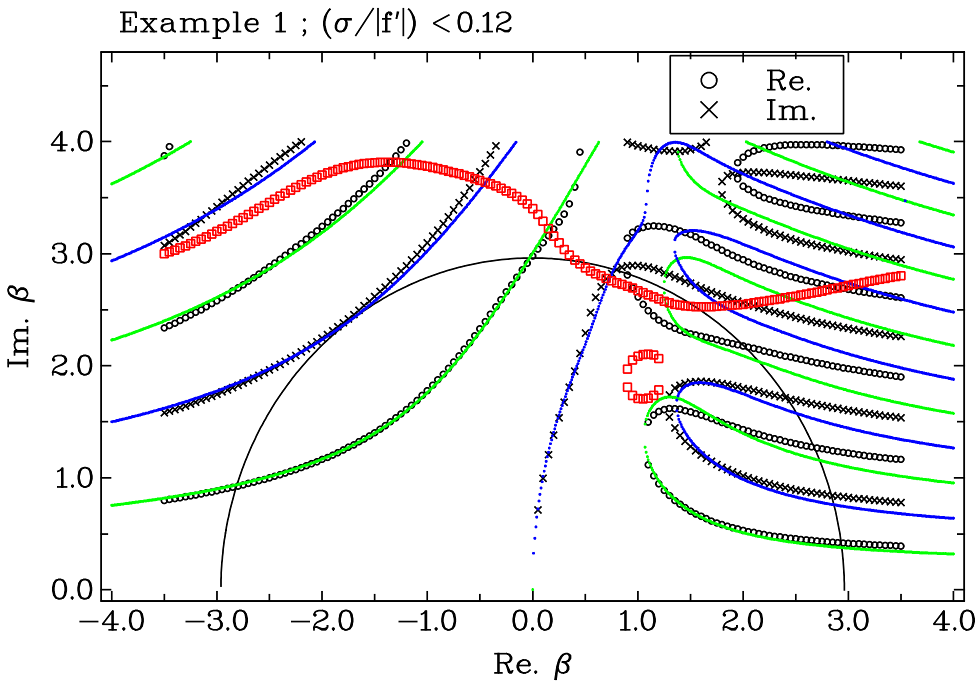
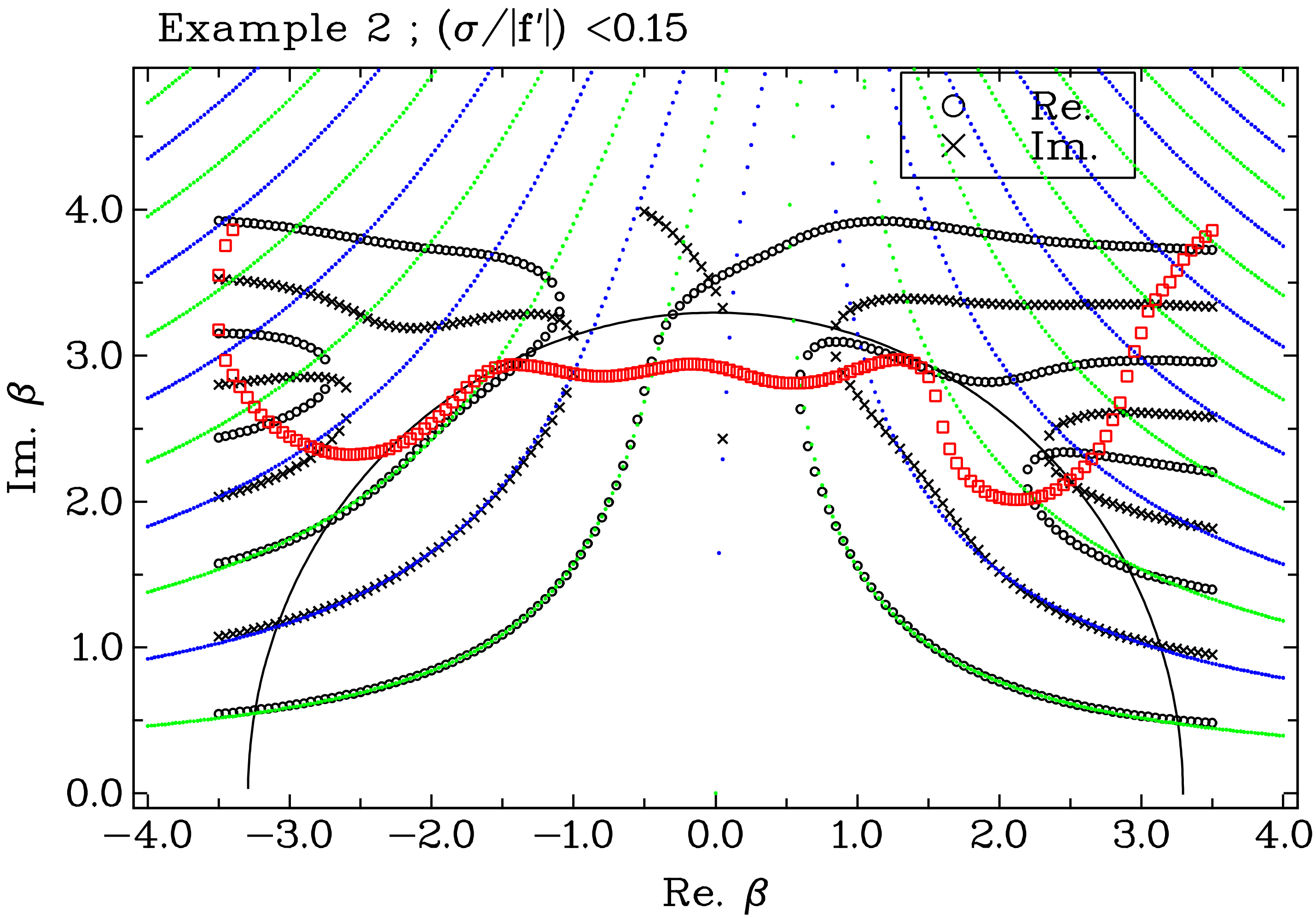


Figure 4: Zeros of the real (crosses) and imaginary (circles) part for 40000 configurations corresponding to the first example. The small dots are the accurate values for the real (green) and imaginary (blue) parts. The exclusion region boundary for $d = 0.12$ is represented by boxes (red). The solid line is the circle of confidence of the Gaussian approximation.


Example 2 ; (σ/|f′|) <0.15
Re.
Im.
Im. β
Re. β
4.0
3.0
2.0
1.0
0.0
−4.0
−3.0
−2.0
−1.0
0.0
1.0
2.0
3.0
4.0

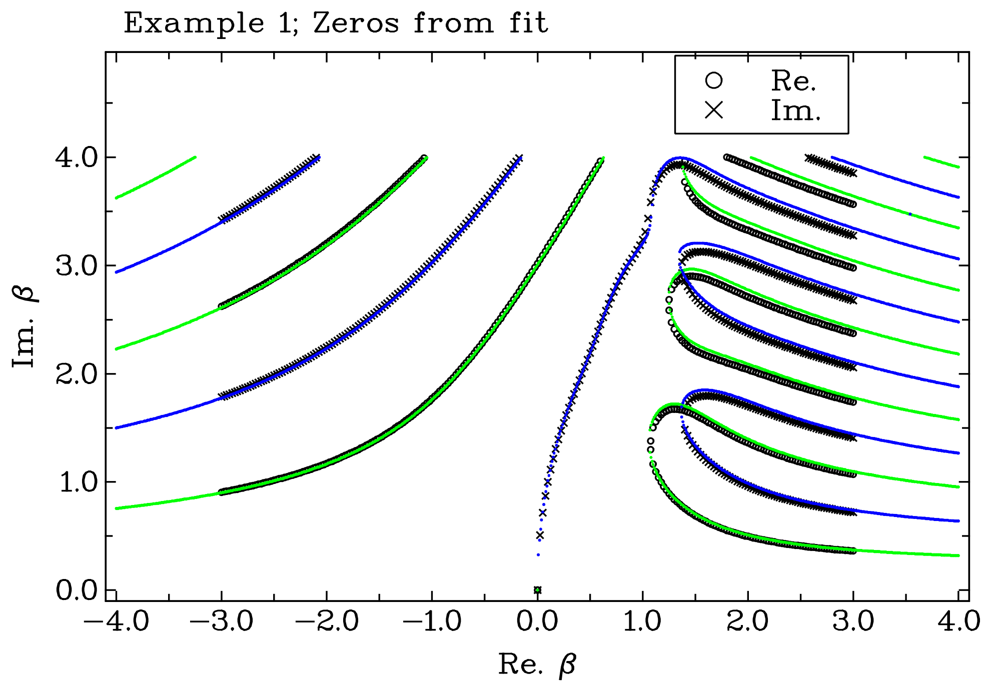
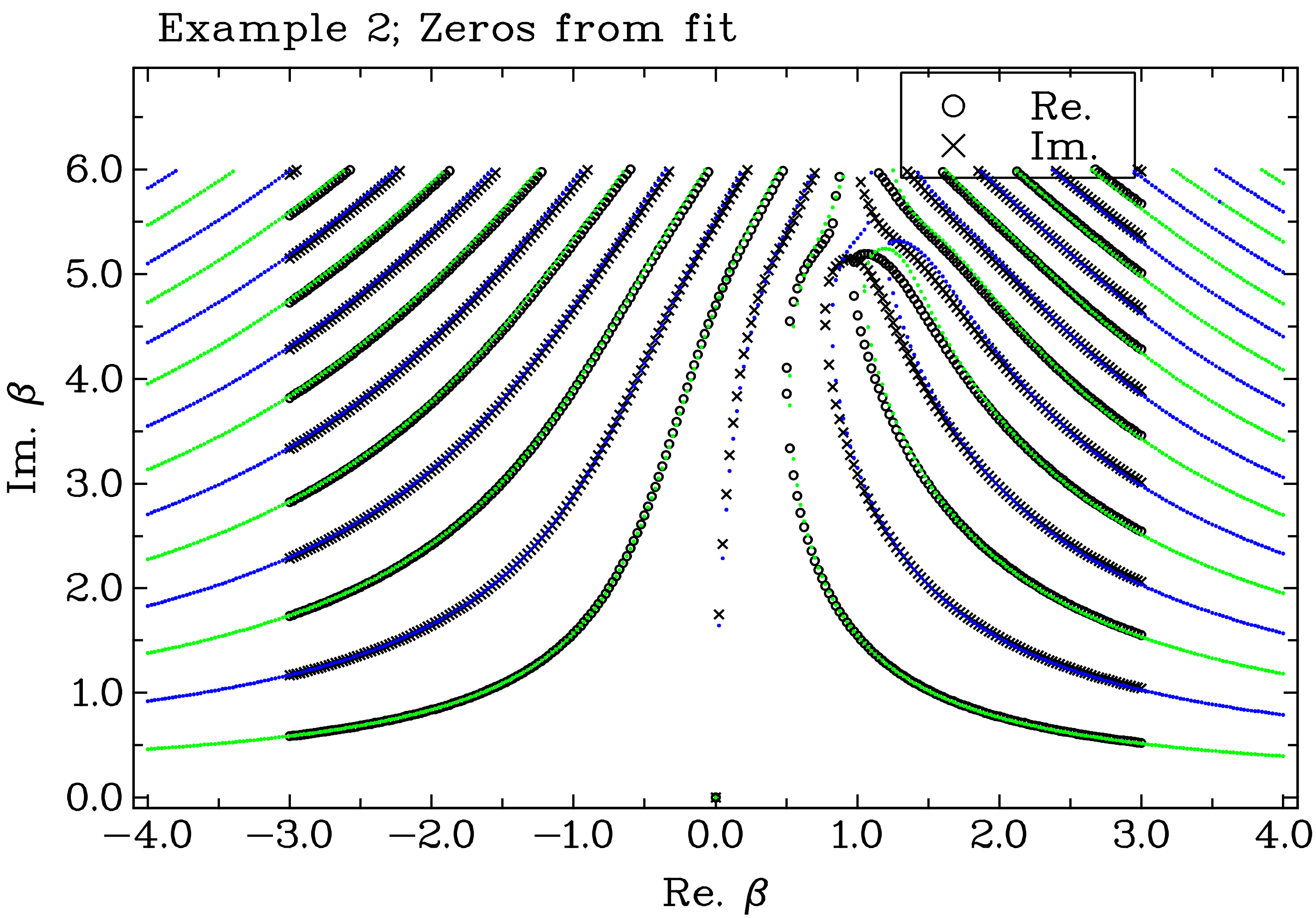


Figure 5: Zeros of the real (crosses) and imaginary (circles) using the approximate λ_i obtained from MC moments. The small dots are the accurate values for the real (green) and imaginary (blue) parts, for example 1.

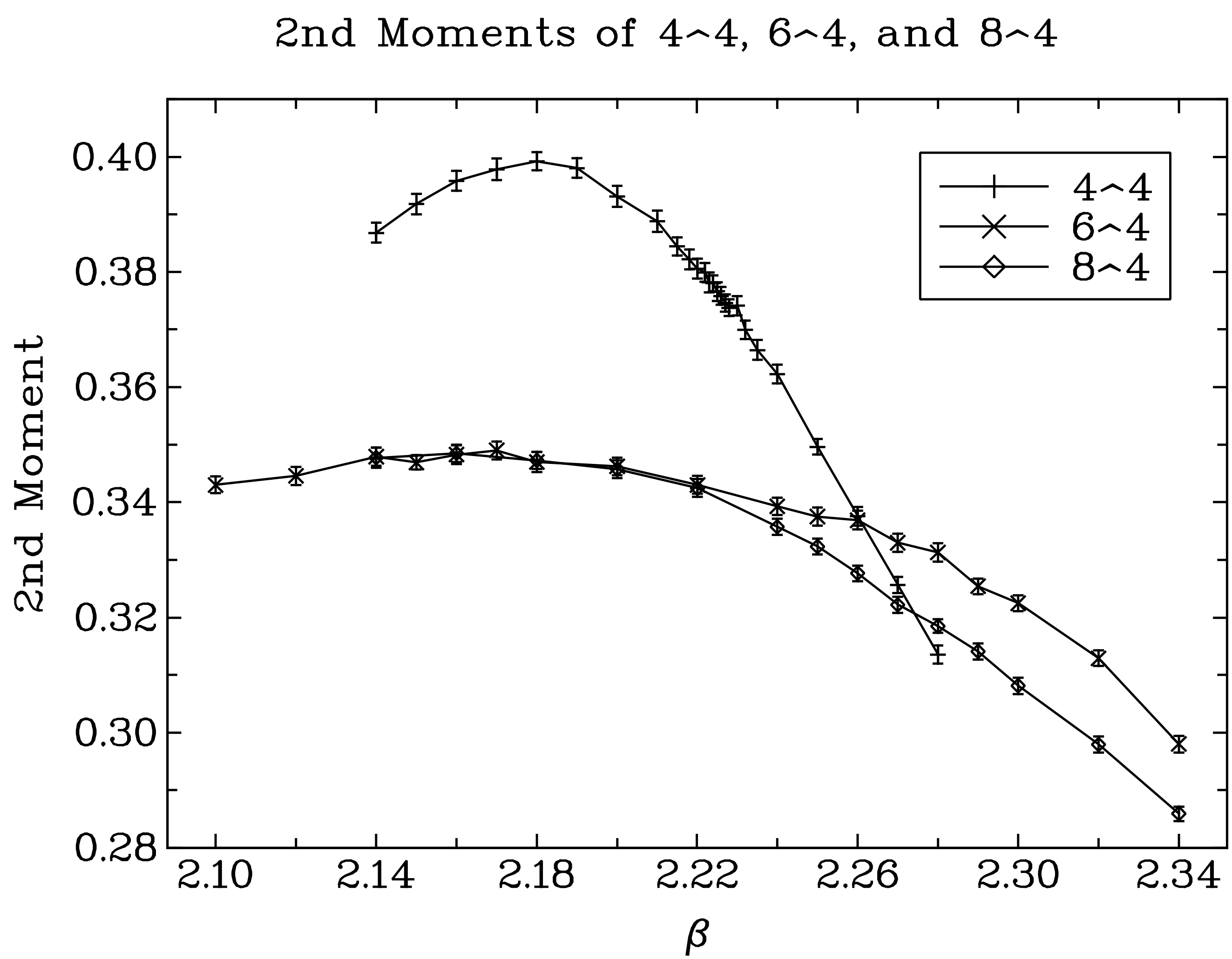

Example 2; Zeros from fit
Re.
Im.
6.0
5.0
4.0
3.0
2.0
1.0
0.0
Im. β
−4.0
−3.0
−2.0
−1.0
0.0
1.0
2.0
3.0
4.0
Re. β

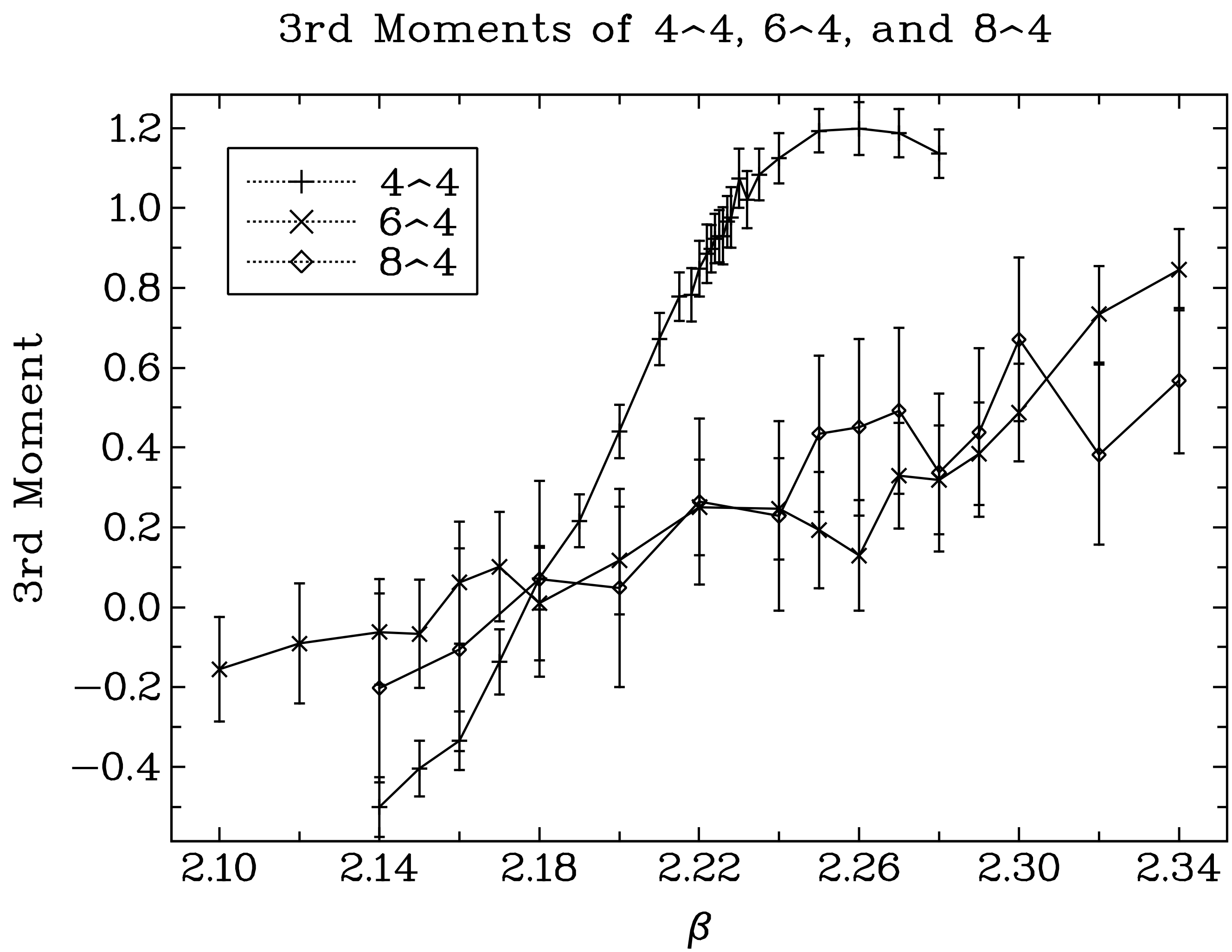
Moments

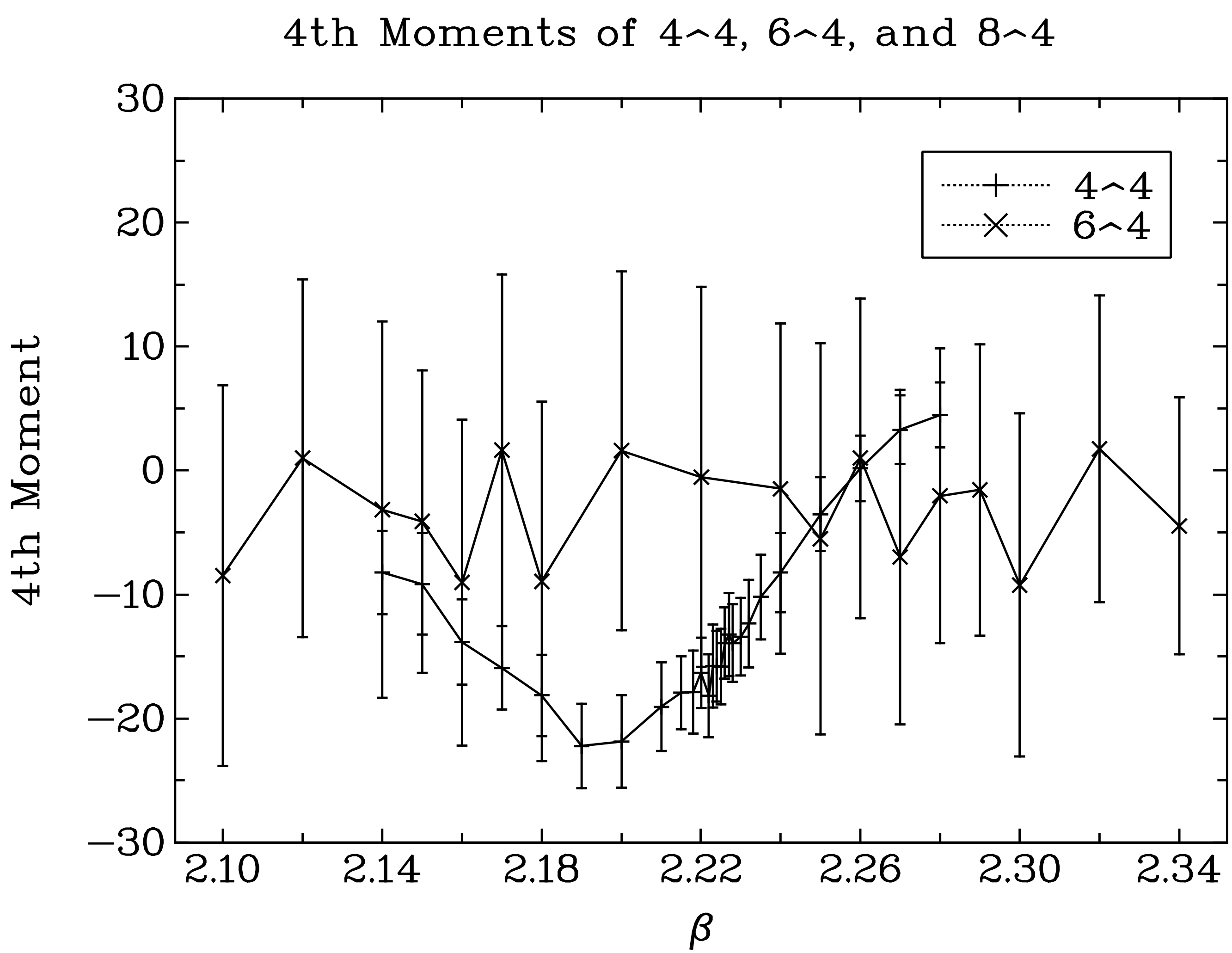
$$
\begin{aligned}
M_2 &= <(S-<S>)^2> \\
M_3 &= <(S-<S>)^3> \\
M_4 &= <(S-<S>)^4> -3<(S-<S>)^2>^2
\end{aligned}
\tag{14}
$$

$\mathcal{N}_p \equiv 6L^4$. Except at a critical value of β, $m_i \equiv M_i/\mathcal{N}_p$ have a finite limit in the infinite volume. Note that in reduced units

$$
\begin{aligned}
M_3/M_2^{3/2} &\propto V^{-1/2} \\
M_4/M_2^2 &\propto V^{-1}
\end{aligned}
$$


2nd Moments of 4~4, 6~4, and 8~4
2nd Moment
0.40
0.38
0.36
0.34
0.32
0.30
0.28
2.10
2.14
2.18
2.22
2.26
2.30
2.34
β
4~4
6~4
8~4


3rd Moments of 4~4, 6~4, and 8~4
4~4
6~4
8~4
3rd Moment
1.2
1.0
0.8
0.6
0.4
0.2
0.0
−0.2
−0.4
2.10
2.14
2.18
2.22
2.26
2.30
2.34
β


4th Moments of 4~4, 6~4, and 8~4
4~4
6~4
4th Moment
30
20
10
0
−10
−20
−30
2.10
2.14
2.18
2.22
2.26
2.30
2.34
β

Perturbative Methods

As the perturbations get smaller, the zeros get a larger imaginary part and the numerical integration becomes more difficult because of the fast oscillations of the integrand. However, it possible to use perturbative methods. When λ_3 and λ_4 are both zero, the problem is Gaussian a solvable analytically. If we calculate $< \exp(-\beta S) >$ at first order in λ_3 and λ_4 and divide by the Gaussian limit (which has no zeros), we obtain a polynomial of order 4 in β:

$$
\begin{aligned}
&< \exp(-\beta S)(1-\lambda_3 S^3 - \lambda_4 S^4) >_G / < \exp(-\beta S) >_G \\
&= Q(\beta) \\
&= 1 + \cdots - \lambda_4 \beta^4 / (16 \lambda_2^4) \ ,
\end{aligned}
$$

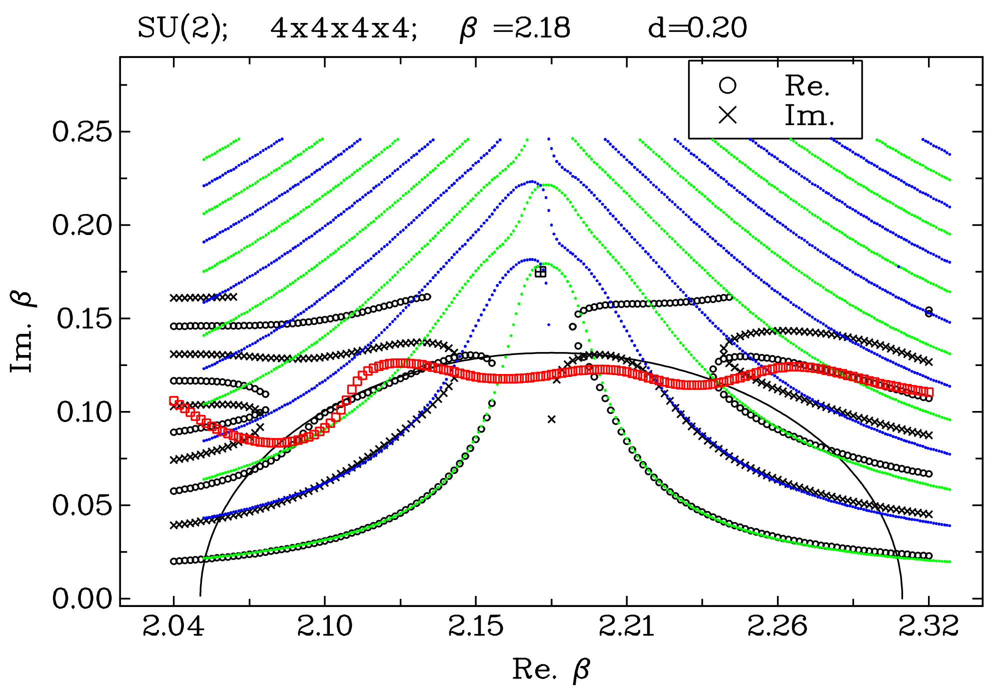
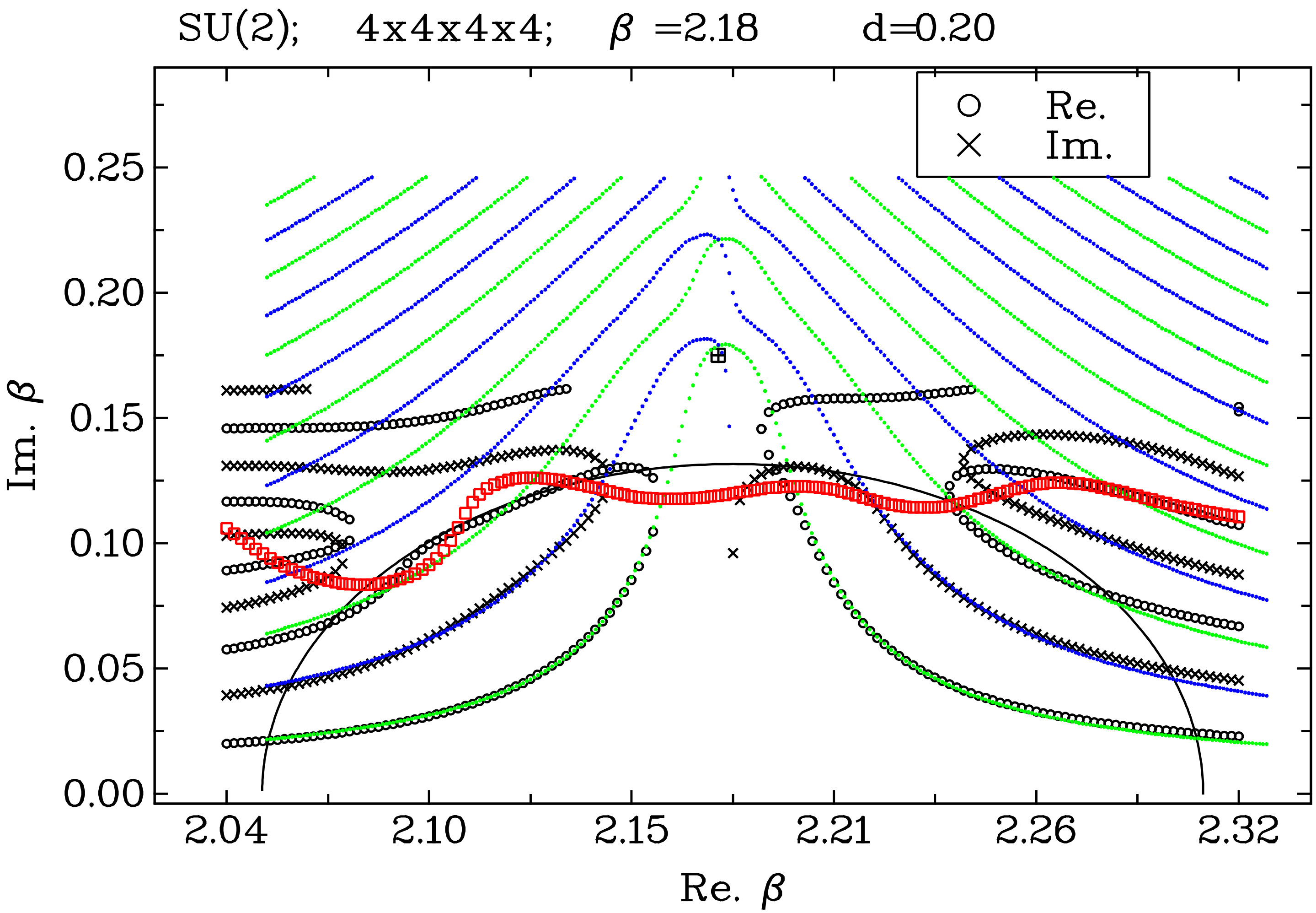
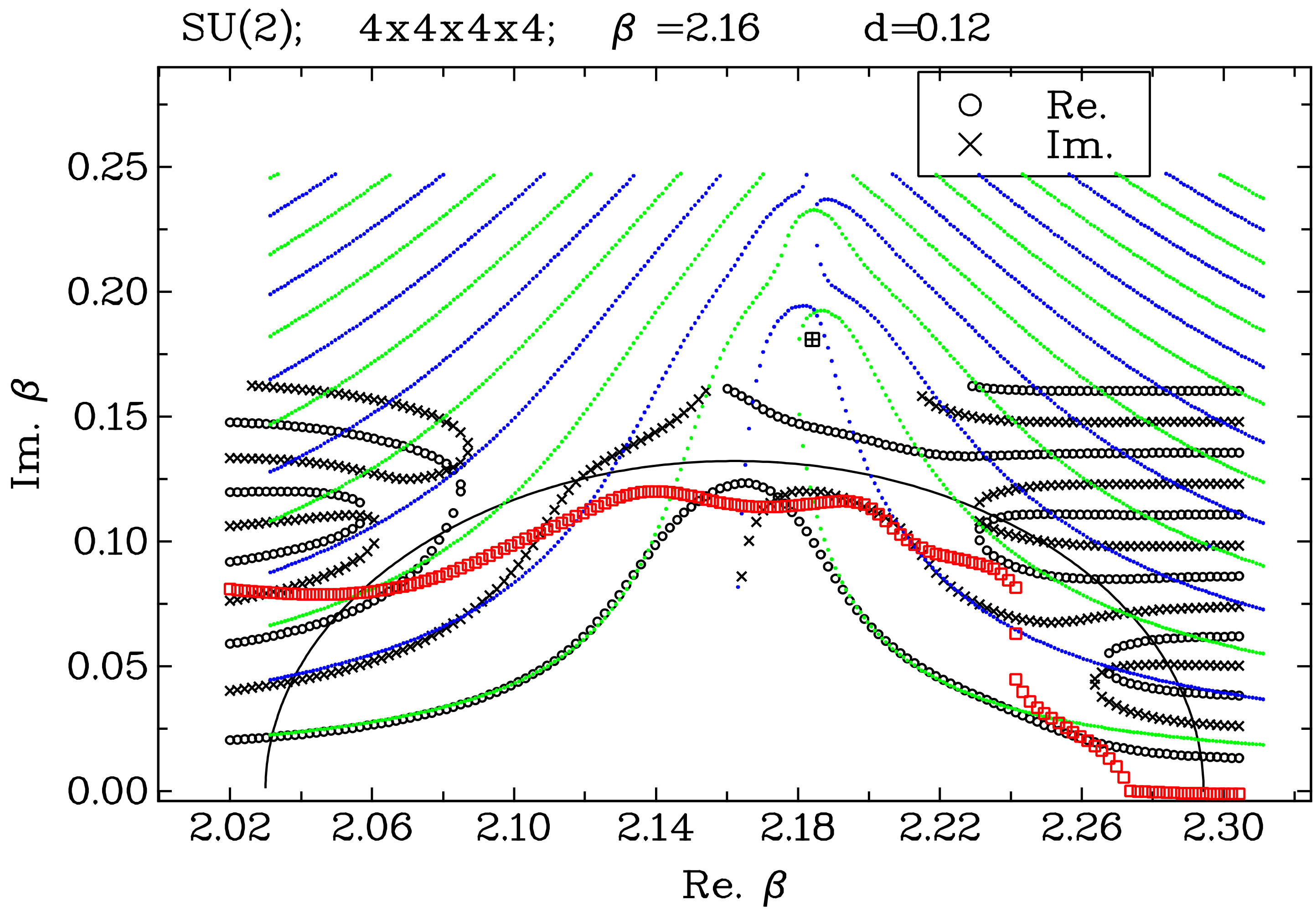
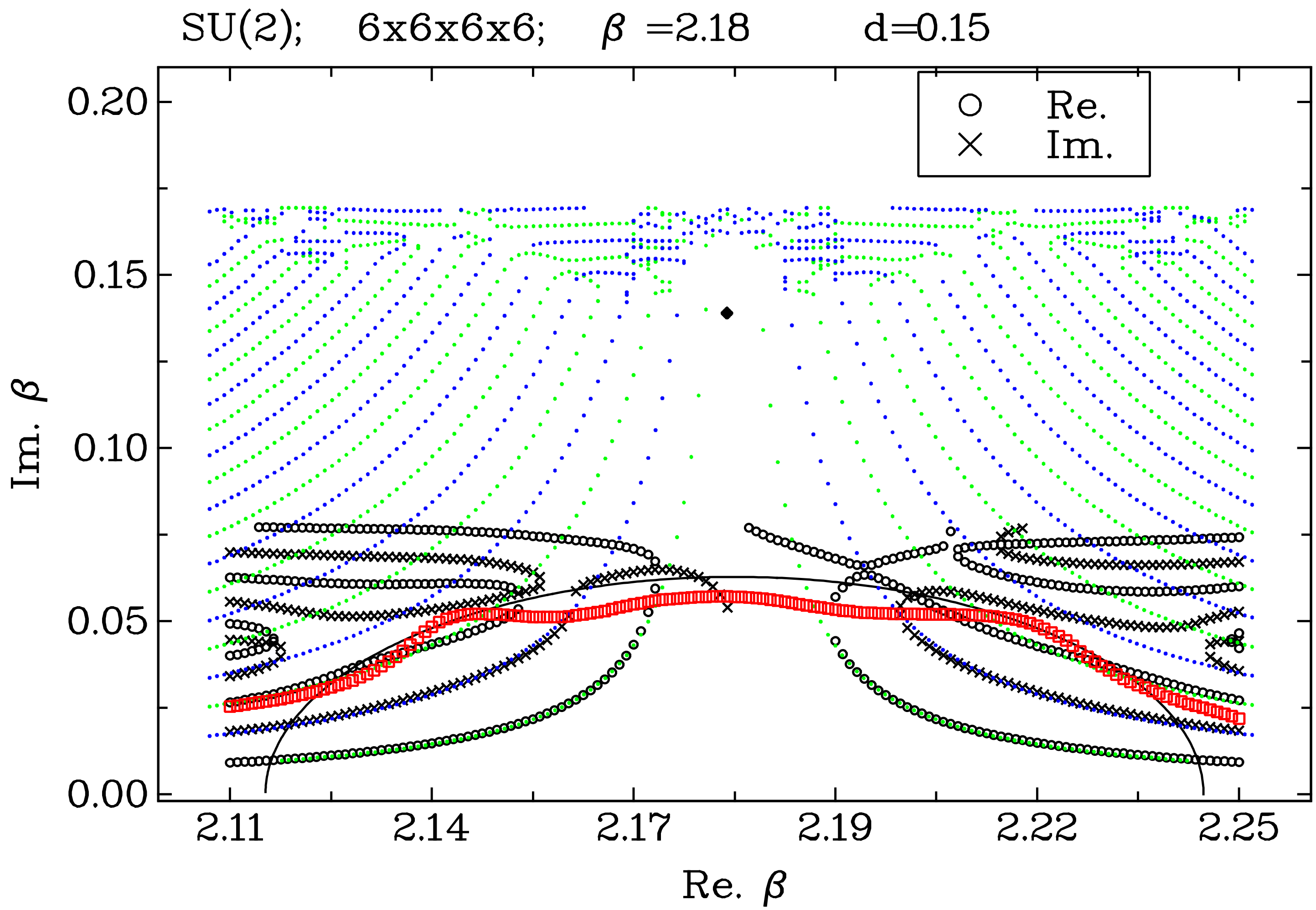


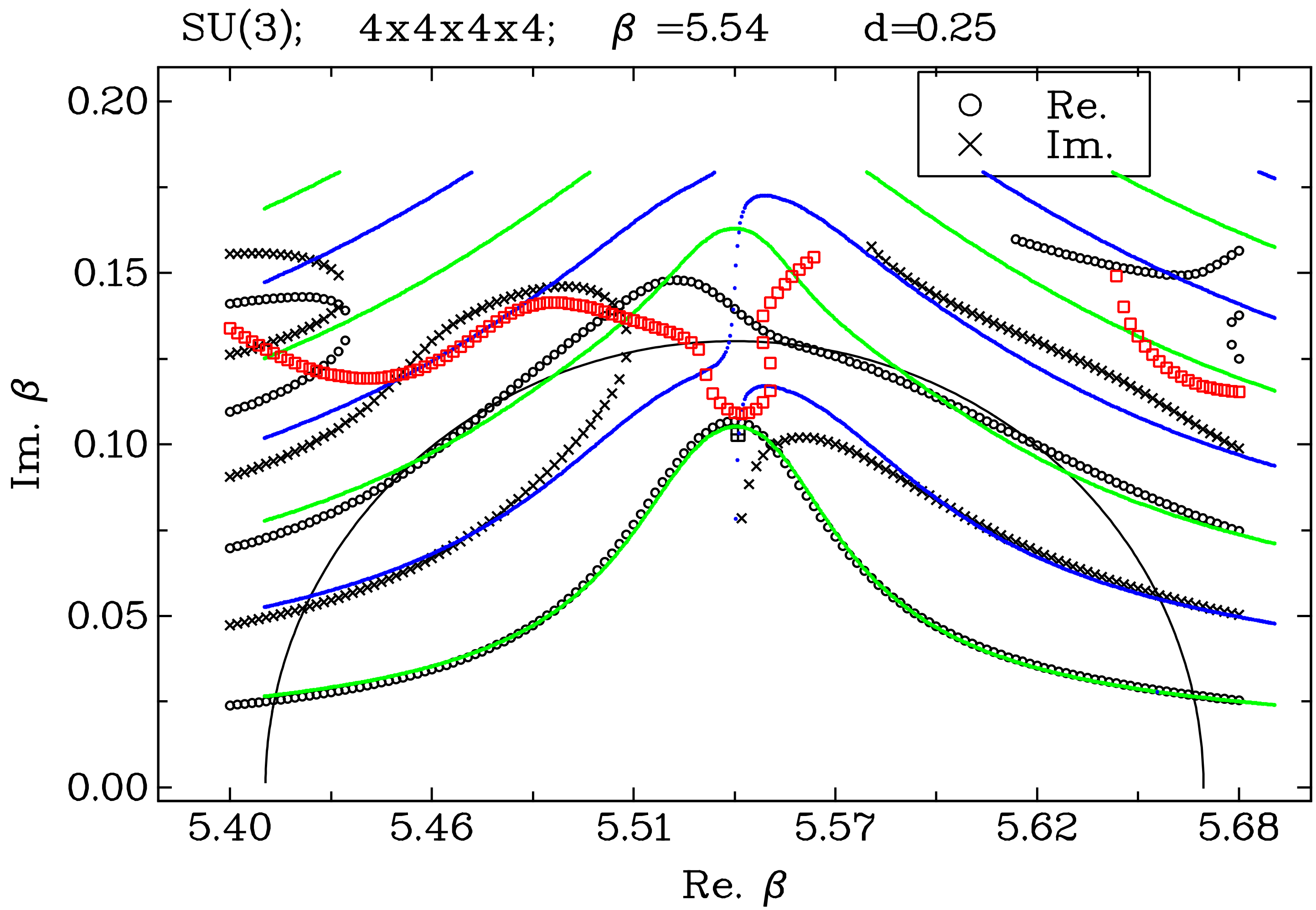
Figure 6: Zeros of the real (crosses) and imaginary (circles) using MC for $SU(2)$ on a 4^4 lattice at $\beta = 2.18$. The small dots are the values for the real (green) and imaginary (blue) parts obtained from the 4 parameter model. The MC exclusion region boundary for $d = 0.2$ is represented by boxes (red). The crossed box at (2.176, 0.175) has been obtained with the perturbative method.


SU(2); 4x4x4x4; β =2.18 d=0.20
Re.
Im.
0.25
0.20
0.15
0.10
0.05
0.00
Im. β
2.04
2.10
2.15
2.21
2.26
2.32
Re. β

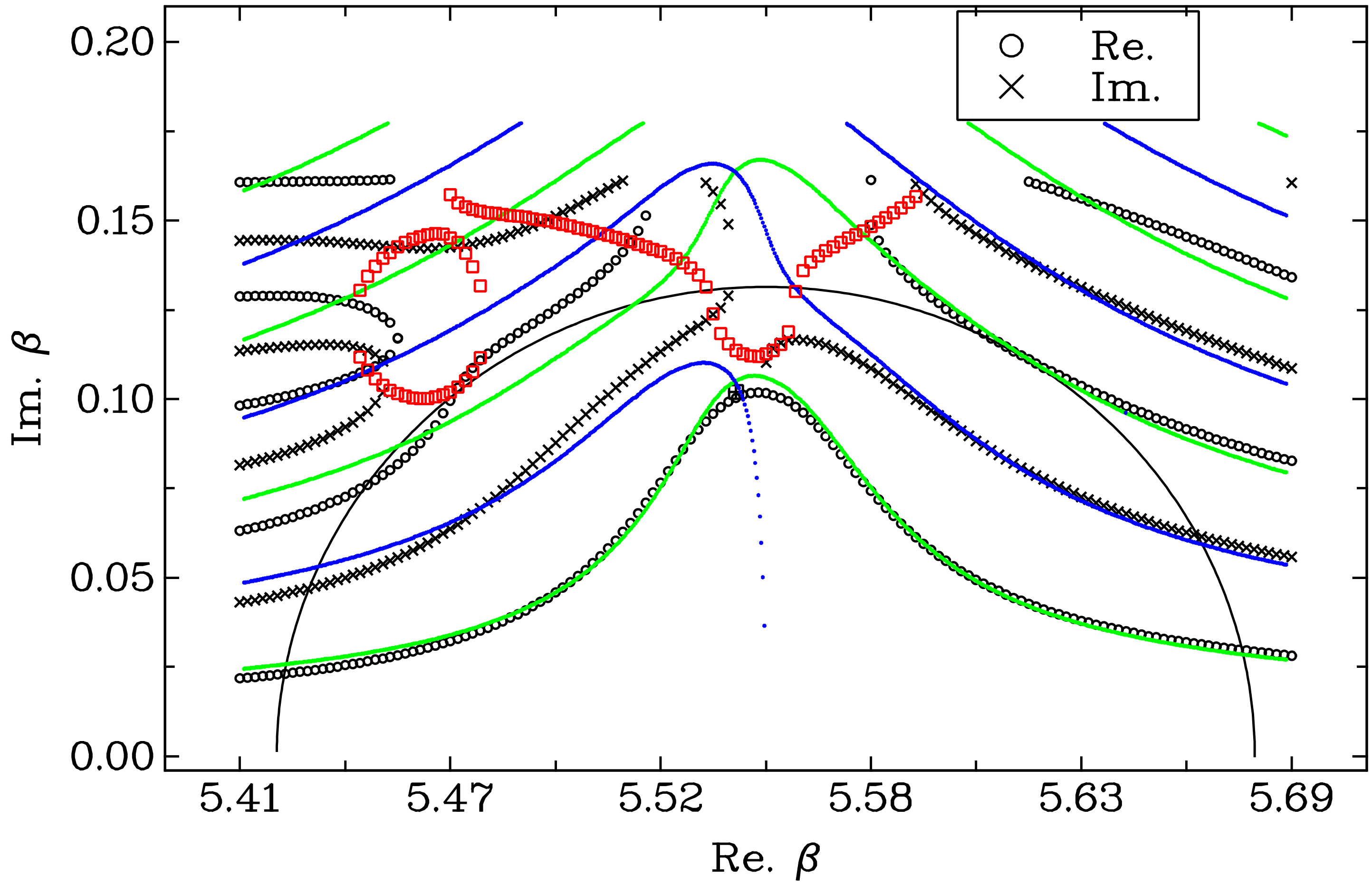


SU(2); 4x4x4x4; β =2.16 d=0.12
Re.
Im.
0.25
0.20
0.15
0.10
0.05
0.00
Im. β
2.02
2.06
2.10
2.14
2.18
2.22
2.26
2.30
Re. β


SU(2); 6x6x6x6; β =2.18 d=0.15
Re.
Im.
Im. β
Re. β
0.20
0.15
0.10
0.05
0.00
2.11
2.14
2.17
2.19
2.22
2.25


SU(3); 4x4x4x4; β =5.54 d=0.25
Re.
Im.
0.20
0.15
0.10
0.05
0.00
Im. β
5.40
5.46
5.51
5.57
5.62
5.68
Re. β

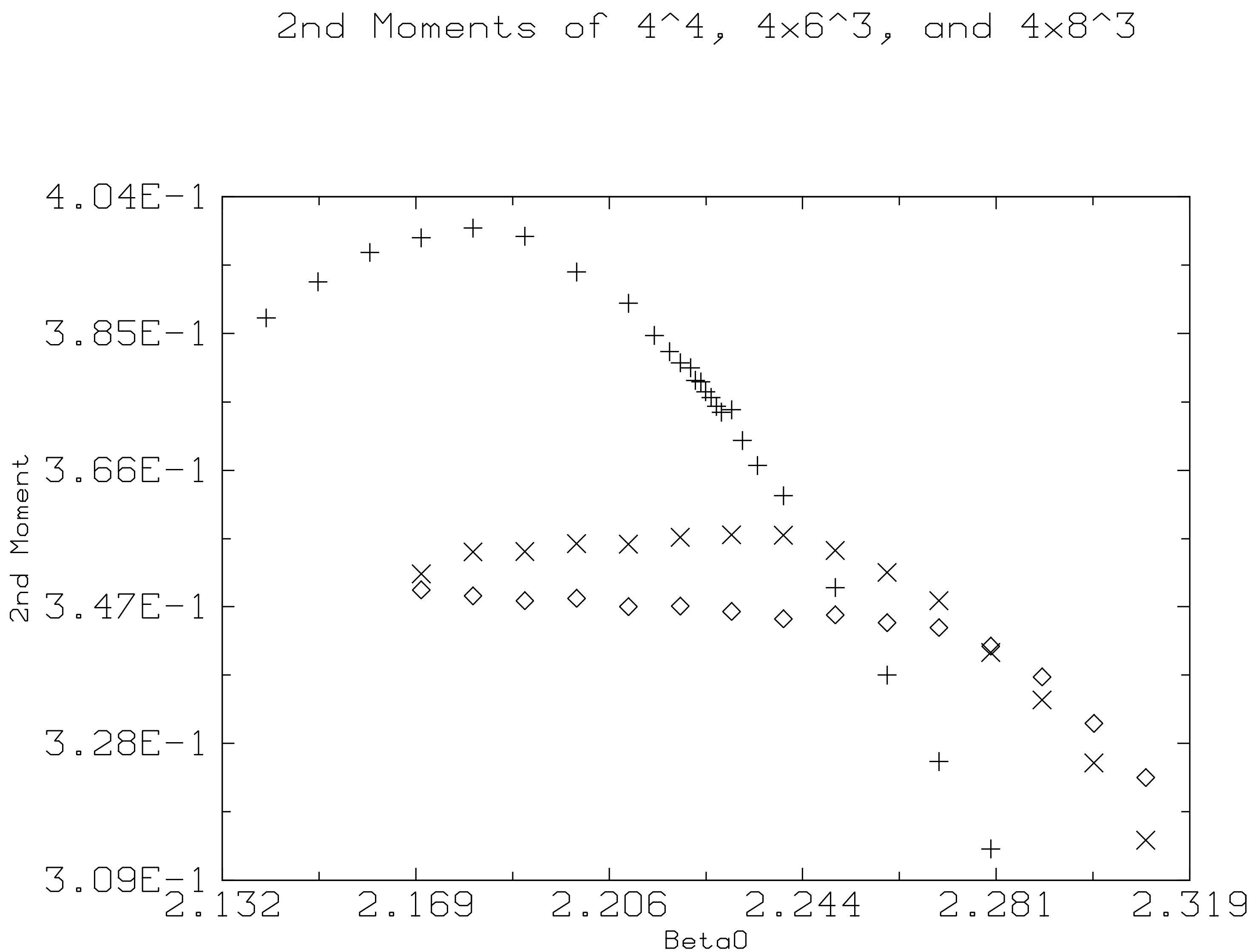
SU(3); 4x4x4x4; β =5.55 d=0.25

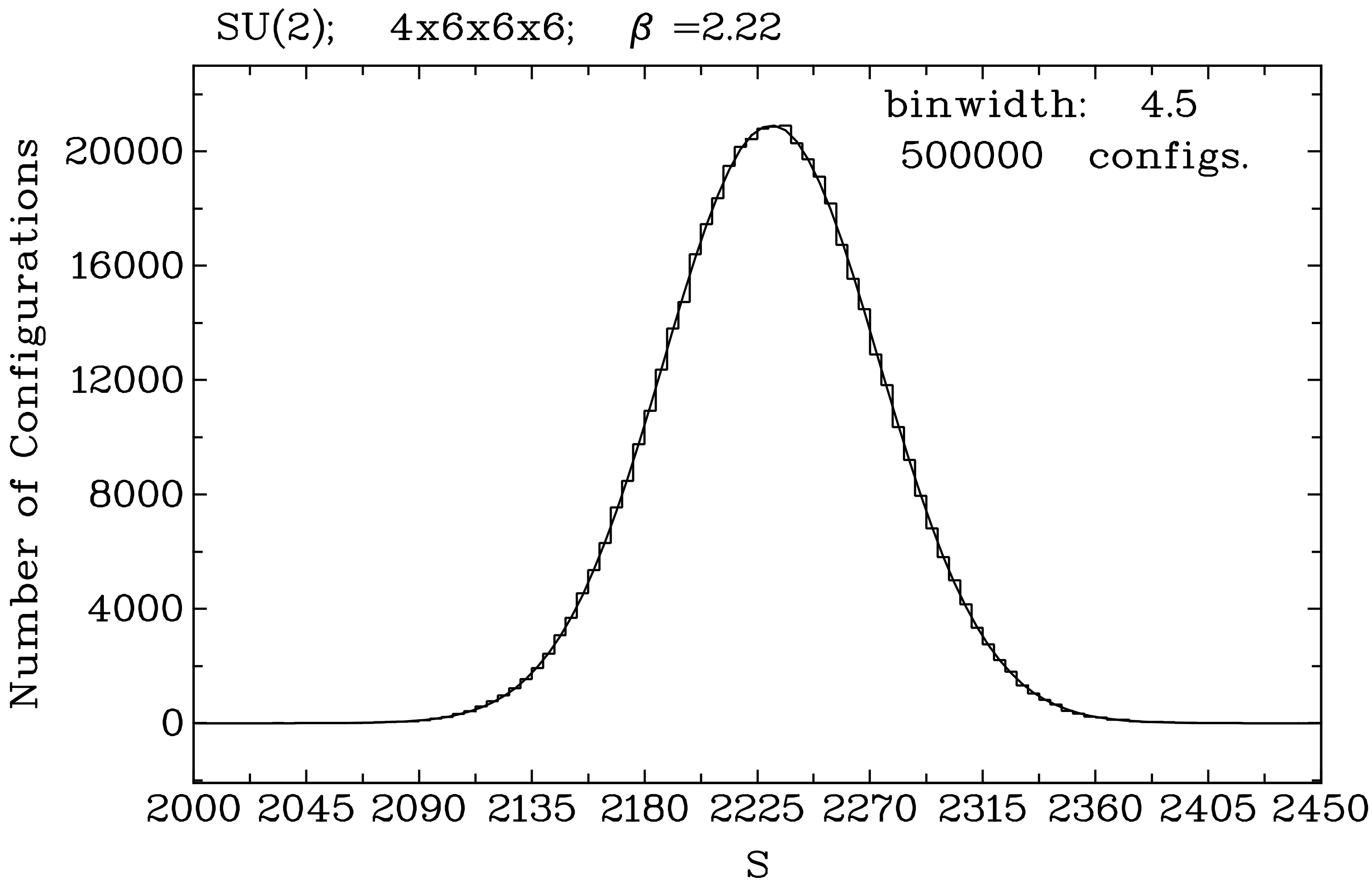


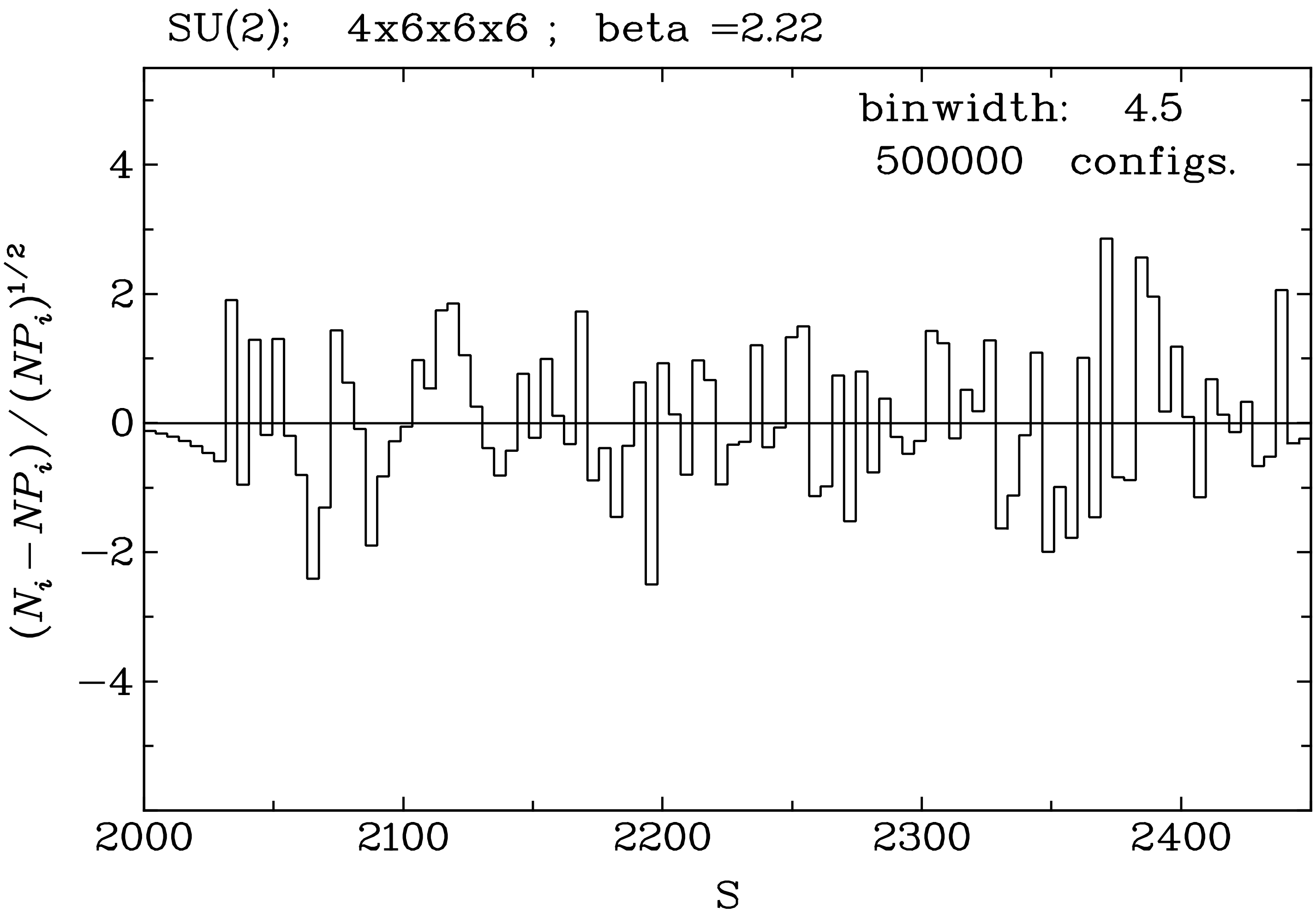
$SU(2)$ **on** 4×6^3

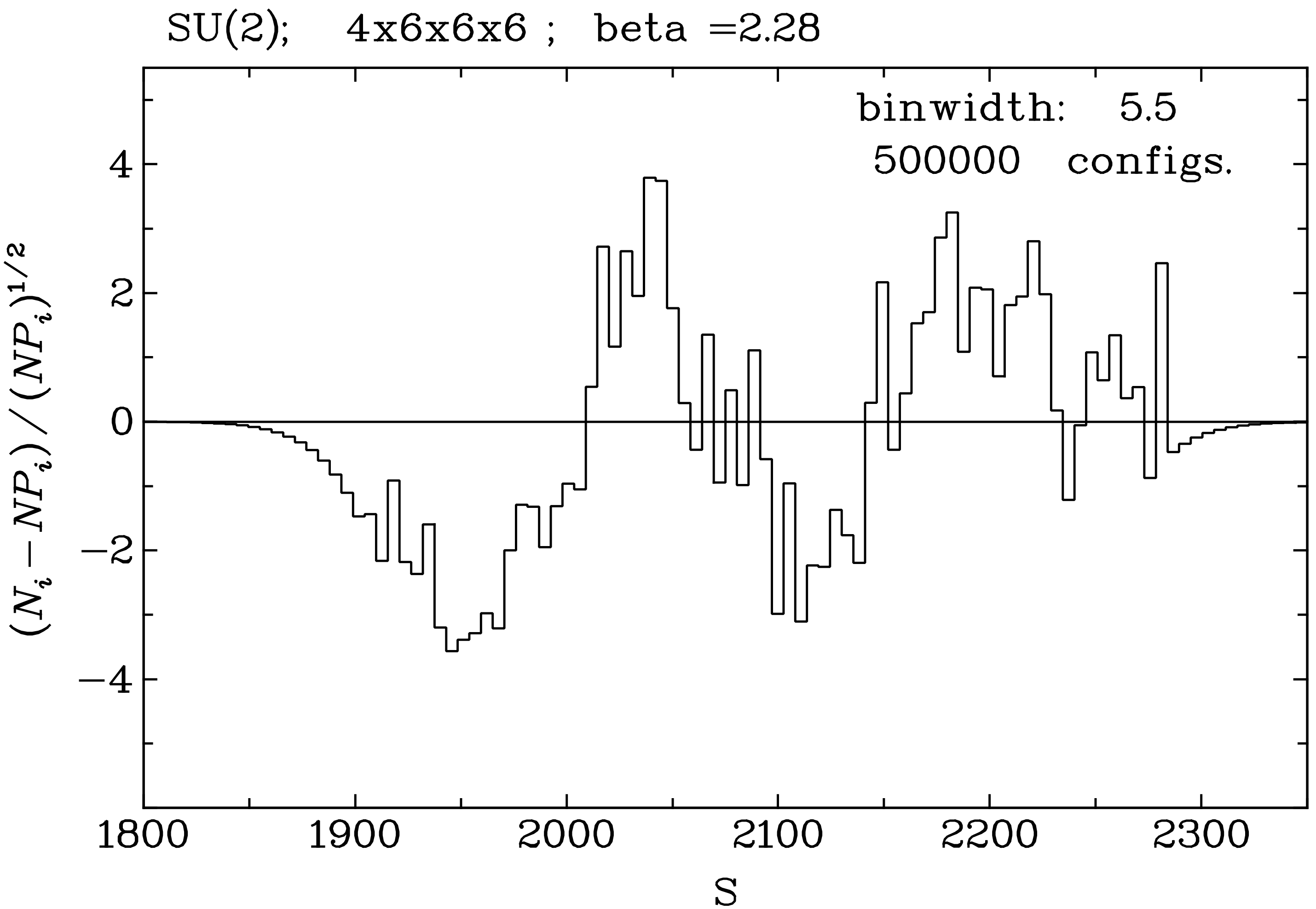
$\beta_c = 2.9686$ (Finnberg, Karsch, Heller, NPB 392)

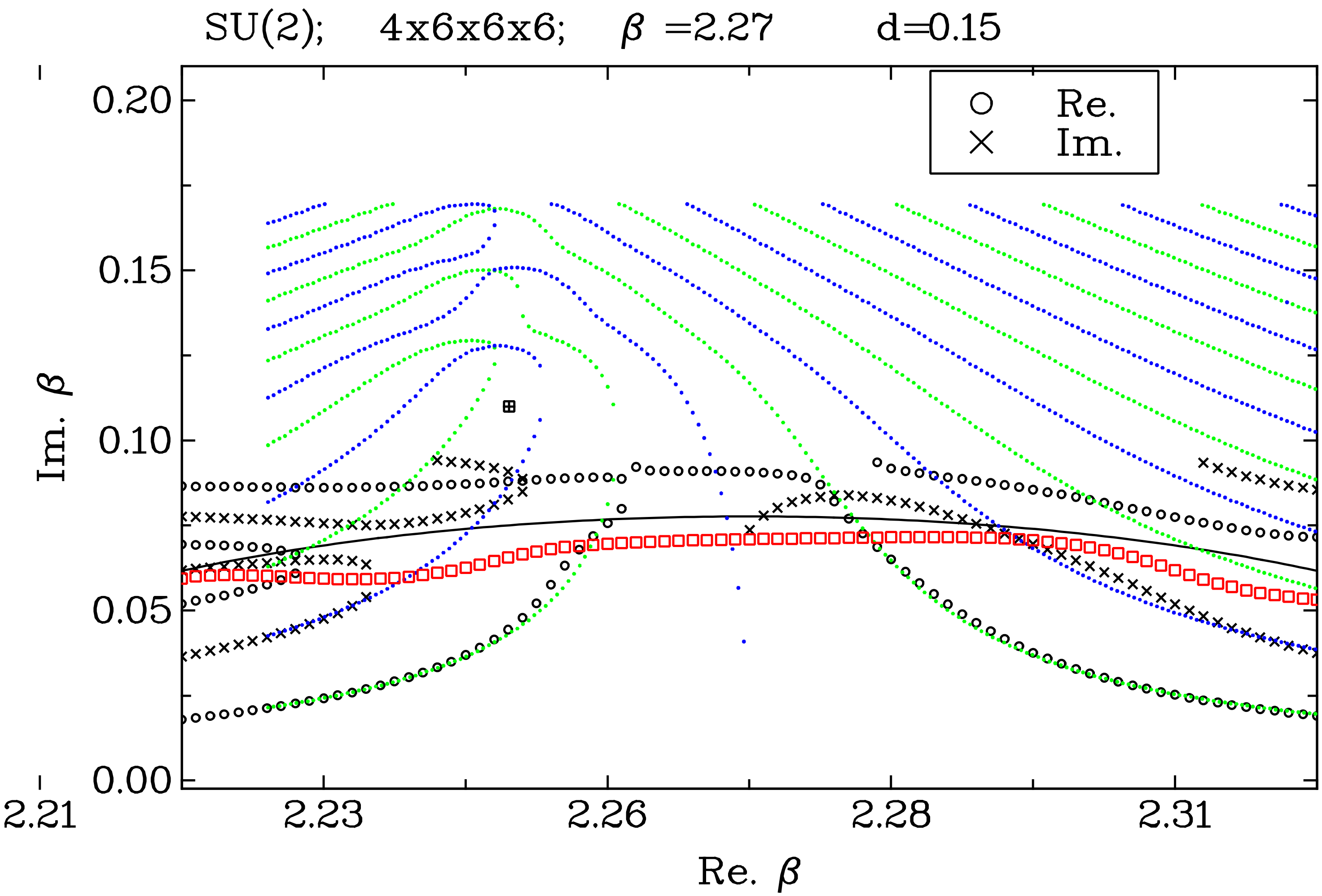
Where does the zero of 4^4 $(\beta = 2.18(1) \pm 0.18(2))$ go?

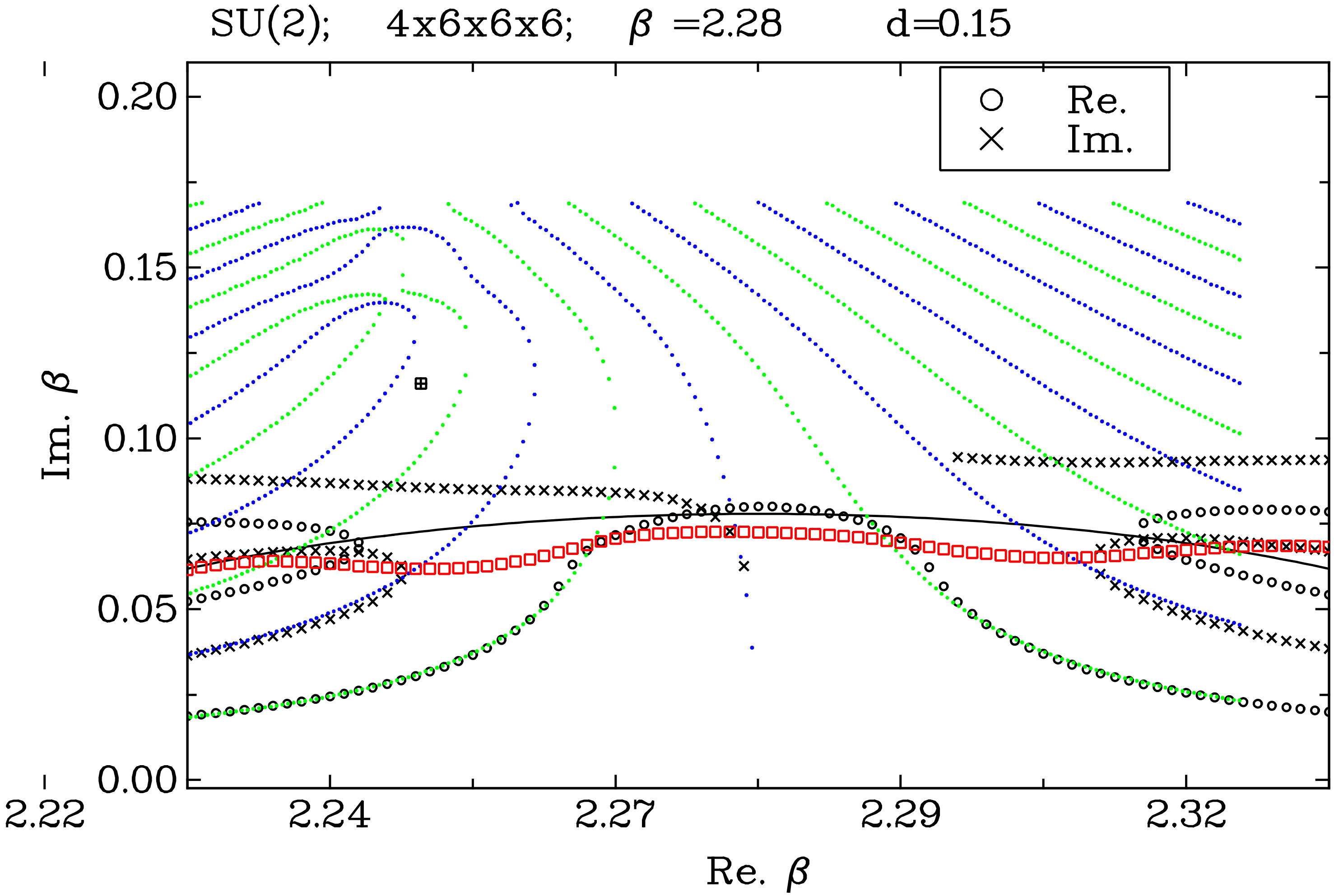

2nd Moments of 4^4, 4x6^3, and 4x8^3
2nd Moment
4.04E-1
3.85E-1
3.66E-1
3.47E-1
3.28E-1
3.09E-1
2.132
2.169
2.206
2.244
2.281
2.319
Beta0


SU(2); 4x6x6x6; β =2.22
binwidth: 4.5
500000 configs.
Number of Configurations
0
4000
8000
12000
16000
20000
2000 2045 2090 2135 2180 2225 2270 2315 2360 2405 2450
S


SU(2); 4x6x6x6 ; beta =2.22
binwidth: 4.5
500000 configs.
4
2
0
−2
−4
$(N_i - NP_i)/(NP_i)^{1/2}$
2000
2100
2200
2300
2400
S


SU(2); 4x6x6x6 ; beta =2.28
binwidth: 5.5
500000 configs.
$(N_i - NP_i)/(NP_i)^{1/2}$
4
2
0
−2
−4
1800
1900
2000
2100
2200
2300
S


SU(2); 4x6x6x6; β =2.27 d=0.15
Re.
Im.
0.20
0.15
0.10
0.05
0.00
Im. β
2.21
2.23
2.26
2.28
2.31
Re. β


SU(2); 4x6x6x6; β =2.28 d=0.15
Re.
Im.
0.20
0.15
0.10
0.05
0.00
Im. β
2.22
2.24
2.27
2.29
2.32
Re. β

Conclusions

- We have build a ”ladder” of methods that can be applied for increasing values of the imaginary part. Note: this is what we need to take the inverse Laplace transform of $Z(\beta)$ and obtain the density of states.

- We found a way to distinguish fake and true MC zeros that works well with non-Gaussian examples.

- Fitting methods based on cubic and quartic perturbations work for larger values of the imaginary part. Perturbative methods work when numerical integration fails.

- Numerical estimates of the zeros for $SU(2)$ and $SU(3)$.

Numerical Estimates of the zeros

4^4 lattices:

$\beta = 2.18(1) \pm 0.18(2)$ for $SU(2)$ (differ from Falcioni 2.23+i0.155 obtained with MC outside regions of confidence).

$\beta = 5.54(2) \pm 0.10(2)$ for $SU(3)$ (agrees with Berg et al.) and another zero at $\beta = 5.54(2) \pm 0.16(2)$.

46^3 lattices:

$\beta = 2.25(1) \pm 0.13(2)$ for $SU(2)$

Estimates on larger lattices require more accurate values of the third and fourth moments.

Work in progress

- Check selfconsistency of the parametrization at different β.
- Estimate the density of states
- Effect of adjoint term, finite-temperature.
- Im/Re larger for $SU(2)$, effects visible at lower order in perturbation theory?