Taylor Expansion Method

in

3-State Potts Model

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should have been

Curvature of the Critical Line

in

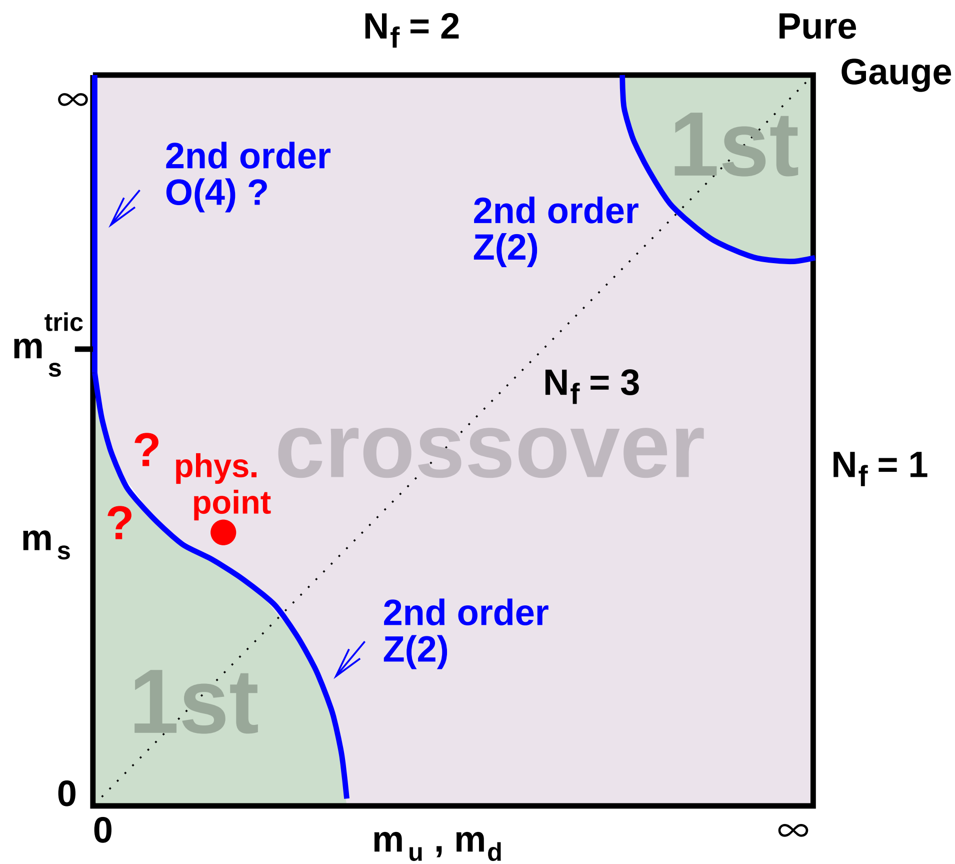
Potts model and QCD

Plan of Talk

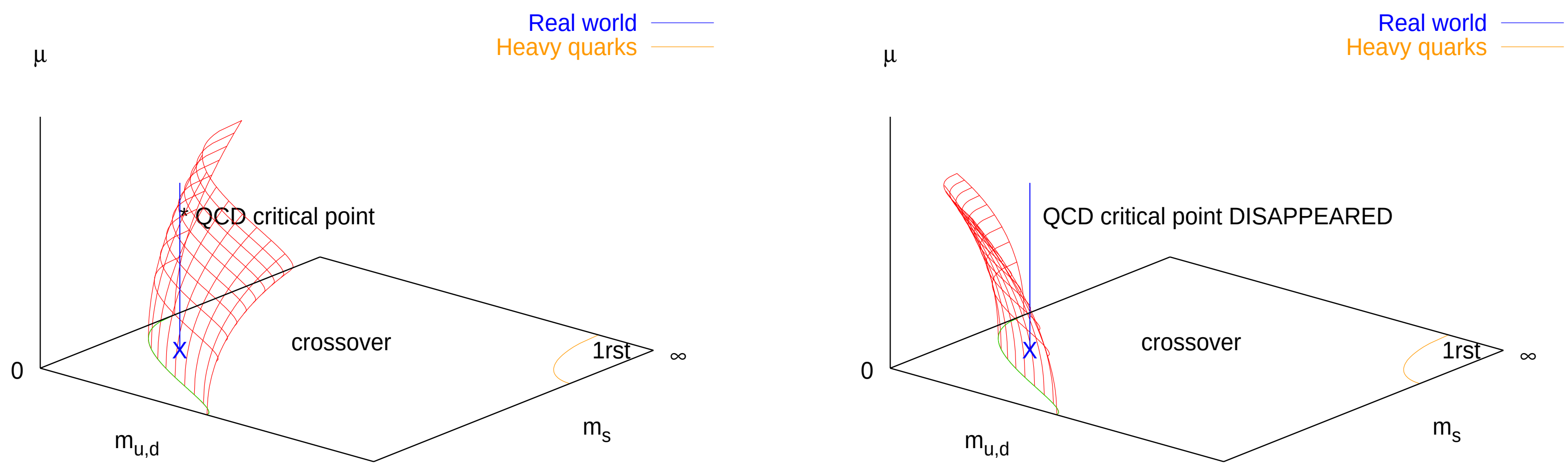
0. Motivation
1. QCD and 3-D 3-state Potts model
2. Taylor Expansion
3. “Numerical Derivative”
4. Summary

0. Motivation

- phase diagram with $\mu = 0$



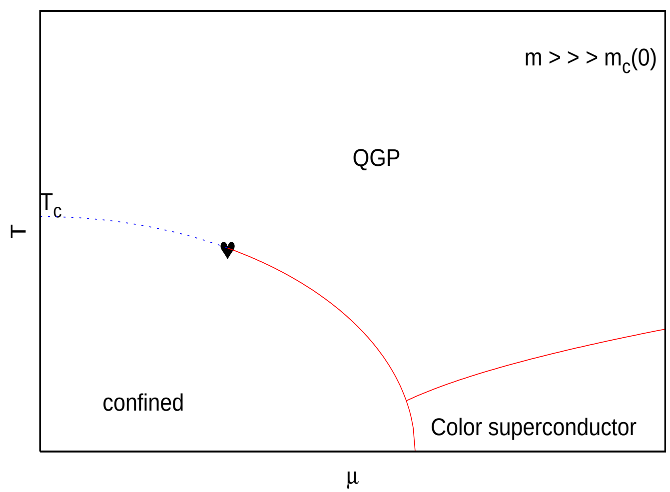
- three parameter space $(\mu, m_{u,d}, m_s)$

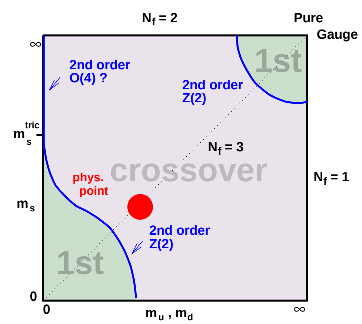


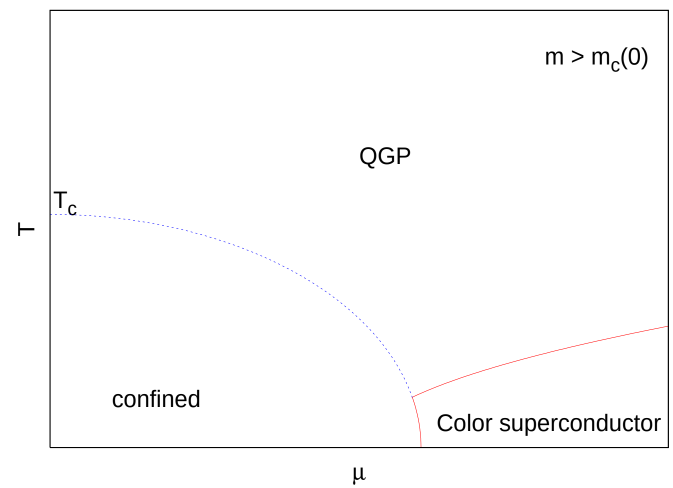
(Ph. de Forcrand and O. Philipsen, hep-lat/0607017)

- $\frac{m_c(\mu)}{m_c(0)} = 1 - 0.7(4) \left(\frac{\mu}{\pi T}\right)^2$ (using 4-th order Binder cumulant)

- standard scenario vs. exotic scenario







- the curvature, $\frac{dm}{d\mu^2}$, at $(m_c, \mu = 0)$ is crucial
- obtain $\frac{dB_4}{dm}$ and $\frac{dB_4}{d{\mu_I}^2}$ at $(m_c, \mu = 0)$ and get the curvature at $(m_c, \mu = 0)$ by taking ratio of these two

[1] find analytic expressions for $\frac{dB_4}{dm}$ and $\frac{dB_4}{d{\mu_I}^2}$ and evaluate them at $(m_c, \mu = 0)$ (a la Swansea-Bielefeld)

[2] or calculate ΔB_4 under Δm and $\Delta {\mu_I}^2$ numerically (reweighting)

1. QCD and 3-D 3-state Potts model

- Finite Temperature QCD with infinitely heavy quark (quenched QCD with the periodic boundary condition for the time direction)

$=$ 3-dimensional 3-state Potts model

in the sense that both have the same global $Z(3)$ symmetry

- First order phase transition in

M_∞ FT QCD and $h = 0$ 3-D $Z(3)$ Potts model

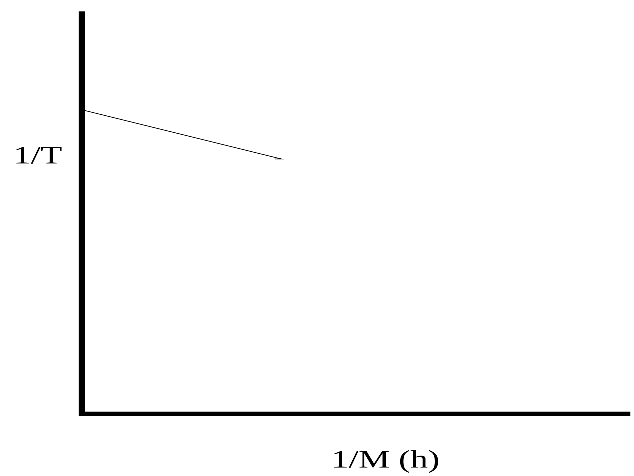
- Heavy quark mass (but not M_∞) FT QCD

$=$ external magnetic field in Potts model

$\rightarrow$ symmetry breaking

• Effect of symmetry breaking term “weakens” the phase transition

• Typical phase diagram



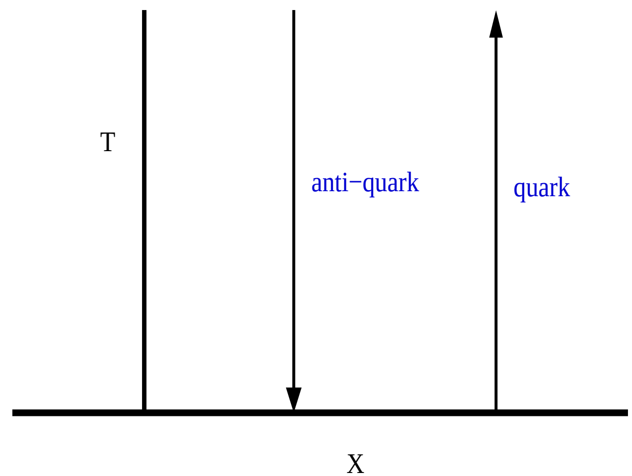
(ex.: P. Hasenfratz et al, PLB 133 (1983) 221;C. Alexandrou et al, PRD 60 (1999) 034504)

• Universality
$\rightarrow$ 3-D Z(3) Potts model and heavy quark FTQCD share the same critical properties

- introducing heavy static quark in FT quenched QCD

$\longrightarrow$ baryon density effect in FT QCD with heavy quark limit

(S. Chandrasekharan, Nucl.Phys.Proc.Suppl.94:71-78,2001; S.K. et al, hep-lat/0510069)



with weight factor $e^{-m/T}$ for a quark and a anti-quark with mass m

• the partition function for n static quark and $\overline{n}$ static anti-quark is

$$Z_{n,\overline{n}} = \int dU e^{-S_g} e^{-m(n+\overline{n})/T} \frac{\Phi^n}{n!} \frac{\Phi^{*\overline{n}}}{\overline{n}!} \qquad (1)$$

where Φ is the Polyakov line (Φ^*, anti-Polyakov line)

• The grand canonical partition function becomes

$$\begin{aligned} Z &= \sum_{n,\overline{n}} Z_{n,\overline{n}} e^{\mu(n-\overline{n})/T} \\ &= \int dU e^{-S_g + \sum_{\vec{x}} [h\Phi(\vec{x}) + h'\Phi^*(\vec{x})]} \end{aligned} \qquad (2)$$

where $h = e^{-(m-\mu)/T}$ and $h' = e^{-(m+\mu)/T}$

• By modeling kinetic energy part of the action, we arrive at

$$S = -k \sum_{i,\vec{x}} \delta_{\Phi(\vec{x}),\Phi(\vec{x}+i)} - \sum_{\vec{x}} [h\Phi(\vec{x}) + h'\Phi^*(\vec{x})] \qquad (3)$$

where Φ is $Z(3)$ Polyakov line and

$$\boxed{h = e^{-(m-\mu)/T} = h_m e^{\mu/T},\ h' = e^{-(m+\mu)/T} = h_m e^{-\mu/T}}$$

when $h \neq h'^*$, S is complex.

• large chemical potential limit ($h' = 0$) of FT QCD with heavy quark has been investigated by many people

(ex.:T.Blum et al (PRL 76 (1996) 1019);Alford et al (NPB 602 (2001), 61))

• first order phase transition changes into 2nd order transition with large m, μ with small μ/m

• with $\mu \neq 0$, action is complex

$\longrightarrow$ sign problem?

$\longrightarrow$ the partition function becomes with $\overline{\mu} = \mu/T$

$$Z = \int \mathcal{D}b(e^{k} - 1)^{N_b} \prod_{C} \Big[e^{2h_M|C|\cosh\overline{\mu}} + 2e^{-h_M|C|\cosh\overline{\mu}} \cos(\sqrt{3}h_M|C|\sinh\overline{\mu}) \Big] \tag{4}$$

• real partition function !!!

• solution of the sign problem is different from that of two color QCD

• explicit summation produces real partition function

• we are interested in small chemical potential region

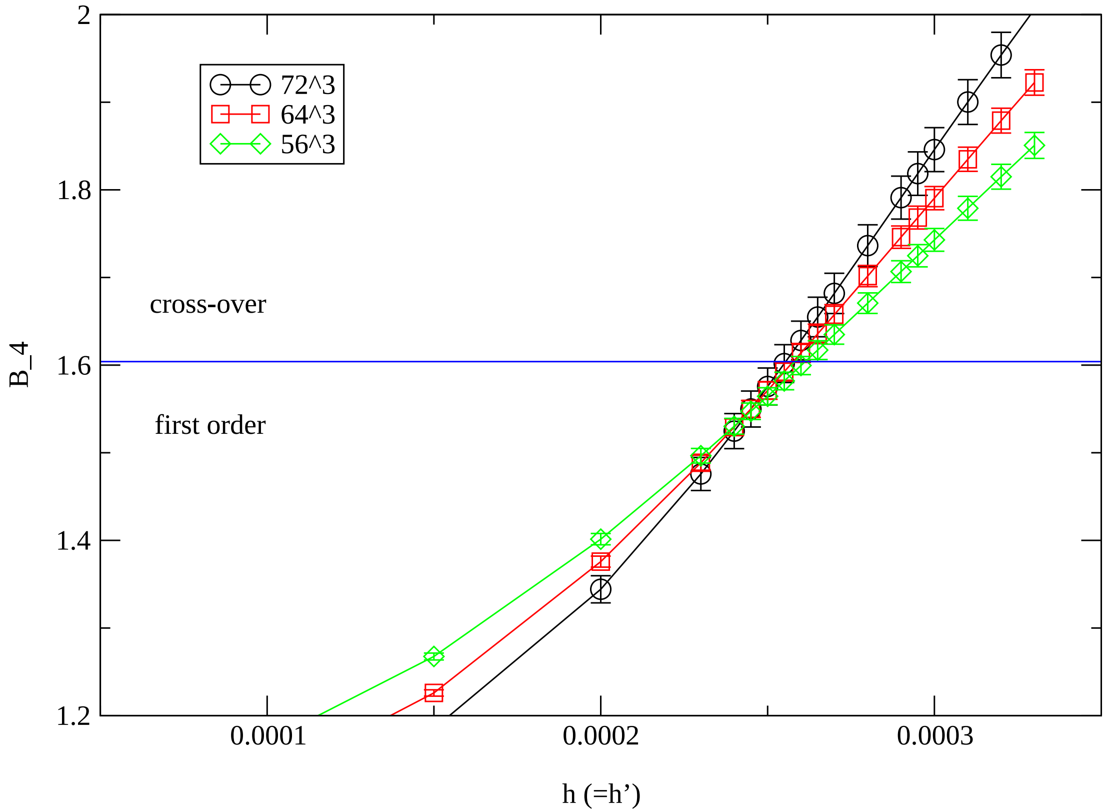
$\longrightarrow$ we need to keep h' term

• for imaginary chemical potential with $\overline{\mu}_I = \mu_I / T$

$$Z_I = \int \mathcal{D}b (e^k - 1)^{N_b} \prod_C \Big[e^{2 h_M |C| \cos \overline{\mu}_I} + 2 e^{-h_M |C| \cos \overline{\mu}_I} \cosh(\sqrt{3} h_M |C| \sin \overline{\mu}_I) \Big] \quad (5)$$

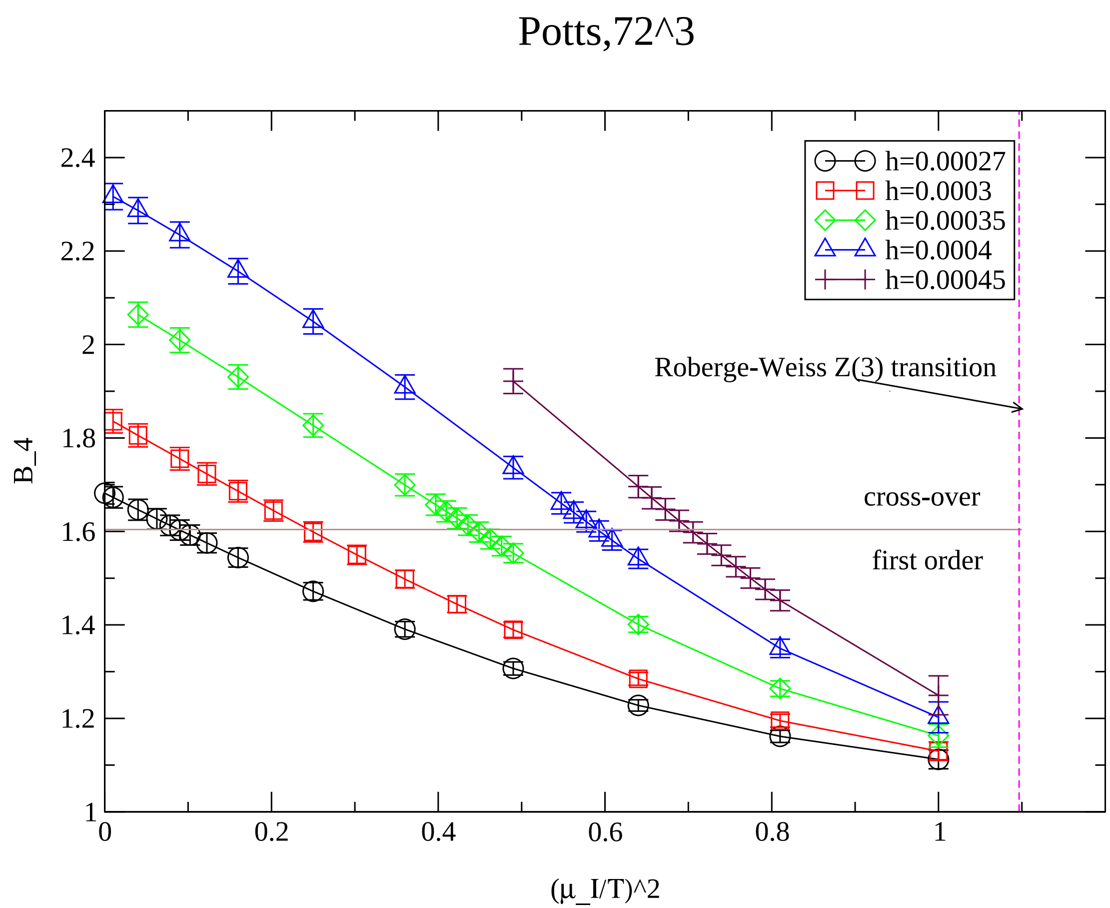
• For the imaginary chemical potential,
$h = e^{-\beta(M - i\mu_I)}$ and $h' = e^{-\beta(M + i\mu_I)} = h^*$
In this case, S is real as expected.

• For both real and imaginary chemical potential, the partition function integrand in cluster formulation is real and positive $\rightarrow$ no sign problem!



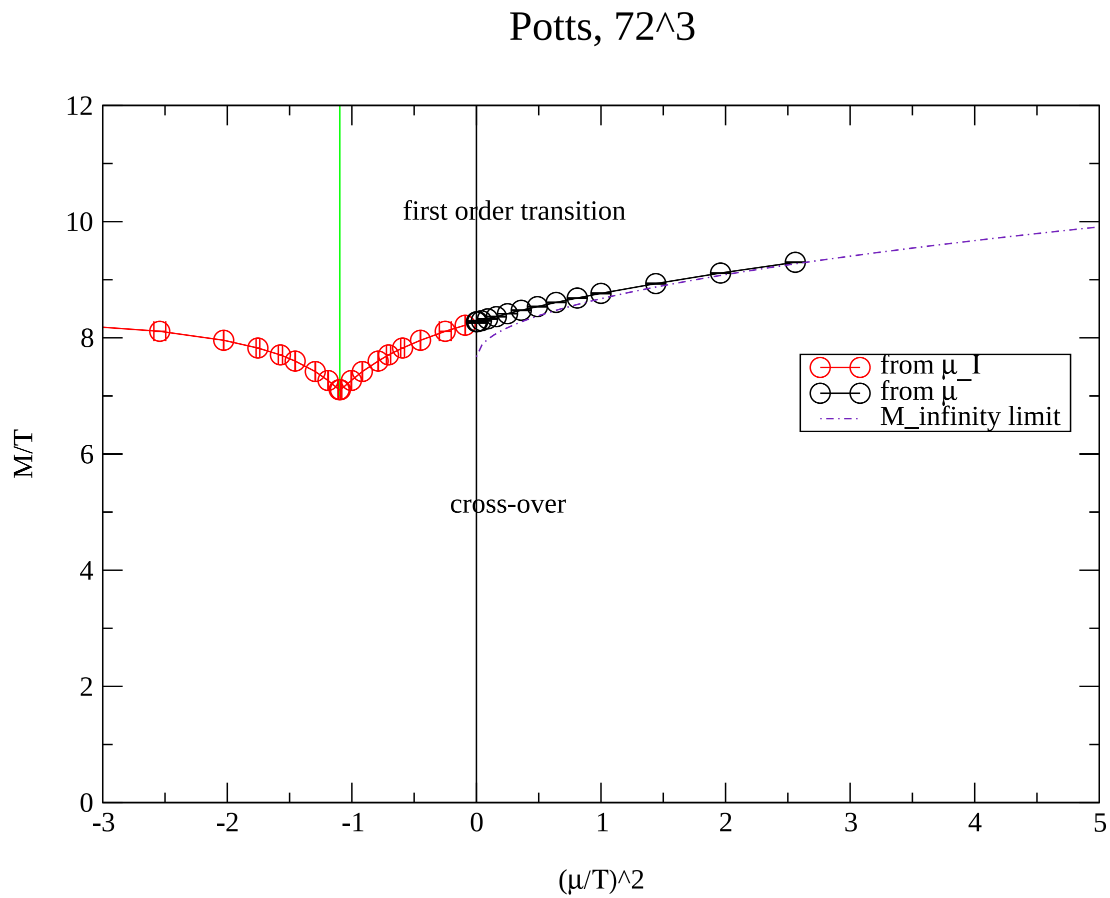
Binder cumulant for magnetization ($h = h'$)

$h_c = 0.000255(5), k_c = 0.54940(4)$ compared to $(0.000258(3), 0.54938(2))$ by Karsch-Stickan



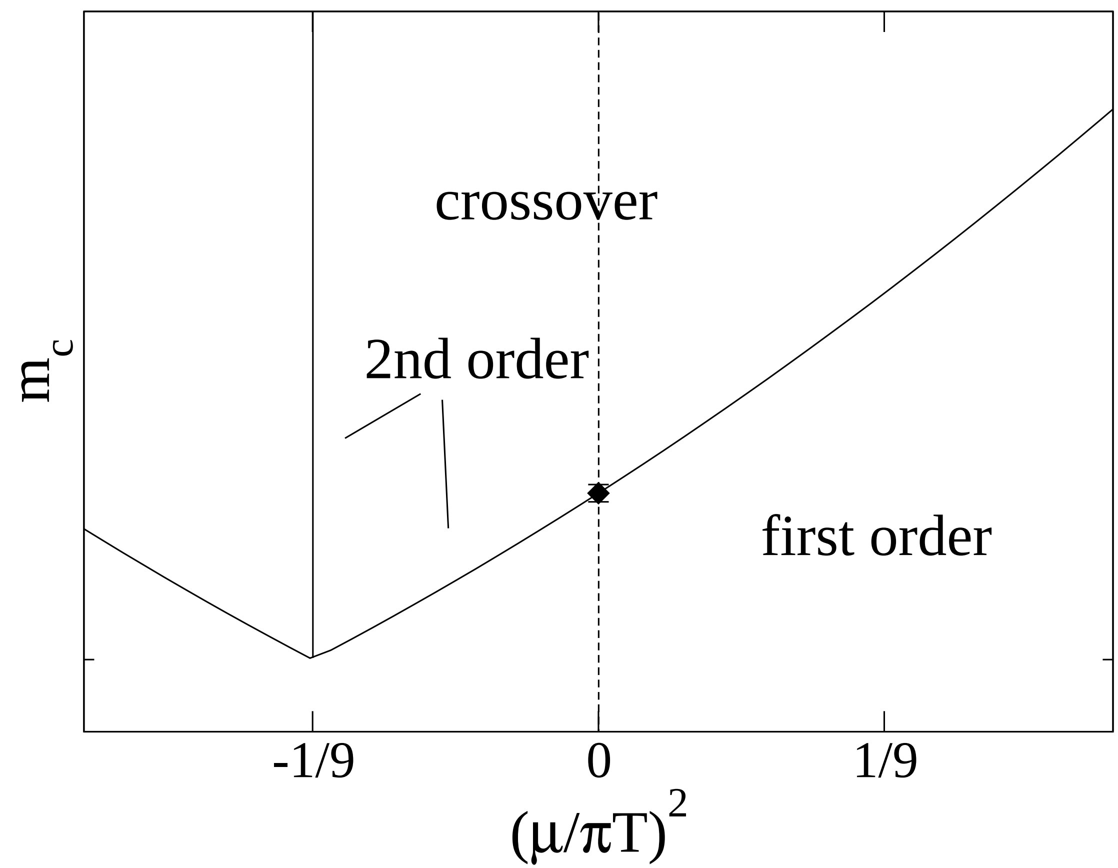
Binder cumulant for magnetization for various μ_I

- curvature depends on the external magnetic field (or mass in QCD case)



M/T for 2nd order transition vs. $(\mu/T)^2$

- the curve is non-linear



schematic three degenerate flavor QCD phase diagram

cf. Ph. de Forcrand and O. Philipsen, NPB 673 (2003), 170

2. Taylor Expansion

- we locate the critical point by

searching parameter space $(k, m/T, \mu/T)$

which gives $B_4 = \frac{\langle \tilde{M}^4 \rangle}{\langle \tilde{M}^2 \rangle^2} = 1.604$ (Ising value)

and $\langle \tilde{M}^3 \rangle = 0$

- we can get the curvature at $\mu/T = 0$ by demanding $\delta B_4 = \delta B_3 = 0$ while varying parameter $(k, \frac{m}{T}, \frac{\mu}{T})$

$$\delta B_4 = A\delta k + B\delta(m/T) + C\delta(\mu/T)^2 = 0$$
$$\delta B_3 = A'\delta k + B'\delta(m/T) + C'\delta(\mu/T)^2 = 0 \tag{6}$$

- In QCD case, similar argument holds for (β, m_i, μ)
- we need to evaluate

$$\frac{\partial}{\partial k}\langle \tilde{M}^n \rangle, \frac{\partial}{\partial m_i}\langle \tilde{M}^n \rangle, \frac{\partial^2}{\partial \mu^2}\langle \tilde{M}^n \rangle, \quad (7)$$

at $(k_c, h_c = h'_c)$ for 3-D 3-state Potts model

- likewise, for QCD

$$\frac{\partial}{\partial \beta}\langle \tilde{X}^n \rangle, \frac{\partial}{\partial m_i}\langle \tilde{X}^n \rangle, \frac{\partial^2}{\partial \mu^2}\langle \tilde{X}^n \rangle, \quad (8)$$

at (β_c, m_c)

- from δB_4 (3-D 3-state Potts model)

$$A = \frac{\langle \tilde{M}^4 \tilde{E} \rangle}{\langle \tilde{M}^2 \rangle^2} - 2 \frac{\langle \tilde{M}^4 \rangle \langle \tilde{M}^2 \tilde{E} \rangle}{\langle \tilde{M}^2 \rangle^3} \tag{9}$$

$$B = -h_m \frac{\langle \tilde{M}^5 \rangle}{\langle \tilde{M}^2 \rangle^2} \tag{10}$$

$$\begin{aligned} C = h_m \frac{\langle \tilde{M}^5 \rangle}{\langle \tilde{M}^2 \rangle^2} + h_m^2 \frac{1}{\langle \tilde{M}^2 \rangle^2} \langle (\tilde{M}^4 - \langle \tilde{M}^4 \rangle)(\Phi - \Phi^*)^2 \rangle \\ -2 h_m^2 \frac{\langle \tilde{M}^4 \rangle}{\langle \tilde{M}^2 \rangle^3} \langle (\tilde{M}^2 - \langle \tilde{M}^2 \rangle)(\Phi - \Phi^*)^2 \rangle \end{aligned} \tag{11}$$

- from δB_3 (3-D 3-state Potts model)

$$A' = \langle \tilde{M}^3 \tilde{E} \rangle - 3\langle \tilde{M}^2 \rangle \langle \tilde{M} \tilde{E} \rangle \tag{12}$$

$$B' = -h_m[\langle \tilde{M}^4 - 3\langle \tilde{M}^2 \rangle^2] \tag{13}$$

$$C' = h_m[\langle \tilde{M}^4 \rangle - 3\langle \tilde{M}^2 \rangle^2] + h_m^2 \langle [\tilde{M}^3 - 3\langle \tilde{M}^2 \rangle \tilde{M}](\Phi - \Phi^*)^2 \rangle] \tag{14}$$

- we get the curvature, 0.528(1), at $k = 0.54938, h = 0.000259$

- need to calculate exactly at $(k_c, {h_m}^c)$ to satisfy the simplified expression

- reweighting ? evaluating 6-th order moment ?

- from δB_4 (QCD)

$$A = \frac{1}{\langle \tilde{X}^2 \rangle^2} [-\langle \tilde{X}^4 \frac{\partial S}{\partial \beta} \rangle - \langle \tilde{X}^4 \rangle \langle \frac{\partial S}{\partial \beta} \rangle + 2 \frac{\langle \tilde{X}^4 \rangle}{\langle \tilde{X}^2 \rangle} \langle \tilde{X}^2 \frac{\partial S}{\partial \beta} \rangle] \qquad (15)$$

$$B = \frac{1}{\langle \tilde{X}^2 \rangle^2} [4 \langle \tilde{X}^3 \frac{\partial X}{\partial m} \rangle - 4 \frac{\langle \tilde{X}^4 \rangle}{\langle \tilde{X}^2 \rangle} \langle \tilde{X} \frac{\partial X}{\partial m} \rangle - \langle \tilde{X}^4 \frac{\partial S}{\partial m} \rangle$$
$$- \langle \tilde{X}^4 \rangle \langle \frac{\partial S}{\partial m} \rangle + 2 \frac{\langle \tilde{X}^4 \rangle}{\langle \tilde{X}^2 \rangle} \langle \tilde{X}^2 \frac{\partial S}{\partial m} \rangle] \qquad (16)$$

$$C = \frac{1}{\langle \tilde{X}^2 \rangle^2} [\langle \tilde{X}^4 \{ (\frac{\partial S}{\partial \mu})^2 - \frac{\partial^2 S}{\partial \mu^2} \} \rangle + \langle \tilde{X}^4 \rangle \langle \{ (\frac{\partial S}{\partial \mu})^2 - \frac{\partial^2 S}{\partial \mu^2} \} \rangle$$

$$-2 \frac{\langle \tilde{X}^4 \rangle}{\langle \tilde{X}^2 \rangle} \langle \tilde{X}^2 \{ (\frac{\partial S}{\partial \mu})^2 - \frac{\partial^2 S}{\partial \mu^2} \} \rangle + 12 \langle \tilde{X}^2 (\frac{\partial X}{\partial \mu})^2 \rangle - 4 \frac{\langle \tilde{X}^4 \rangle}{\langle \tilde{X}^2 \rangle} \langle (\frac{\partial X}{\partial \mu})^2 \rangle$$

$$+4 \langle \tilde{X}^3 (\frac{\partial^2 X}{\partial \mu^2} - 2 \frac{\partial X}{\partial \mu} \frac{\partial S}{\partial \mu}) \rangle - 4 \frac{\langle \tilde{X}^4 \rangle}{\langle \tilde{X}^2 \rangle} \langle \tilde{X} (\frac{\partial^2 X}{\partial \mu^2} - 2 \frac{\partial X}{\partial \mu} \frac{\partial S}{\partial \mu}) \rangle \tag{17}$$

• from δB_3 (QCD)

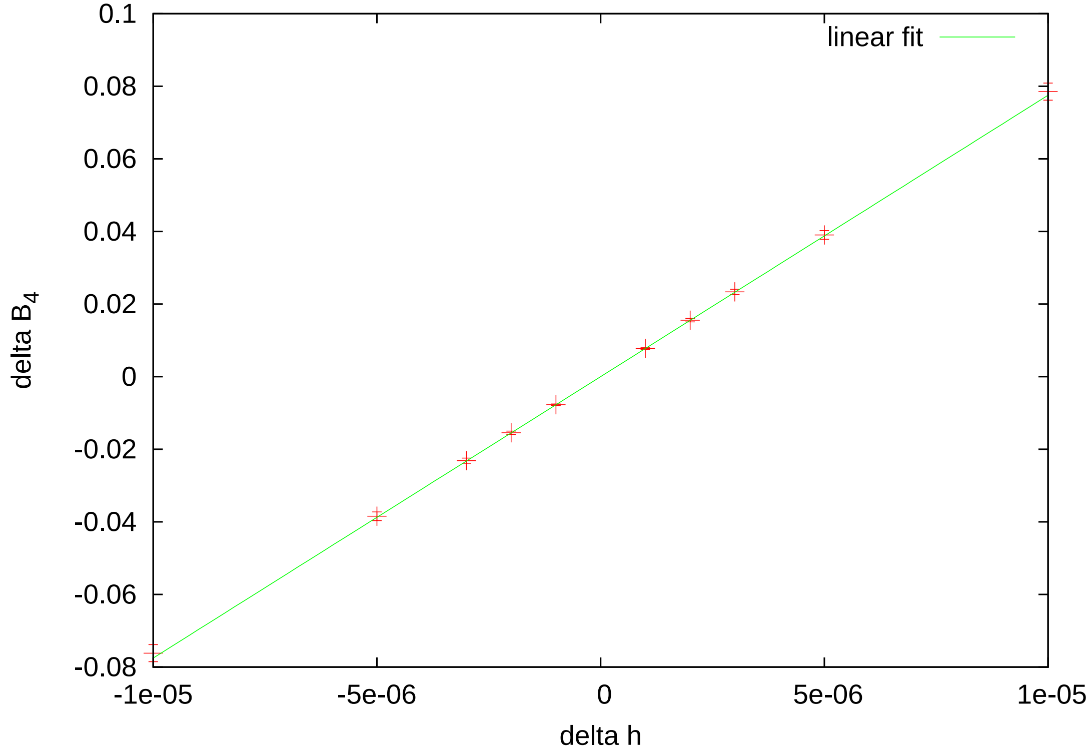
$$A' = 3\langle\tilde{X}^2\rangle\langle\tilde{X}\frac{\partial S}{\partial\beta}\rangle - \langle\tilde{X}^3\frac{\partial S}{\partial\beta}\rangle \tag{18}$$

$$B' = 3\langle[\tilde{X}^2 - \langle\tilde{X}^2\rangle]\frac{\partial X}{\partial m}\rangle + 3\langle\tilde{X}^2\rangle\langle\tilde{X}\frac{\partial S}{\partial m}\rangle - \langle\tilde{X}^3\frac{\partial S}{\partial m}\rangle \tag{19}$$

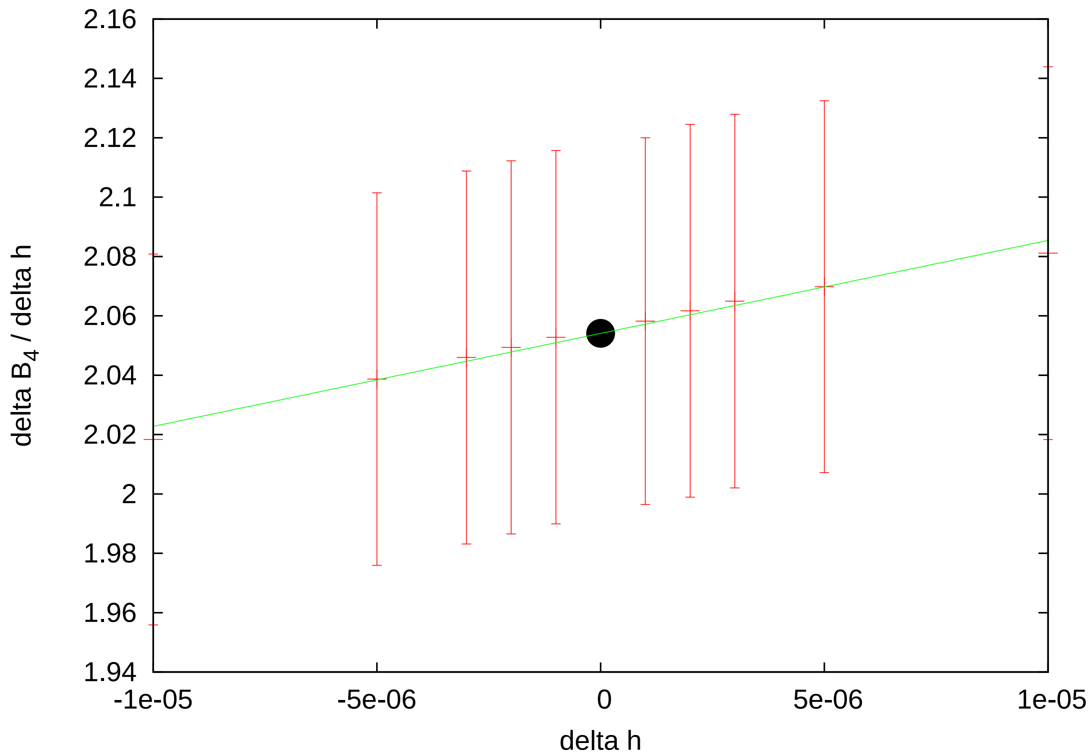
$$C' = \langle\tilde{X}^3\{(\frac{\partial S}{\partial\mu})^2 - \frac{\partial^2 S}{\partial\mu^2}\}\rangle - 3\langle\tilde{X}^2\rangle\langle\tilde{X}\{(\frac{\partial S}{\partial\mu})^2 - \frac{\partial^2 S}{\partial\mu^2}\}\rangle$$
$$+6\langle\tilde{X}(\frac{\partial X}{\partial\mu})^2\rangle + 3\langle\tilde{X}^2(\frac{\partial^2 S}{\partial\mu^2} - \frac{\partial X}{\partial\mu}\frac{\partial S}{\partial\mu})\rangle - 3\langle\tilde{X}^2\rangle\rangle(\frac{\partial^2 S}{\partial\mu^2} - \frac{\partial X}{\partial\mu}\frac{\partial S}{\partial\mu})\rangle \tag{20}$$

3. Numerical Derivative

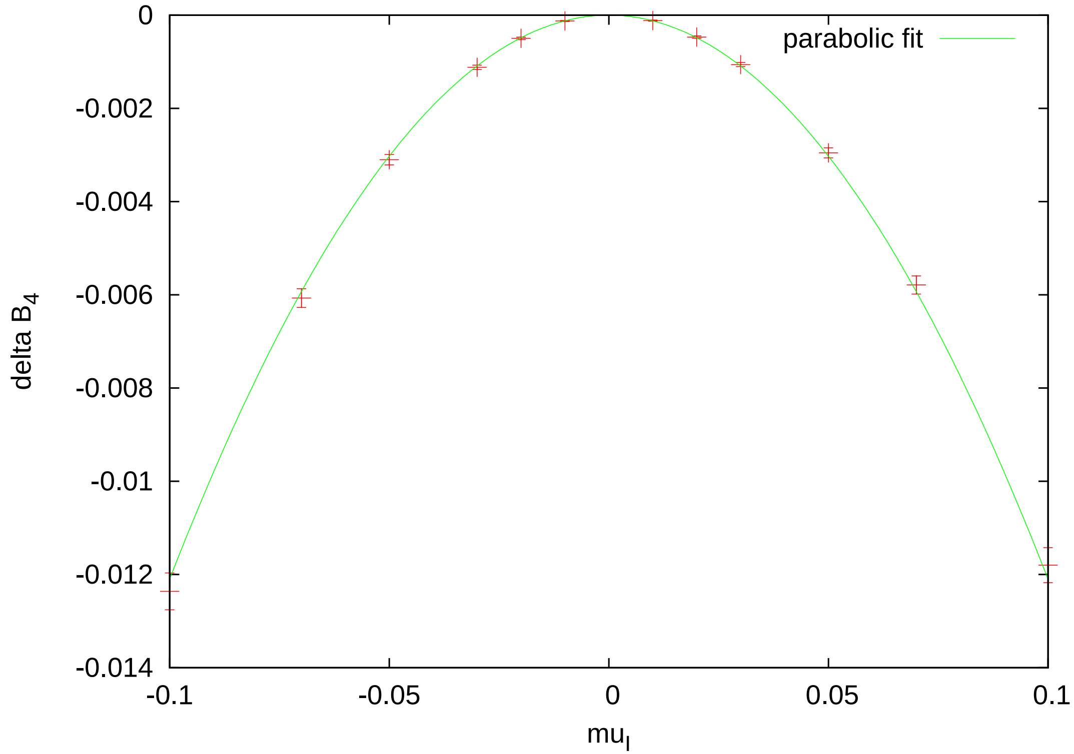
- ΔB_4 vs. Δh with linear fit



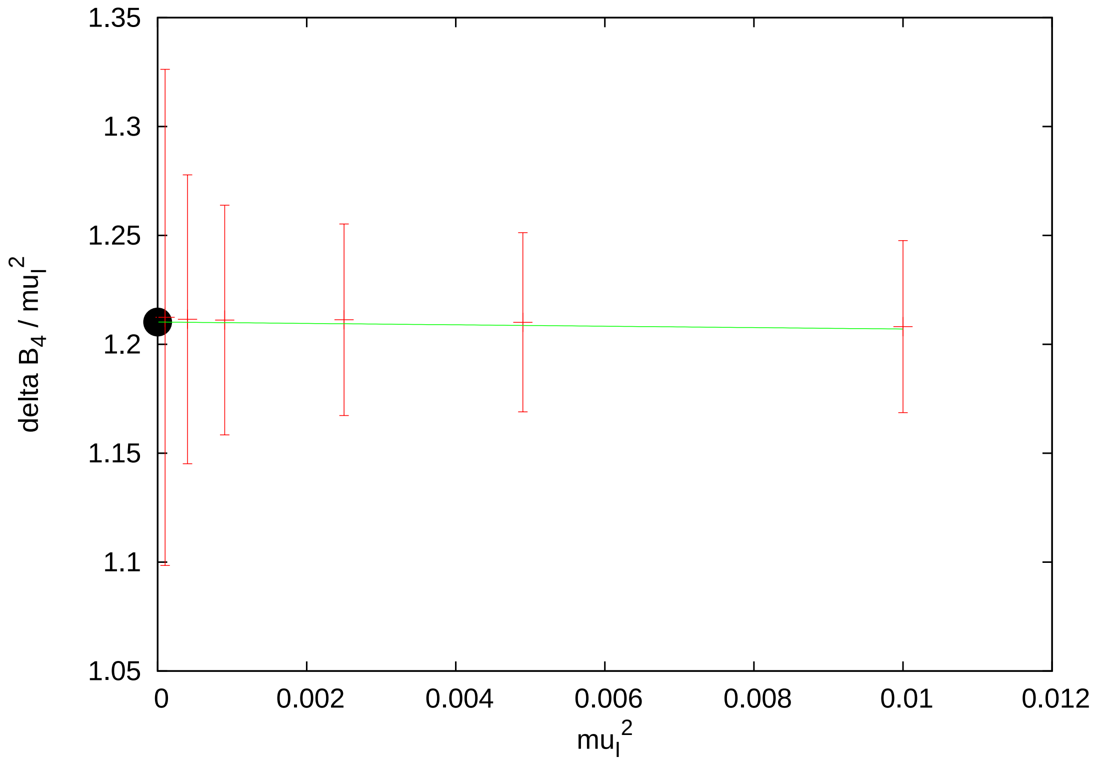
- $\frac{dB_4}{dh}$ vs. Δh



- ΔB_4 vs. Δmu_I



- $\frac{dB_4}{d\mu_I^2}$ vs. μ_I^2



$$\frac{M}{T} = 8.273 + 0.585 \left(\frac{\mu}{T}\right)^2 - 0.174 \left(\frac{\mu}{T}\right)^4 + 0.160 \left(\frac{\mu}{T}\right)^6 - 0.071 \left(\frac{\mu}{T}\right)^8$$

(hep-lat/0510069)

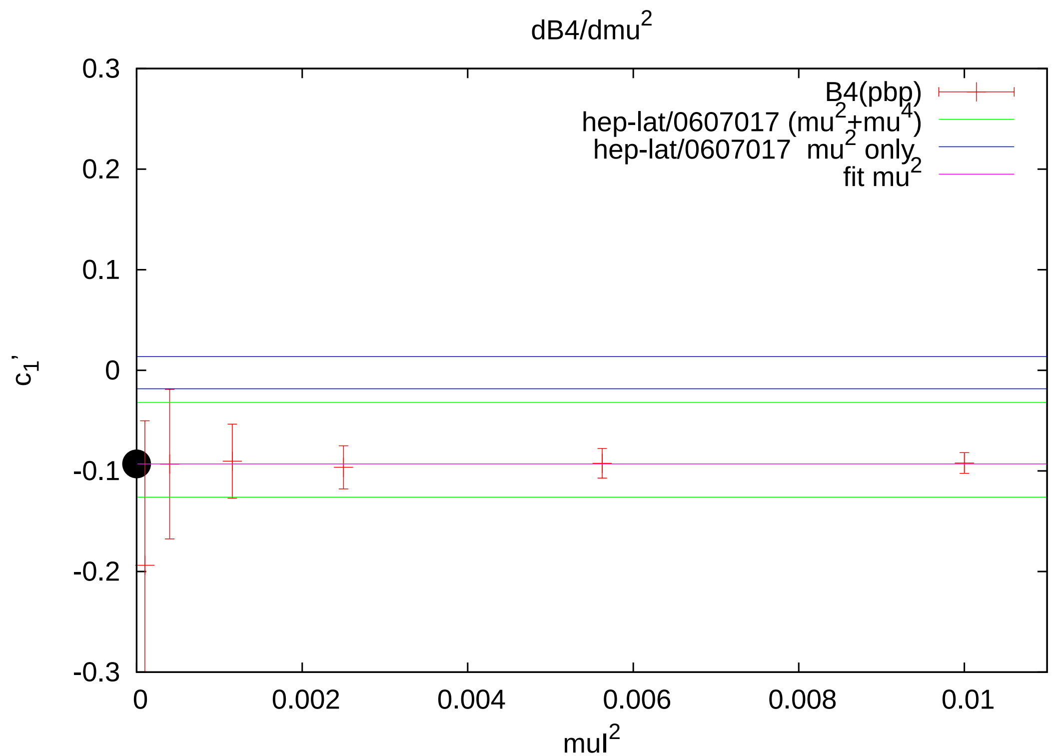
numerical derivative method gives $0.589(8)$ for the curvature

QCD case (Lat07 Talk by Ph. de Forcrand)

- $B_4(am, a\mu) = 1.604 + b_{10}\left[am - am_0^c - c_1'(a\mu)^2\right]$

$\frac{dB_4}{d(am)} = b_{10} \sim 13.6(6)$ well determined from hep-lat/0607017

$c_1' \sim -0.09(1)$, **negative beyond doubt**



3. Summary

• Curvature at $((m/T)_c, (\mu/T) = 0)$ can be obtained by fitting from numerical simulations from global fit(=0.585(3)) or from numerical differentiaion (=0.589(8))

• or Taylor expansion method (=0.528(1)) which is cumbersome and is not competitive enough compared to direct numerical evaluation of the curvature

• QCD with heavy quark in (m, μ) parameter space has positive curvature at the critical point

• QCD with light 3 quark in (m, μ) parameter space has negative curvature at the critical point (Ph. de Forcrand's and O. Philipsen's talks at Lat07)

• full phase diagram