# Recent progress in finite density QCD via an imaginary chemical potential

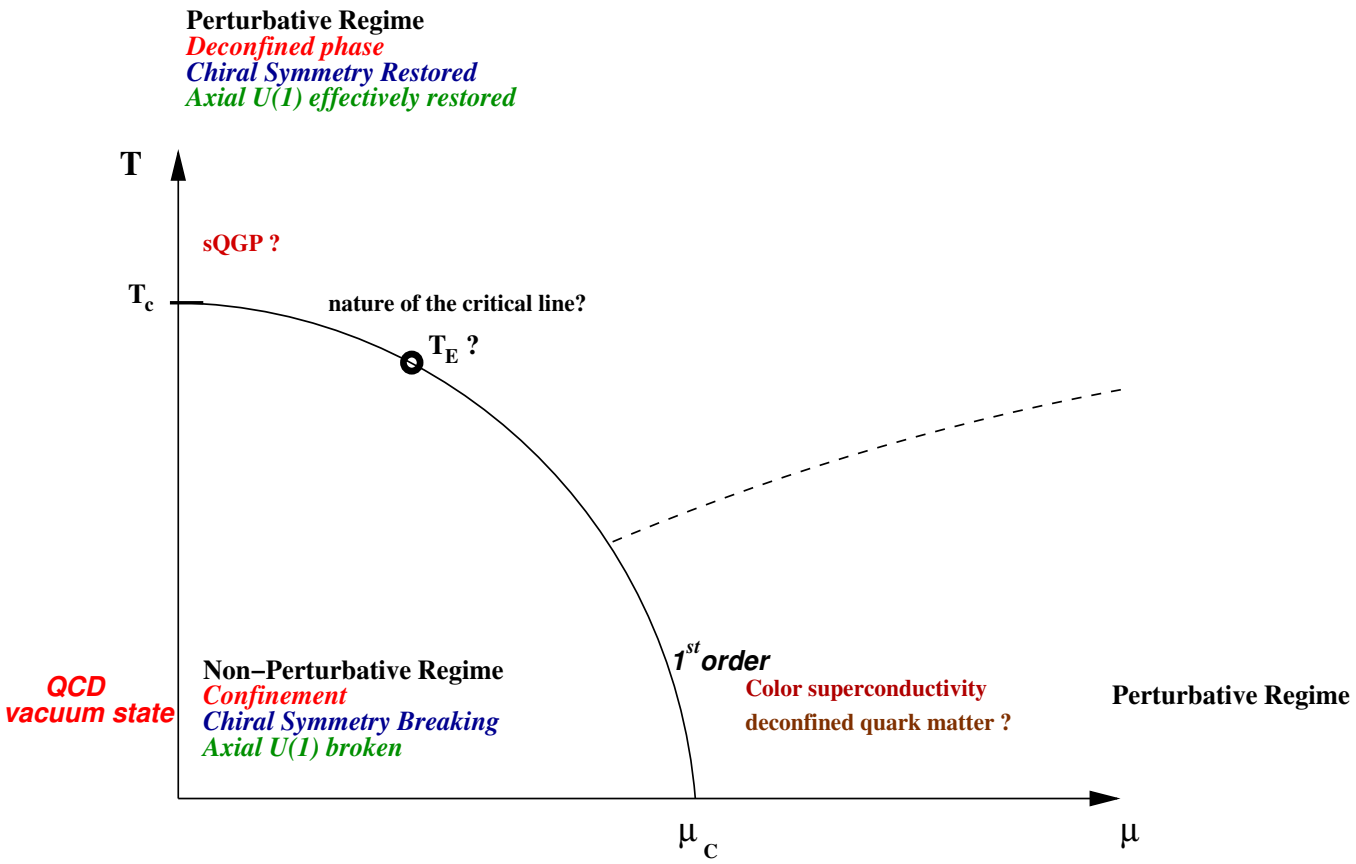
**M. D'Elia (Genova)**

QCD in Extreme Conditions 2007 - Frascati, August 7 2007

# 1 – OUTLINE

- QCD at finite density and imaginary chemical potentials
- Analytic continuation and the phase diagram in the $T - \mu^2$ plane.
- Physical results
- Analytic continuation in QCD-like models
- Imaginary $\mu$ and the phase of the determinant

# 2 – QCD at finite density and the sign problem




**Our knowledge of the QCD phase diagram and of the dynamics of strongly interacting matter at finite temperature and baryonic density is still partial.**

**Lattice QCD simulations, which are the best non-perturbative computational tool, are hindered at finite baryonic density by the well known "sign problem".**

**The correct way to discretize the QCD partition function at finite chemical potential**

$$Z(\mu) = \mathrm{Tr}\left(e^{-\frac{H_{\mathrm{QCD}}-\mu N}{T}}\right)$$

**is to consider $\mu$ as being part of the covariant derivative, like the temporal component of a $U(1)$ imaginary background field**

**P. Hasenfratz F. Karsch,1983; J.B. Kogut et al., 1983; R. V. Gavai 1985.**

**This is implemented by modifying the temporal links appearing in the fermion matrix:**

$$U_t \to e^{a\mu} U_t \qquad\qquad (U_t)^\dagger \to e^{-a\mu} (U_t)^\dagger$$

**The hermiticity properties of the fermion matrix are lost, the residual symmetry being**

$$(\det M(\mu))^* = \det M(-\mu)$$

**which means that $\det M$ is in general complex and thermal expectations values**

$$\langle O \rangle = \frac{\int DU \, \det M[U] \, e^{-S_g[U]} \, O[U]}{\int DU \, \det M[U] \, e^{-S_g[U]}}$$

**cannot be evaluated by standard Monte Carlo simulations.**

**Apart from studying sign-problem-free similar theories (2-color QCD, finite isospin density QCD, ...), the problem can be partially circumvented by a variety of methods, mostly in a limited region of high $T$, small $\mu$**

**Reweighting** $\langle O \rangle = \left\langle O \frac{\det(M(\mu))}{\det(M(0))} \right\rangle_{\mu=0} / \left\langle \frac{\det(M(\mu))}{\det(M(0))} \right\rangle_{\mu=0}$ I.M. Barbour *et al*, 1998

**Multiparameter reweighting** (reweighting also in $\beta$) Z. Fodor and S.D. Katz, 2002 →

**Taylor expansion** Physical quantities expanded as Taylor series around $\mu = 0$

Coefficients obtained as expectation values of local operators at $\mu = 0$

Bielefeld-Swansea Collaboration 2002 → Gavai and Gupta 2003 →

**Density of states** partition function rewritten in terms of the density of states for some quantity (plaquette, quark number, . . . ), sign problem might be better under control

Bhanot, Bitar, Salvador (1987), Karliner, Sharpe, Chang (1988), Azcoiti, Di Carlo, Grillo (1990), Luo (2001), Ambjorn *et al* (2002), Fodor, Katz, Schmidt (2006)

**Heavy quark expansions** sign problem still present but easier simulations

T. C. Blum, J. E. Hetrick, D. Toussaint, 1996, J. Engels, O. Kaczmarek, F. Karsch, E. Laermann, 1999

R. De Pietri, A. Feo, E. Seiler, I. O. Stamatescu, 2006

**Imaginary chemical potential** ...

# Imaginary chemical potential

**Consider the partition function $Z(i\mu_I)$ defined by an imaginary chemical potential**

$$U_t \to e^{ia\mu_I} U_t \qquad\qquad U_{-t} \to e^{-ia\mu_I} U_{-t} = (e^{ia\mu_I} U_t)^\dagger$$

**this is like adding a constant and real $U(1)$ background field.** $\det M[U] > 0 \implies$

**Monte Carlo simulations are feasible. What can we learn from imaginary $\mu's$?**

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## Reconstruction of the canonical partition function

**$Z(i\mu_I)$ can be used to reconstruct the canonical partition function $Z(n)$ at fixed quark number $n$ (Roberge, Weiss, 1986)** ($\theta = \mu_I/T$)

$$Z(n) = \mathrm{Tr}\left((e^{-\frac{H_{\mathrm{QCD}}}{T}}\delta(N-n)\right) = \frac{1}{2\pi}\mathrm{Tr}\left(e^{-\frac{H_{\mathrm{QCD}}}{T}}\int_0^{2\pi} \mathrm{d}\theta e^{i\theta(N-n)}\right) = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}\theta e^{-i\theta n} Z(i\theta T)$$

**A. Hasenfratz, D. Toussaint, 1990; Alford *et al.*, 1992 (2-d Hubbard model); de Forcrand, Kratochvila, 2004, 2006; A. Alexandru *et al.*, 2005 Thermodynamical limit not easily reachable.**

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## Analytic continuation to real $\mu$

**Away from critical points $Z(T,\mu)$ is a regular function of $\mu^2$. Results at $\mu_I$ ($\mu^2 < 0$) can be used to fit the expected dependence continued from real $\mu' s$ ($\mu^2 > 0$).**

**M.P. Lombardo, 2000, A. Hart *et al*, 2000, de Forcrand, Philipsen, 2002, D'E., Lombardo, 2003**

**The method of analytic continuation from an imaginary chemical potential has some advantages over other methods:**

- **it does not suffer much from increasing the spatial volume $\longrightarrow$ easier thermodynamical limit**

- **it gathers information from a wider region than just at $\mu = 0$**

**It must be used with a grain of salt**

- **the phase structure of the theory in the whole $T - \mu^2$ is needed to understand to what extent analytic continuation can be applied**

- **the choice of the best interpolating functions to be continued may be non-trivial and physical intuition may serve as a guidance**

# Phase structure in the $T - \mu_I$ plane

A. Roberge and N. Weiss, 1986

**The fermionic determinant in the QCD partition function can be interpreted in terms of an effective action**

$$Z = \int DU \, \det M[U] \, e^{-S_g[U]} = \int DU \, e^{\mathrm{Tr} \ln M[U]} \, e^{-S_g[U]}$$

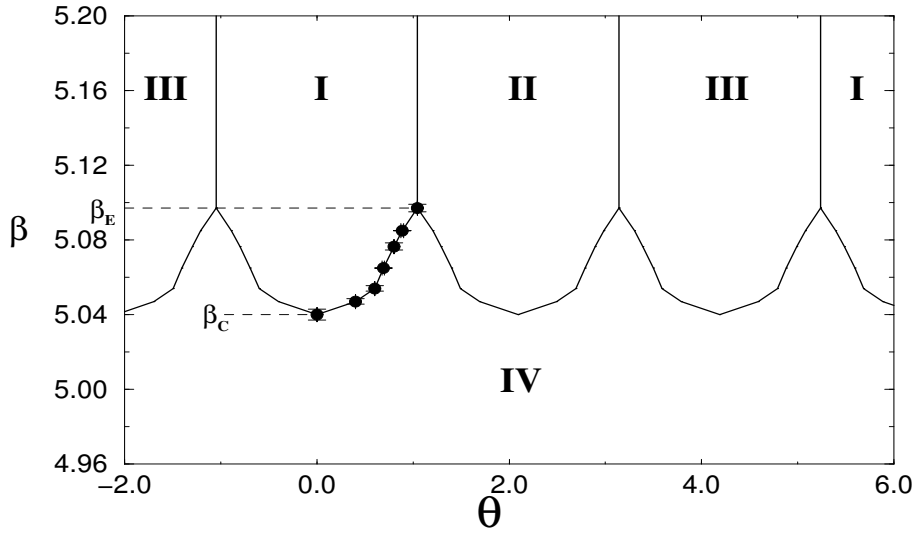
**a loop expansion clearly shows that this introduces couplings to the Polyakov loop $L$, precisely to $(\mathrm{Tr}L + \mathrm{Tr}L^\dagger)$, which explicitly breaks the $Z_3$ symmetry, acting like a magnetic field pointing along the real $Z_3$ root:**

- **at low temperatures $\langle L \rangle \neq 0$ and real, even if small**
- **at high temperatures the symmetry breaking term removes the $Z_3$ degeneracy selecting one particular $Z_3$ vacuum (the real one)**

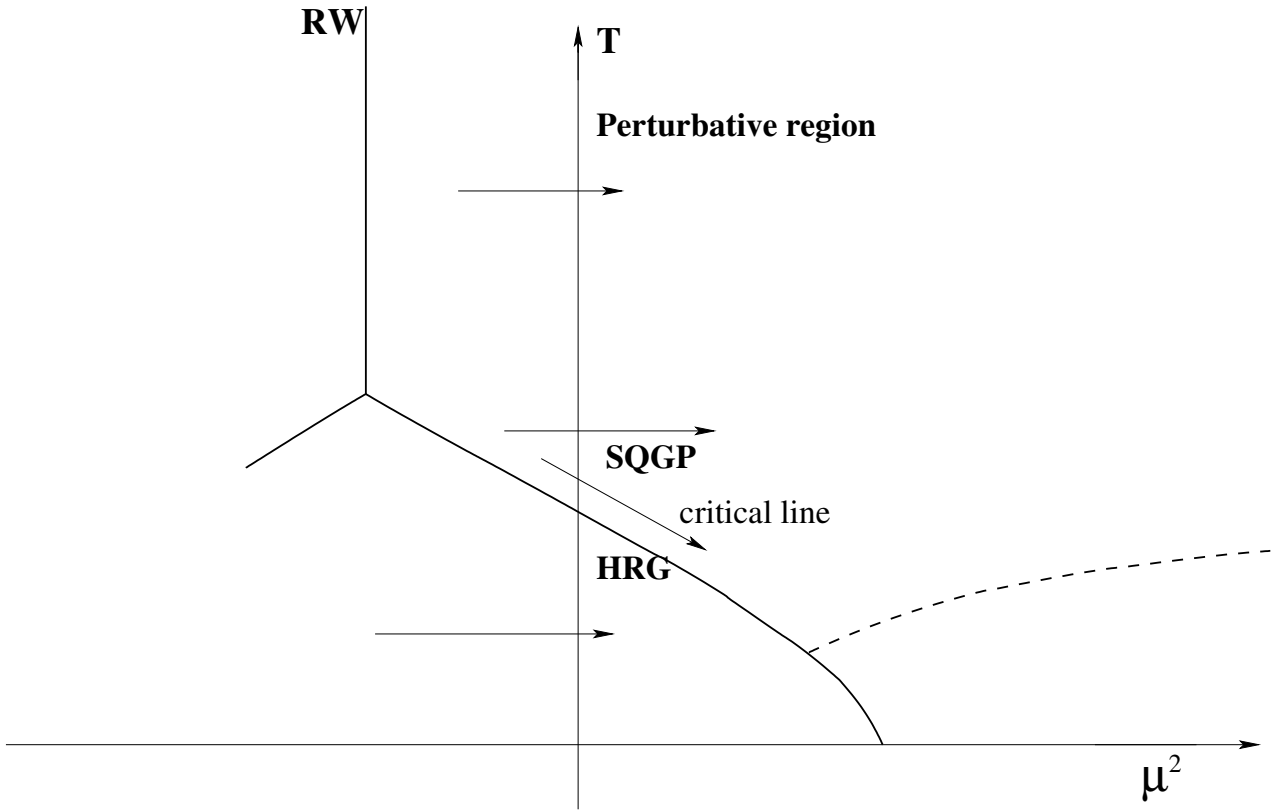
**The introduction of an imaginary chemical potential modifies the loop expansion by multiplying all loops winding around the temporal direction by $\exp(in_w\mu_I/T)$, where $n_w$ is the winding number. That is equivalent to a rotation of the effective magnetic field by an angle $-\mu_I/T$.**

- **At low temperatures $\langle L\rangle$ rotates following the external field. $Z(i\mu_I)$ is a smooth periodic function of $\mu_I/T$ with period $2\pi/N_c$**
- **At high temperatures $\langle L\rangle$ is constrained to one of the $Z_3$ vacua, but the orientation of the external field selects the true vacuum: first order phase transitions are present for $\mu_I/T = 2(k+1/2)\pi/N_c$ where $k$ is integer. $\Longrightarrow$ RW transitions**

**A typical phase diagram in the $T-\mu_I$ plane is then the following. As a matter of fact, endpoints of RW transition lines cannot be resolved from the continuation of the chiral (pseudo)critical line.**



**The following issues can be therefore analyzed within the analytic continuation approach**




- **continuation of the critical line for small chemical potentials**
- **physical properties of hadronic matter below the transition (HRG model)**
- **physical properties of deconfined matter right above the transition (strongly interacting QGP) and at very high T (connection with perturbation theory).**

## The imaginary chemical potential method allows generalizations

V. Azcoiti, G. Di Carlo, A. Galante, V. Laliena, JHEP 1204:010 and Nucl. Phys. B 723, 77 (2005).

$$S_F = ma \sum_n \bar{\psi}_n \psi_n + \frac{1}{2} \sum_n \sum_{i=1}^{3} \bar{\psi}_n \eta_i(n) \left( U_{n,i} \psi_{n+i} - U^{\dagger}_{n-i,i} \psi_{n-i} \right) + S_\tau(x,y) \,,$$

$$\begin{aligned} S_\tau(x,y) &= x \frac{1}{2} \sum_n \bar{\psi}_n \eta_0(n) \left( U_{n,0} \psi_{n+0} - U^{\dagger}_{n-0,0} \psi_{n-0} \right) \\ &+ y \frac{1}{2} \sum_n \bar{\psi}_n \eta_0(n) \left( U_{n,0} \psi_{n+0} + U^{\dagger}_{n-0,0} \psi_{n-0} \right) \,, \end{aligned}$$

$x \pm y = e^{\pm a\mu} \quad \Longrightarrow$ **usual real chemical potentials**

$y = iy_I$ **and** $x \pm iy_I = e^{\pm ia\mu_I} \quad \Longrightarrow$ **imaginary chemical potential**

**The sign problem disappears for imaginary values of $y = iy_I$. For $x^2 + y_I^2 \neq 1$ this is like using an imaginary chemical potential plus a variable prefactor $\sqrt{x^2 + y_I^2}$ in front of the temporal covariant derivative in the fermion matrix.**

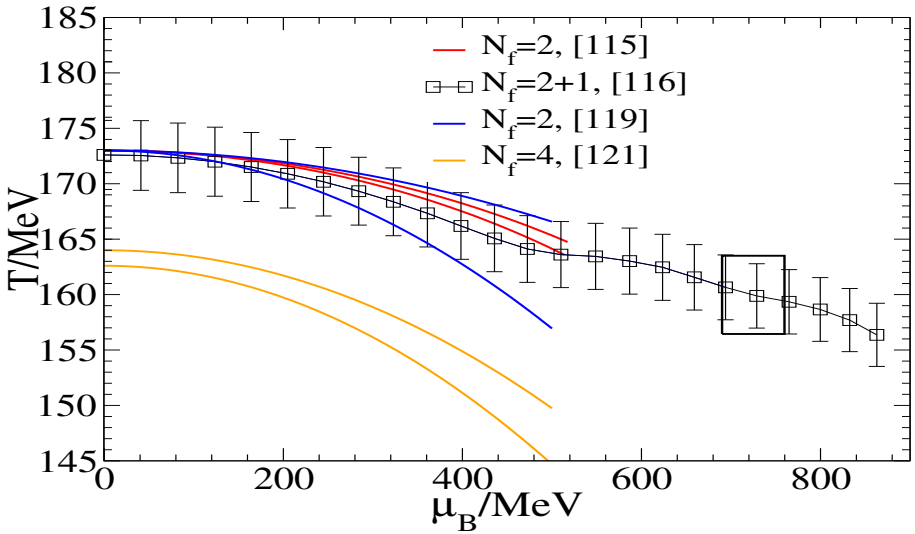
**The idea is to compute quantities as a function of $iy_I$ and to continue them to the physical point given by $x^2 - y^2 = 1$ ($x + y = e^{a\mu}$, $x - y = e^{-a\mu}$)**

**An extension to lower temperatures should be possible.**

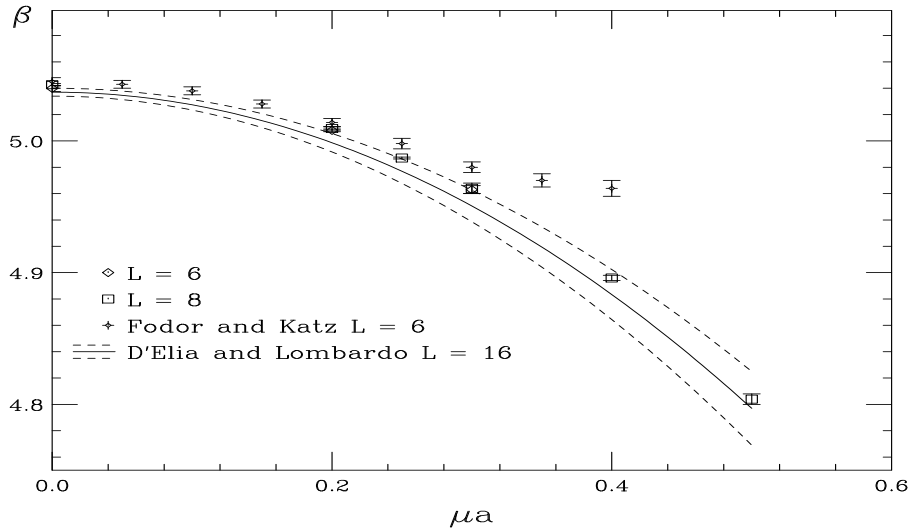
# 3 – Continuation of the critical line

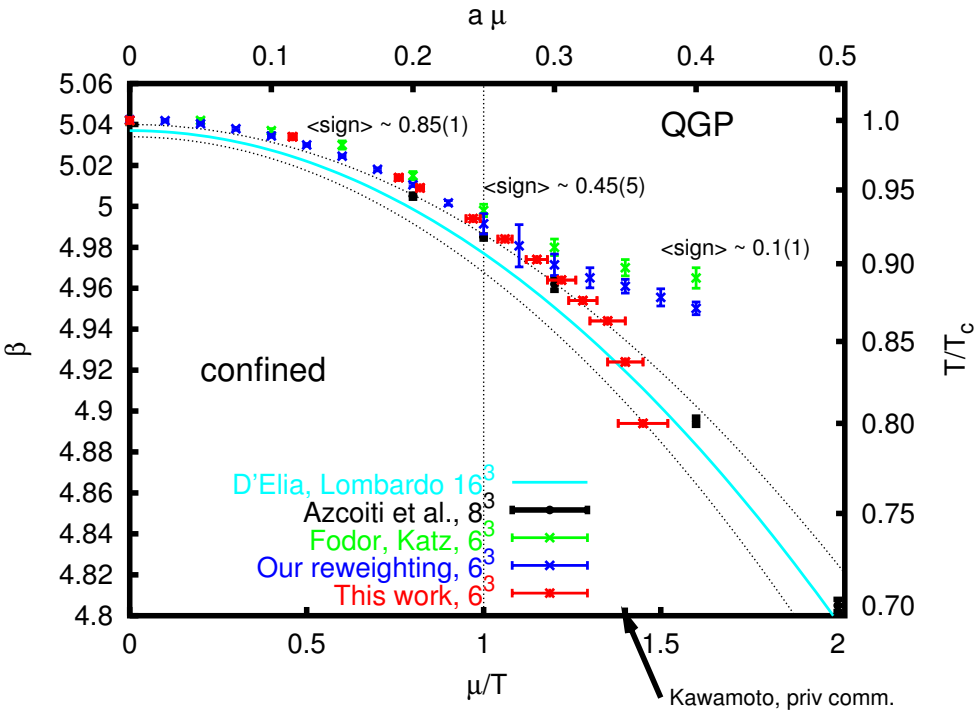
**Several results have been obtained allowing a detailed comparison among several methods:**

**The critical line obtained for $N_f = 2$ by de Forcrand and Philipsen by analytic continuation is compared to determination by Taylor expansion (Bielefeld-Swansea) and reweighting (Fodor and Katz).** **from E. Laermann and O. Philipsen, hep-ph/0303042.**

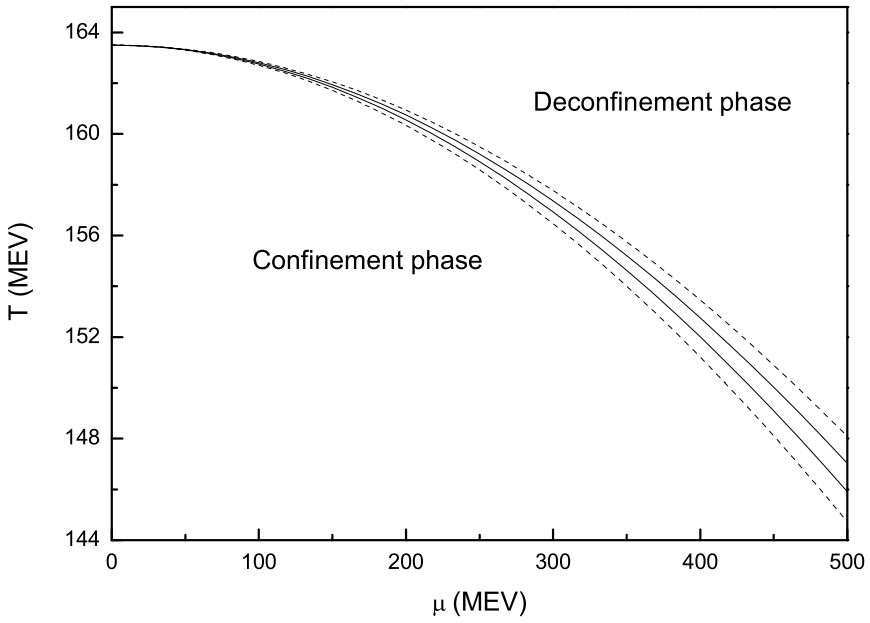



**The critical line obtained for $N_f = 4$ obtained by analytic continuation (D'E. and Lombardo) is compared to that obtained by Azcoiti *et al* and by reweighting (Fodor and Katz).** **from V. Azcoiti *et al*, Nucl. Phys. B 723, 77 (2005).**






**The critical line for $N_f = 4$ compared to that obtained with the canonical approach.** **from S. Kratochvila and P. de Forcrand, PoS LAT2005, 167 (2006).**




**The critical line obtained for $N_f = 4$ for Wilson and staggered fermions.** **from H. S. Chen and X. Q. Luo, Phys. Rev. D 72, 034504 (2005).**

# Current issues about the critical line

**1) Within the analytic continuation approach, de Forcrand and Philipsen have studied the behaviour as a function of the chemical potential of the critical quark mass $m_c(\mu)$ corresponding to the critical end point of the first order line for $N_f = 3$.**

**Last results (talk by de Forcrand at Lattice 2007) clearly show that $dm_c/d\mu^2 < 0$ for small $\mu$ $\Longrightarrow$ the transition weakens as $\mu$ is increased from zero $\Longrightarrow$ if crossover at $\mu = 0$ no critical endpoint is expected for relatively small chemical potentials.**

**Result supported by**

- **studies at finite isospin density (Kogut and Sinclair)**
- **model study of the 3d 3-state Potts model (de Forcrand, Kim, Kratochvila, Takaishi)**

**2) A more careful study of the analytic continuation of the critical line (continuum limit and physical quark masses) could permit an interesting comparison with freeze-out experimental curves (talk by O. Philipsen at Lattice 2007).**

## 4 – Analytic continuation below $T_c$: testing the HRG model

**In the Hadron Resonance Gas model the free energy density is described in a very simple way in terms of a free gas of resonances**

$$\frac{F(T,\mu)}{VT^4} = \frac{F(T,0)}{VT^4} + f(T)\left(1-\cosh(\frac{3\mu}{T})\right) \ ,$$

**hence the baryon number density is**

$$n_B(\mu) \equiv -\frac{1}{\mu}\frac{\partial}{\partial\mu}F(T,\mu) \propto \sinh(\frac{3\mu}{T})$$

.

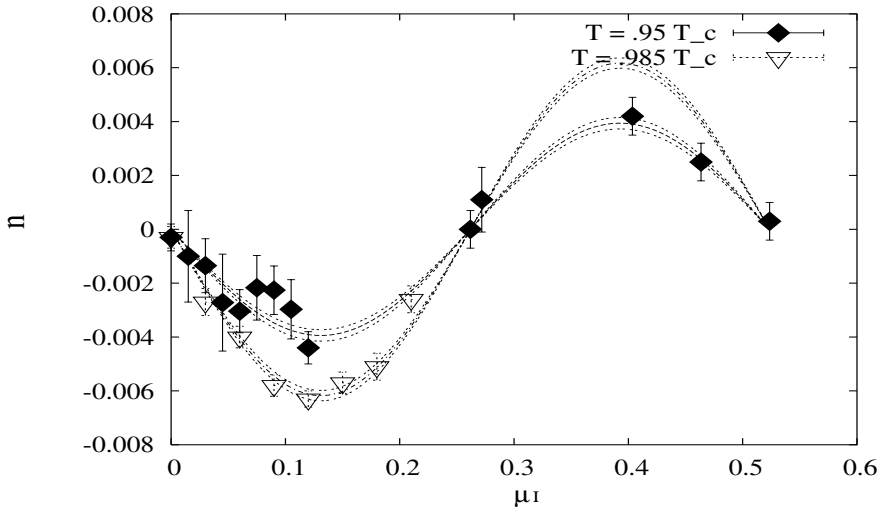
**that dependence can be naturally continued to imaginary chemical potential leading to the following expression for the free energy density**

$$\frac{\Delta F(T,\mu_I)}{VT^4} \equiv \frac{F(T,\mu_I)}{VT^4} - \frac{F(T,0)}{VT^4} = f(T)\left(1-\cos(\frac{3\mu}{T})\right) \ ,$$

**and for the imaginary part of the quark number density**

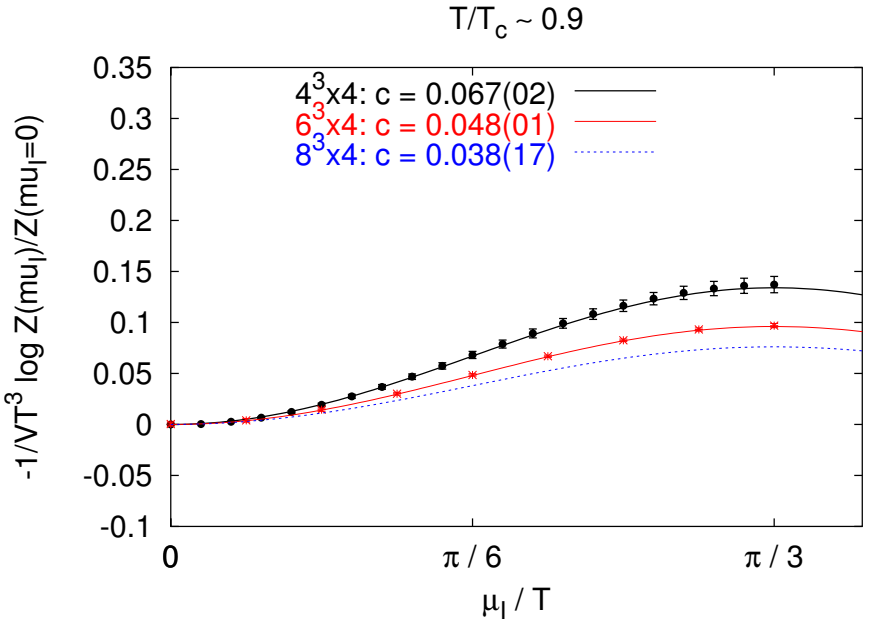
$$\mathrm{Im}(n_B(\mu_I)) \propto \sin(\frac{3\mu_I}{T})$$

**These predicitions are nicely tested at $\mu = i\mu_I$: single Fourier terms well fit the data**

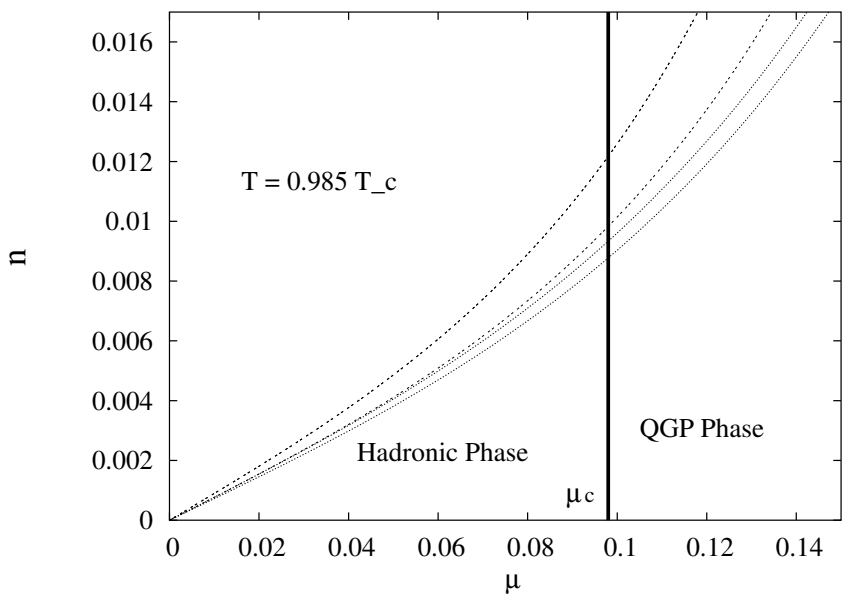



**The baryon number density is well described by a single sine term. Results obtained for $N_f = 4$.**

**M.D'E. and M.P. Lombardo, 2004**




**The free energy density is well described by (1 - cosine). Results for $N_f = 4$. Ph. de Forcrand and S. Kratochvila, 2006**




**For $T < T_c$ analytic continuation is limited at $\mu^2 > 0$ by the chiral line. We can however obtain interesting information regarding the transition. For instance the baryon number density continued to real values of $\mu$ can be used to estimate the critical density at the phase transition.** **(M.D'E. and M.P. Lombardo, 2004 for $N_f = 4$)**

**Similar result are obtained with other methods (e.g. the Taylor expansion method, C. R. Allton *et al*, 2005).**

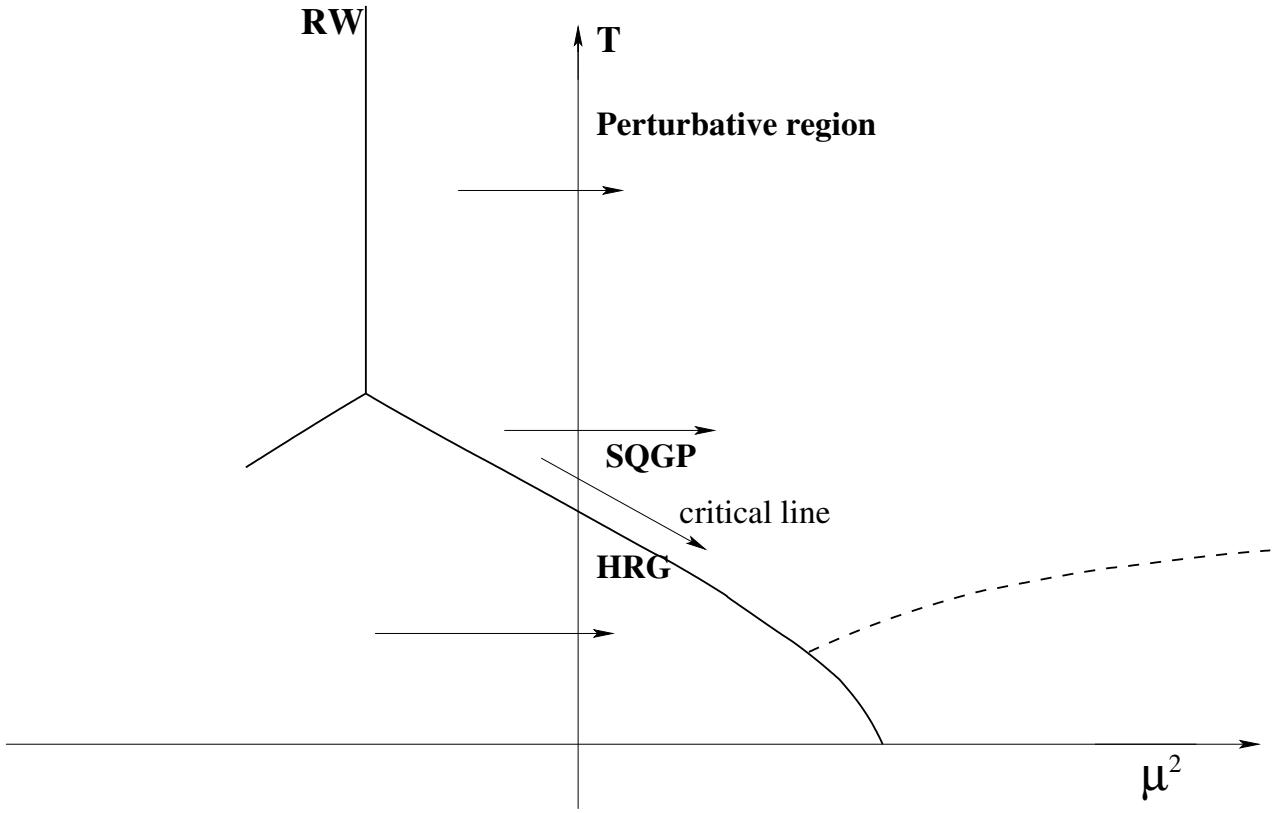
**Future high precision studies in the region $T < T_c$ could help finding violations to the simple one-Fourier-term behaviour (these are clearly visible in SU(2), see later ...)**

**That could clarify the extent of validity of the HRG model.**

## 5 – Exploring the Deconfined Phase

**At high $T$ the region of $\mu^2 < 0$ available for analytic continuation is limited either by the RW transition or by the continued physical (pseudo)critical line.**

**There is no apriori limit for analytic continuation at $\mu^2 > 0$, however the cut at $\mu^2 < 0$ implies at least a practical problem for continuation to large chemical potentials, because of the limited information available.**



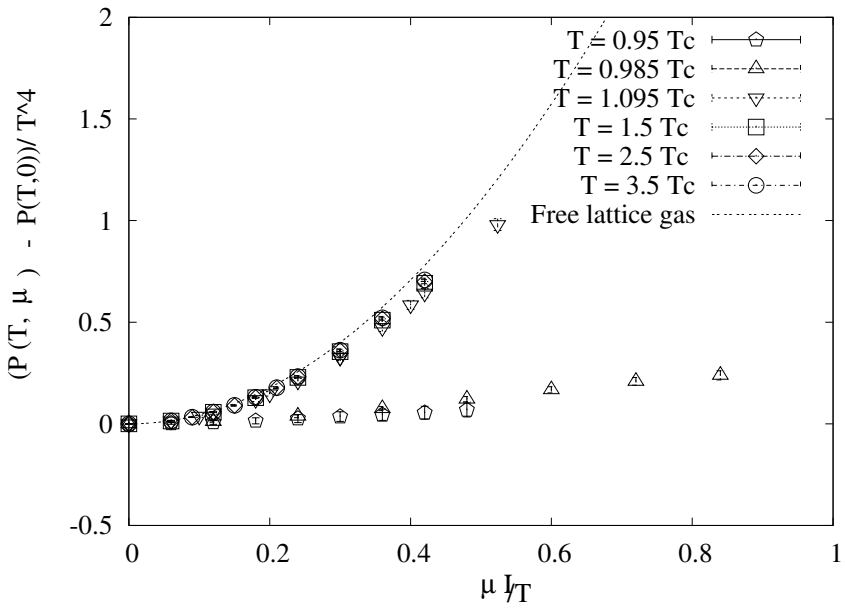
**A number of quantities can be studied and confronted with model predictions:**

- **baryon number density**
- **pressure density**

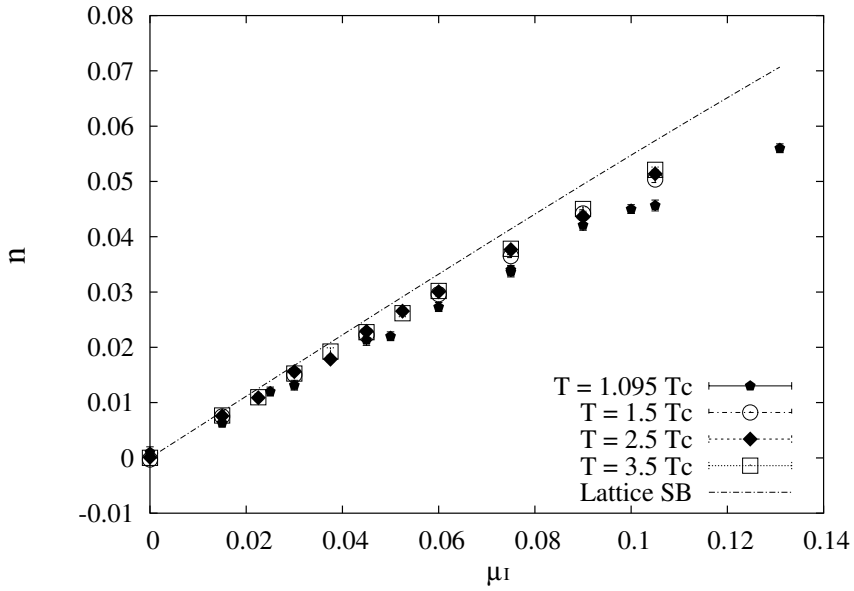
$$\Delta P/T^4 = (P(T,\mu,m_q) - P(T,\mu=0,m_q))/T^4$$

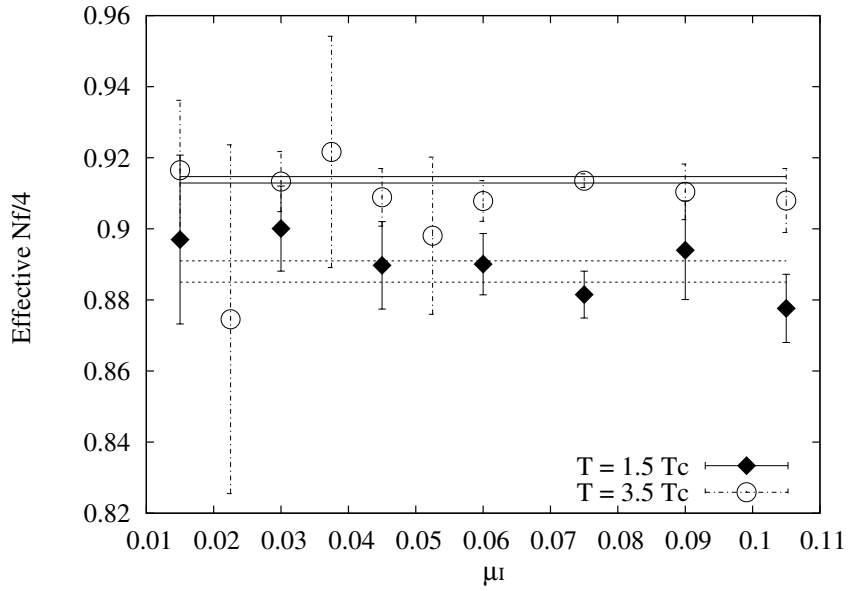
**obtained for instance by integrating the baryon number**

$$n(T,\mu,m_q) = \frac{\partial P(T,\mu,m_q)}{\partial \mu}\,; \qquad (P(T,\mu,m_q) - P(T,\mu=0,m_q))/T^4 = N_t^4 \int n(\mu) d\mu$$



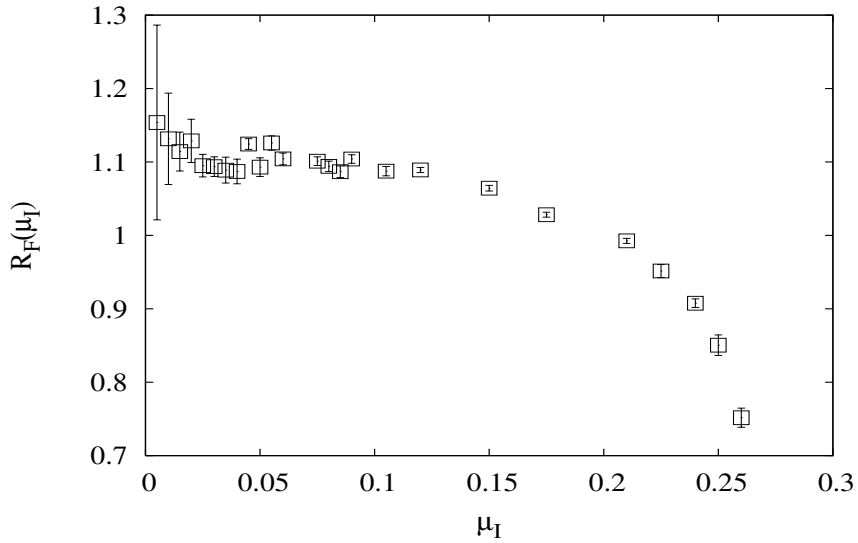

**At high temperatures the pressure approaches asymptotically free lattice gas predicition. (M.D'E. and M.P. Lombardo, 2004 for $N_f = 4$). Similar result have been obtained by de Forcrand and Kratochvila.**




**The baryon density is well reproduced by the free lattice gas results if an effective number of flavors, depending on temperature, is taken.**

**A more refined analysis can be tried by matching to the predictions of perturbation theory (M. D'E., F. Di Renzo, M.P. Lombardo, A. Vuorinen, in progress)**




**As we lower the temperature non-perturbative effects show up and the simple free field behavior is ruled out.** **The figure shows results for $n(\mu_I)/n_{free}(\mu_I)$ at $T \simeq 1.1T_c$: we are indeed approaching the region where the QGP is thought to be strongly interacting.**

**Can we investigate the properties of the sQGP?**

**We have started a study aimed at that (M. D'E., F. Di Renzo, M.P. Lombardo, 2007) .**

- **Can we get any information about the bound states (colored and colorless) populating the sQGP?**
- **Do the critical properties at $\mu^2 < 0$ (chiral critical line, endpoint of the RW transition) have any influence at $\mu^2 > 0$?**

**We have studied $N_f = 4$ QCD at $T \simeq 1.1T_c$, corresponding almost exactly to the endpoint of the RW transition. Studies based on the second and fourth order cumulant of number densities (Karsch, Ejiri, Redlich, 2005) show that states with the quantum number of quarks are dominating for $T \geq 1.5T_c$.**

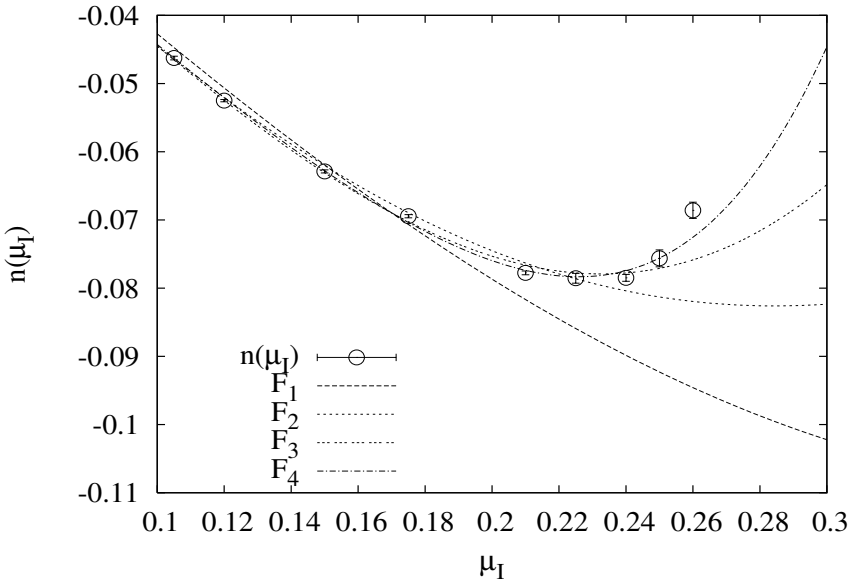
## Bound state analysis

**Following the simple prescription of the HRG model, we can try describing the sQGP as a sum of non-interacting bound states:**

$$\frac{\Delta P}{T^4} = F_q(T)(1-\cos(\mu/T))+F_{qq}(T)(1-\cos(2\mu/T))+F_{qqq}(T)(1-\cos(3\mu/T))+F_{qqqq}(T)(1-\cos(4\mu/T))$$

**giving in turn**

$$n(\mu_I,T) = F_q(T)\sin(\mu_I/T)+2F_{qq}(T)\sin(2\mu_I/T)+3F_{qqq}(T)\sin(3\mu_I/T)+4F_{qqq}(T)\sin(3\mu_I/T)+\ldots$$

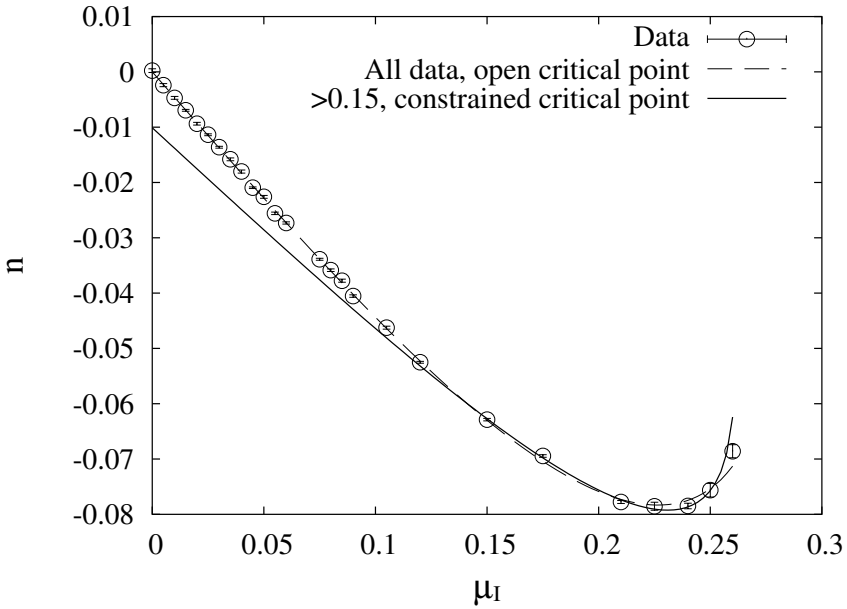



**Data are fitted better and better as more terms are introduced, however fit parameters are unstable and one cannot get a** $\chi^2/\mathrm{d.o.f.}$ **better that** $\sim 3$**.**

- **Hypothesis of non-interacting bound states could be too strong**
- **Bound state masses could depend on** $\mu$ **as well, thus modifying the simple trigonometric behaviour**

# A different hypothesis:

**Can we interpret results in terms of a critical behaviour induced by the nearby end-point of the Roberge-Weiss transition line happening at $\mu_I/T = \pi/3$?**




**A critical behaviour for the fermion number density (imaginary part) like**

$$n(\mu_I) = A\mu_I(\mu_I^{c\,2} - \mu_I^2)^\alpha \rightarrow n(\mu) = A\mu(\mu_I^{c\,2} + \mu^2)^\alpha$$

**with** $\alpha \sim 0.3$, $\mu_I^{c\,2} \sim 0.08$ **and** $\tilde{\chi}^2 \sim 1.8$ **well reproduces the numerical data.**

**Can the physical properties of the sQGP be described in terms of this critical behaviour at $\mu^2 < 0$? Further studies are needed to test this "strong" hypothesis.**

# ANALYTIC CONTINUATION IN QCD-LIKE THEORIES

## 6 – Analytic continuation under scrutiny

**In QCD-like theories where simulations are feasible both at imaginary and real chemical potentials, analytic continuation can be carefully tested. A few examples:**

- **Two color QCD** SU(2) gauge group is real $\Longrightarrow$ the determinant is real
  P. Giudice and A. Papa, 2004; P. Cea, L. Cosmai, M. D'E. and A. Papa, 2006
- **3d 3-state Potts models** sign problem is still present but well under control S. Kim, P. de Forcrand, S. Kratochvila, and T. Takaishi, 2005
- **QCD at finite isospin chemical potential**
  yet not done

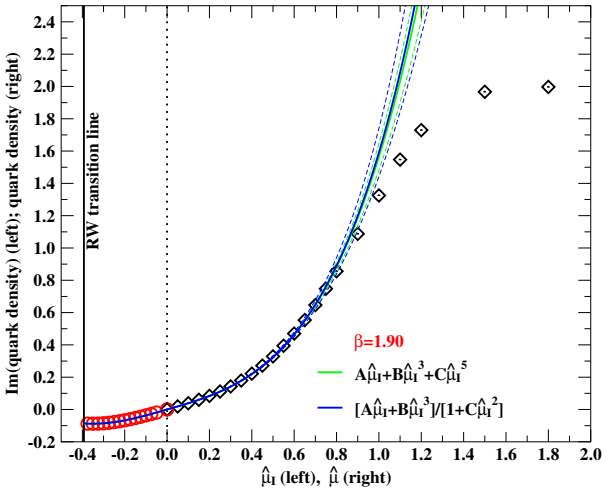
**A few specific points to address:**

- **To what extent can we trust analytic continuation?**
- **What is the optimal way to extract information? (best interpolation = polynomial, ratio of polynomials, Fourier . . . ?)**

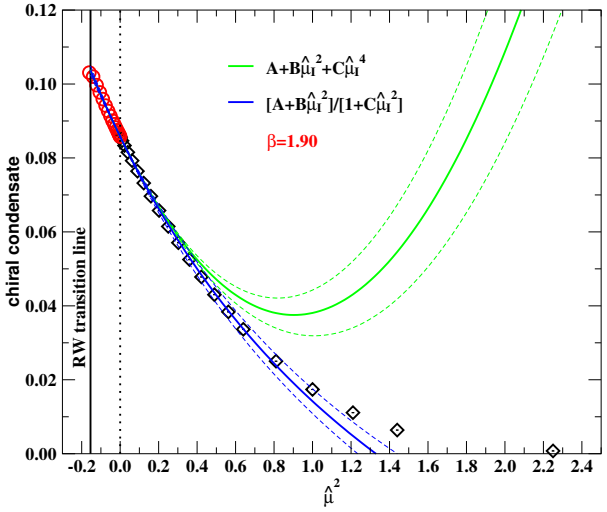
**Let us look at some results obtained in two-color QCD ...**

$N_f = 8$, $N_t = 4$ **staggered fermions;** P. Cea, L. Cosmai, M. D'E. and A. Papa, JHEP 0702 (2007) 066

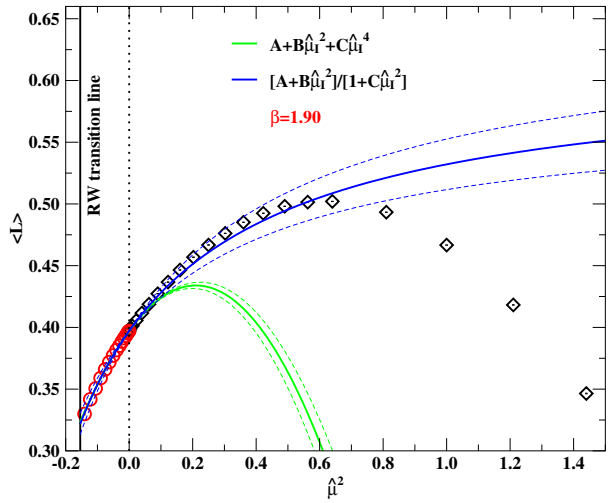
## High Temperature Region




quark density



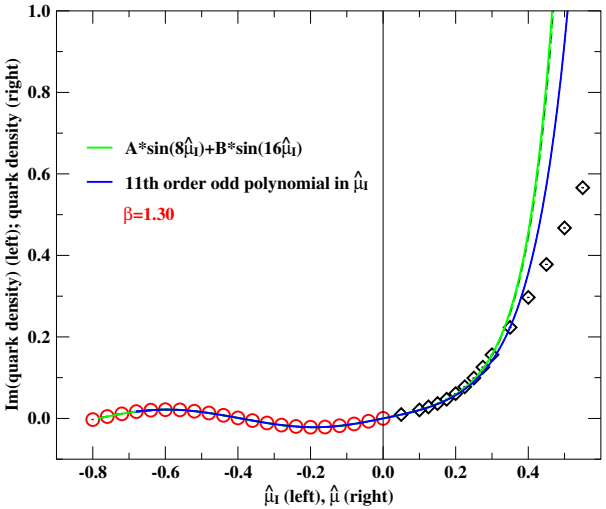

chiral condensate



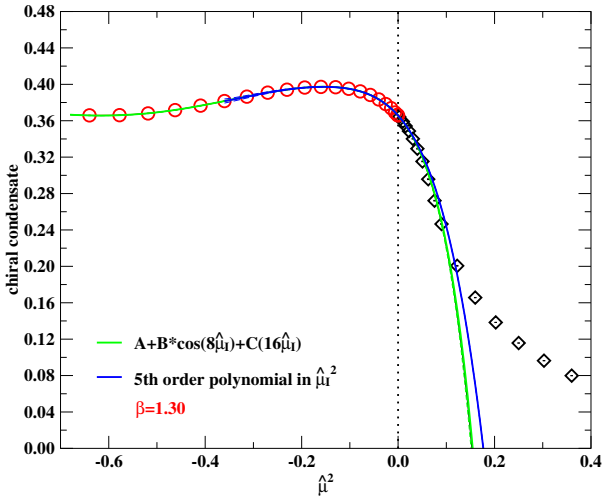

Polyakov loop

- **Extrapolations from $\mu^2 < 0$ are consistent with real data in a wide range of $\mu^2 > 0$**
- **A considerable improvement is obtained if ratio of polynomials (Padè approximants) are used to interpolate data at $\mu^2 < 0$ instead of simple Taylor expansions. That fits well with recent proposals (M.P. Lombardo, 2005)**
- **Deviations at large real chemical potentials can be ascribed to the onset of saturation, an unphysical lattice artifact induced by the Pauli exclusion principle.**
  **fermions fill all available levels $\Longrightarrow$ fermion dynamics gets quenched**
  **analytic continuation better than direct simulations at real $\mu$ ??**

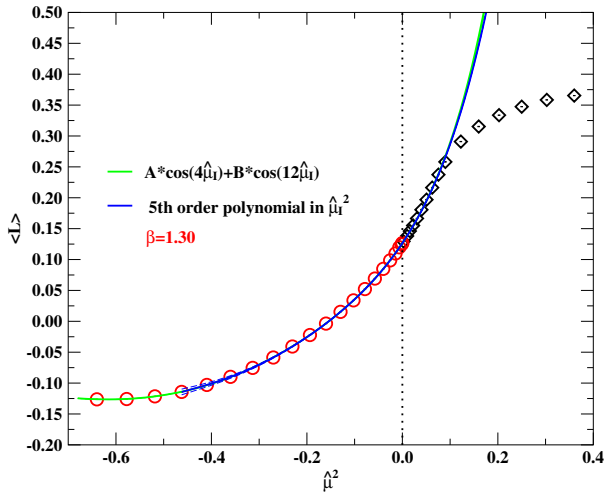
## Low Temperature Region $T < T_c$




quark density




chiral condensate

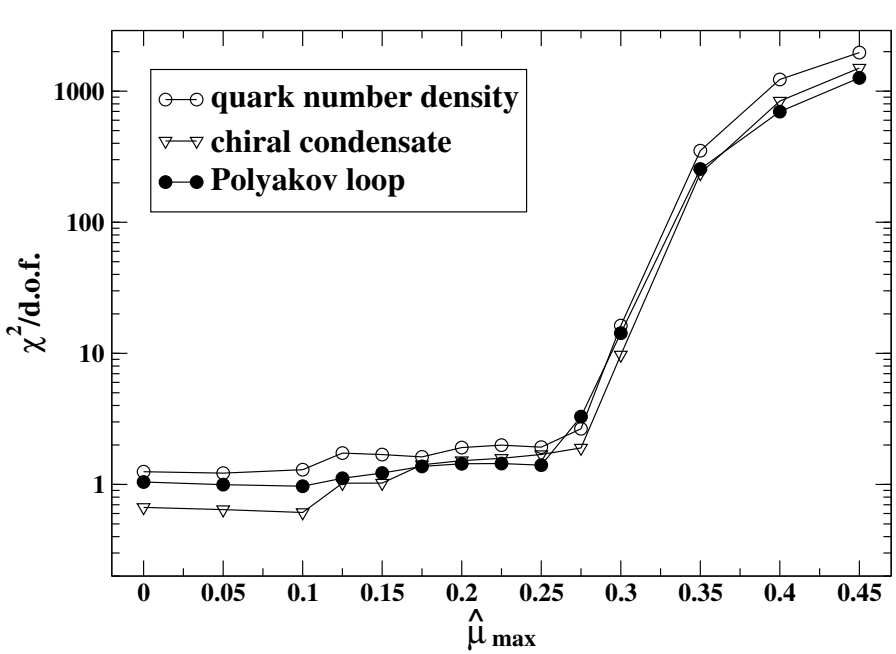


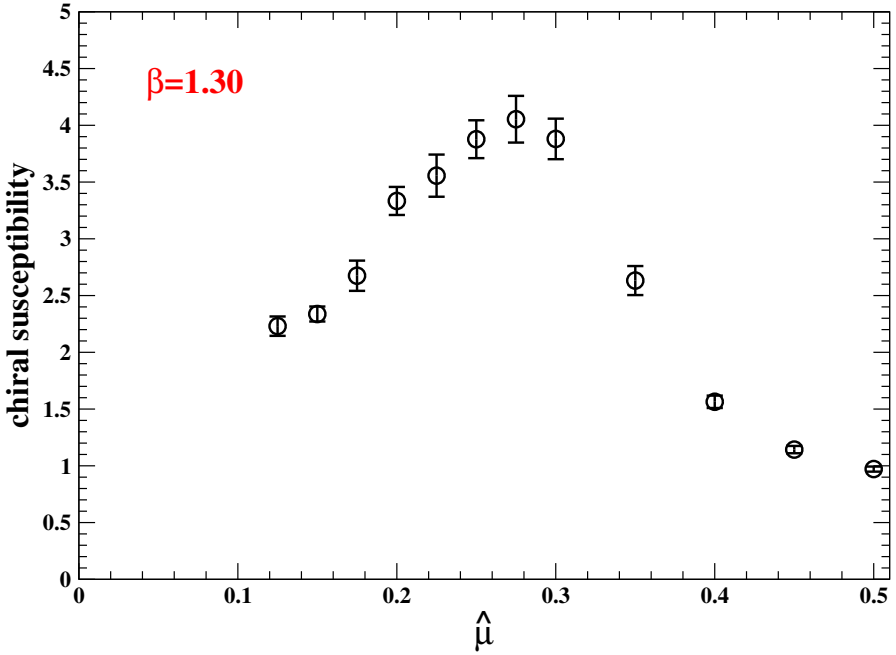

Polyakov loop

- **Trigonometric functions are the natural interpolating functions for data at $\mu^2 < 0$ in this case. They are continued to hyperbolic functions for real chemical potentials.**
- **The high precision of our data permits to fit deviations from the hadron resonance gas model, e.g. for the quark number density**

$$A\sin(2\mu_I/T) + B\sin(4\mu_I/T)\,; A = -0.021582(37)\,,\ B = -0.000611(35)\,,\ B/A < 1/30$$

- **Deviations at real $\mu$ can be ascribed in this case to the presence of the chiral transition line.**

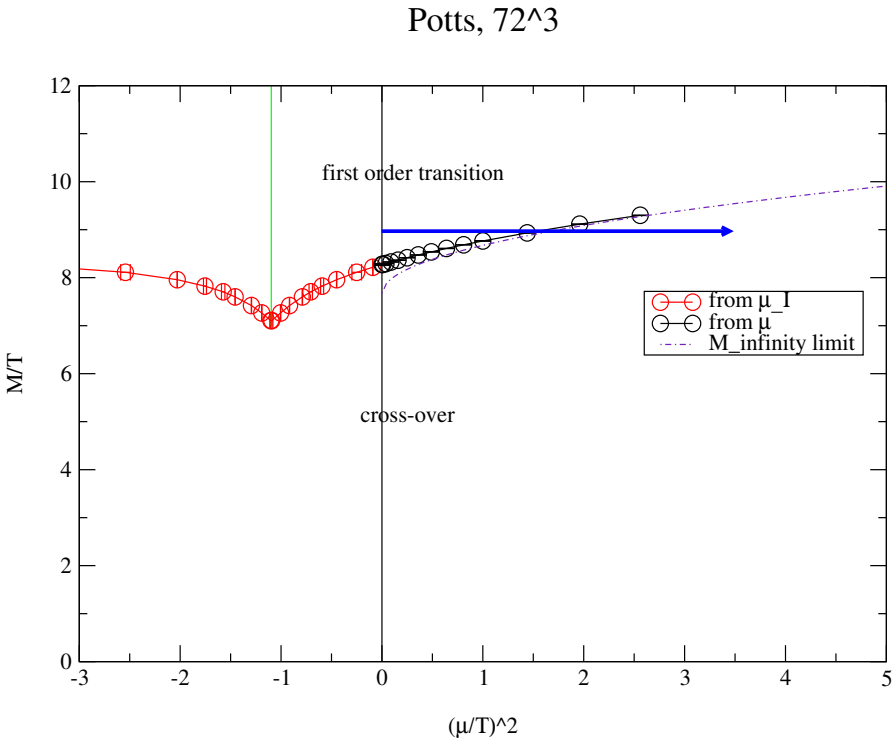



**It is even possible to try global fits to the whole set of data at real and imaginary chemical potentials, thus directly testing the hypothesis that they can be described by a single analytic function Fits are successfull till the chiral transition at real $\mu$ is reached, as signalled also by the peak of the chiral susceptibility.**

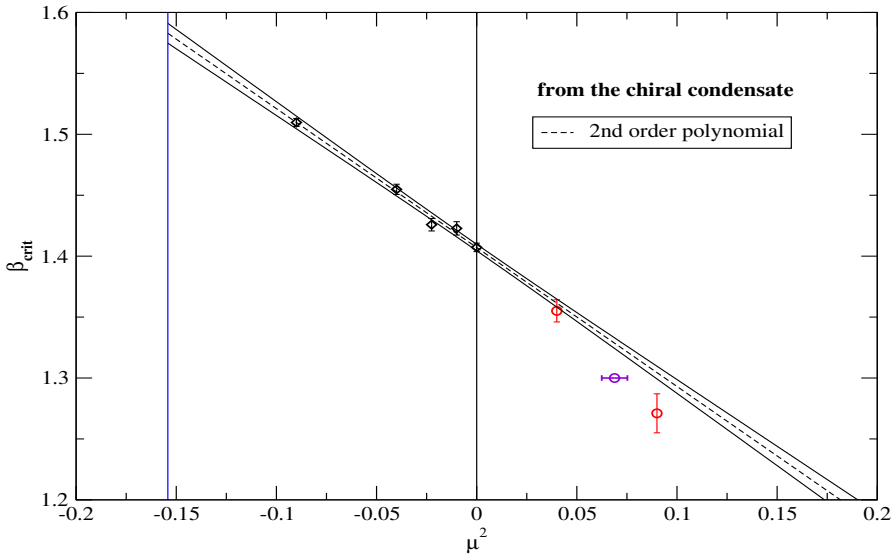
# Continuation of the critical line

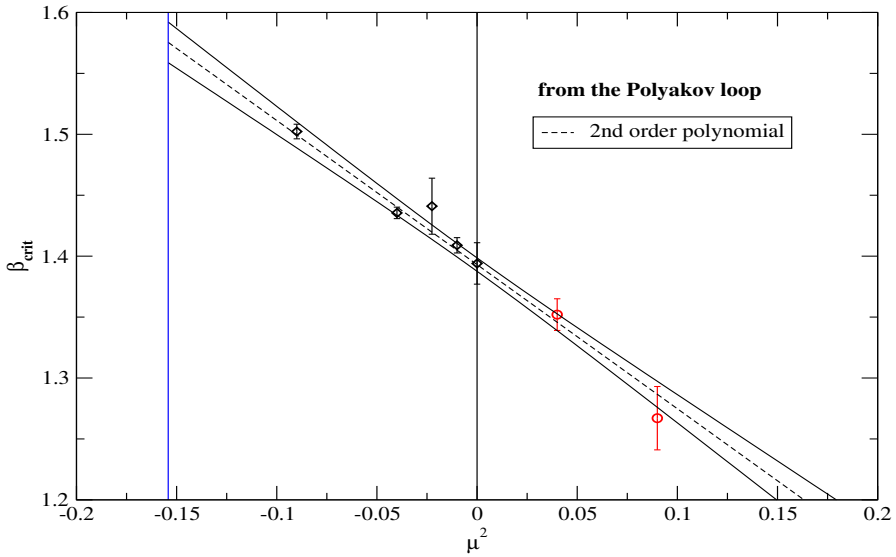
**The analytic continuation of the critical line $T_c(\mu)$ is more subtle and careful tests are even more important in this case.**




**S. Kim, P. de Forcrand, S. Kratochvila, and T. Takaishi (2005) have studied the critical endpoint of the first order transition in the 3d 3state Potts model as a function of $\mu^2$. Analyticity is well verified but a fourth order polynomial in $\mu^2$ is needed.**

**We are currently extending our investigation of analytic continuation in two-color QCD to the critical line issue (poster by A. Papa at Lattice07)**




**we determine the critical line from various susceptibilities: chiral, Polyakov, plaquette**

- **different susceptibilities give perfectly compatible results**
- **The critical line at $\mu^2 < 0$ is well described by a simpler linear behaviour in $\mu^2$ up to the endpoint of the RW line: no trace of further terms is still present.**
- **Location of the critical line at real $\mu$ show a "tension" with respect to analytic continuation in the case of the chiral susceptibility, where the error is smaller**
- **The issue must be fully clarified:**
  - **Larger statistics to see if the tension becomes disagreement or not**
  - **Careful study of the phase structure at $\mu^2 > 0$**

# WHERE HAS THE PHASE GONE?

**does the theory at $\mu^2 < 0$ remember of the sign problem?**

**An observable which is strictly related to the phase of the fermionic determinant is the Polyakov loop:**

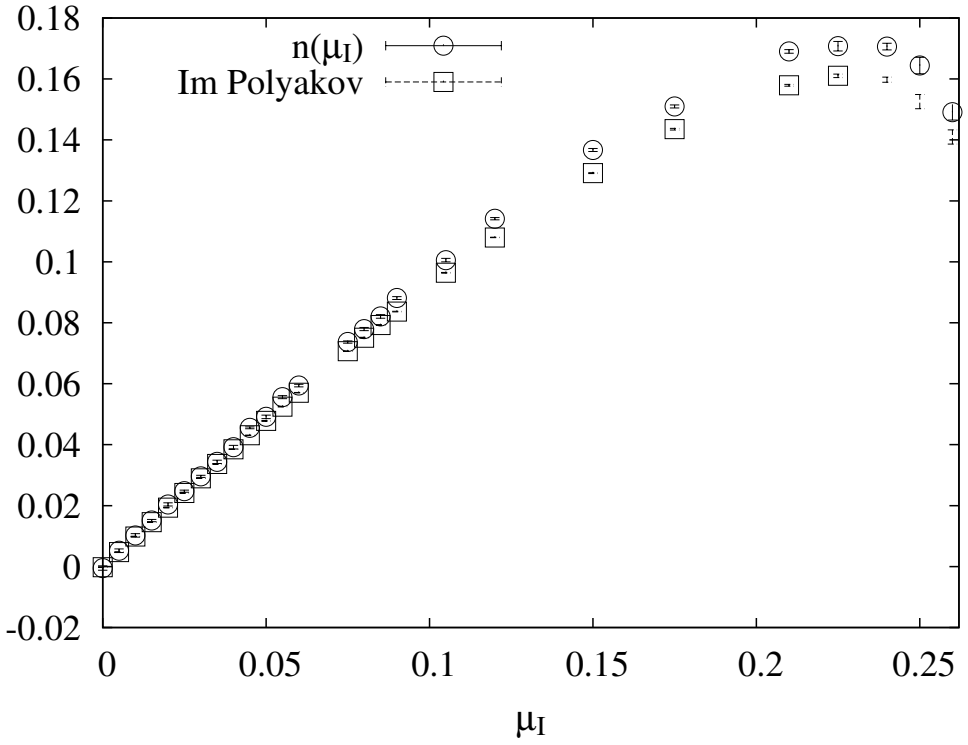
- **it is known that $L = \langle \mathrm{Tr} P \rangle$ and $\bar{L} = \langle \mathrm{Tr} P^\dagger \rangle$ are both real at real chemical potential, but $L \neq \bar{L}$, and in particular $L(\mu) = \bar{L}(-\mu)$ (see e.g. F. Karsch and H. W. Wyld, 1985).**

- **configuration by configuration $\mathrm{Tr} P$ and $\mathrm{Tr} P^\dagger$ are always the complex conjugate of each other, the fact that $\mathrm{Re}\langle \mathrm{Tr} P \rangle_\mu \neq \mathrm{Re}\langle \mathrm{Tr} P^\dagger \rangle_\mu$ is strictly related to the complex measure which is meant $\langle \cdot \rangle_\mu$**

**what about imaginary chemical potentials? M.D'E., F. Di Renzo, M.P. Lombardo, 2007**

- **consider the continuation of $L_{o/e}(\mu) \equiv L(\mu) \pm \bar{L}(\mu) = L(\mu) \pm L(-\mu)$.**
  - **$L_{o/e}(\mu)$ is odd/even in $\mu \implies L_{o/e}(i\mu_I)$ is purely imaginary/real**
  - **$L(i\mu_I) = (L_o(i\mu_I) + L_e(i\mu_I))/2 \implies L_o/2$ is the imaginary part of the Polyakov loop at imaginary chemical potentials.**

- **hence it is the imaginary part of the Polyakov loop which keeps memory of $\bar{L} \neq L$, hence of the complex nature of the measure at real $\mu$.**




**Imaginary part of the Polyakov loop confronted with the imaginary part of the quark number density, showing that $\mathrm{Im}(L)$ is directly related to the finite baryonic density. The correspondence should become exact in the infinite mass limit. (M.D'E., F. Di Renzo, M.P. Lombardo, 2007).**

# ANALYTIC CONTINUATION OF THE PHASE OF THE FERMIONIC DETERMINANT

**The QCD partition function with two flavours of equal chemical potential $\mu$ is:**

$$Z(\mu,\mu) \equiv \int \mathcal{D}U e^{-S_G[U]} (\det M[U,\mu])^2 = \int \mathcal{D}U e^{-S_G[U]} |\det M[U,\mu]|^2 e^{i2\theta} \ ,$$

$$\det M[U,\mu] = |\det M[U,\mu]| e^{i\theta}$$

**If the two flavours have opposite $\mu$ $\Longrightarrow$ no sign problem** ($\det M(-\mu) = \det M(\mu)^*$)

$$Z(\mu,-\mu) = \int \mathcal{D}U e^{-S_G[U]} |\det M[U,\mu]|^2$$

**This corresponds to finite isospin density or phase quenched QCD**

**The average phase factor $\langle e^{i2\theta} \rangle_{(\mu,-\mu)}$ gives an indication of the severeness of the sign problem and of the difference between finite isospin and baryon density.**

$$\langle e^{i2\theta} \rangle_\mu \equiv \left\langle \frac{\det M(\mu)}{\det M(-\mu)} \right\rangle_{(\mu,-\mu)} = \frac{Z(\mu,\mu)}{Z(\mu,-\mu)} \ .$$

**Recently it has been proposed to continue $\langle e^{i2\theta}\rangle_\mu$ to imaginary $\mu's$**

**(K. Splittorff and J. J. M. Verbaarschot, 2007 ; K. Splittorff and B. Svetitsky, 2007)**

$$\langle e^{i2\theta}\rangle_{i\mu} \equiv \left\langle \frac{\det M(i\mu)}{\det M(-i\mu)} \right\rangle_{(i\mu,-i\mu)} = \frac{Z(i\mu,i\mu)}{Z(i\mu,-i\mu)} = \frac{\int \mathcal{D}U e^{-S_G[U]} \det M(i\mu) \det M(i\mu)}{\int \mathcal{D}U e^{-S_G[U]} \det M(i\mu) \det M(-i\mu)}$$

$\det M[U,\pm i\mu]$ **is real** $\Longrightarrow$ **both partition functions are suitable for simulations.**

**That has been studied in the $\varepsilon$ regime by evaluating $\det M(\pm i\mu)$ on the basis of the lowest lying eigenvalues of the Dirac matrix and confronted with analytical predictions. (K. Splittorff and B. Svetitsky, 2007)**

**At imaginary chemical potentials one can define a generalized partition function**

$$Z(i\mu,i\nu) \equiv \int \mathcal{D}U e^{-S_G[U]} \det M[U,i\mu] \det M[U,i\nu]\,.$$

**which is suitable for numerical simulations for every $(i\mu,i\nu)$. That allows numerical techniques for an exact evaluation of average phase factor also outside the $\varepsilon$ regime.**

**(S. Conradi and M. D'E., 2007)**

**Let us consider**

$$R_\mu(\nu) = \frac{Z(i\mu, i\nu)}{Z(i\mu, -i\mu)}$$

$R_\mu(\mu) = \langle e^{i2\theta} \rangle_{i\mu}$ **while** $R_\mu(-\mu) = 1$**. The idea is to compute the derivative of** $\ln R_\mu(\nu)$ **and then integrate it (similar to the integral method for computing of** $\Delta P/T^4$**).**

$$\rho(\nu) \equiv \frac{d}{d\nu} \ln R_\mu(\nu) = \frac{d}{d\nu} \ln Z(i\mu, i\nu) = \left\langle i \, \mathrm{Tr} \left( M^{-1}(i\nu) \frac{d}{d(i\nu)} M(i\nu) \right) \right\rangle_{(i\mu, i\nu)} .$$

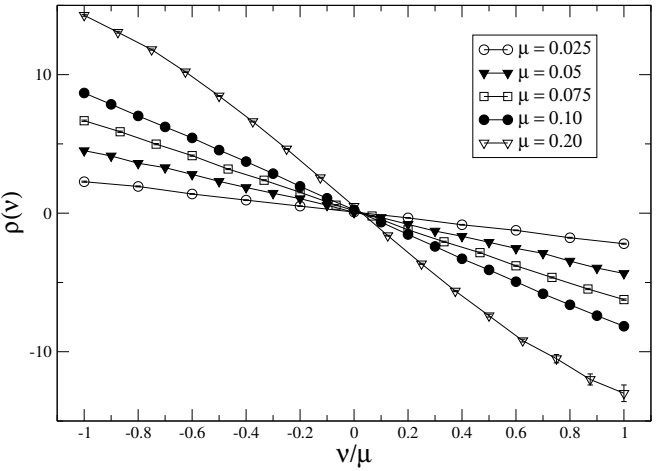
**The trace gives the quark number (imaginary) coupled to the chemical potential** $i\nu$**.**

**The average phase factor is then given by**

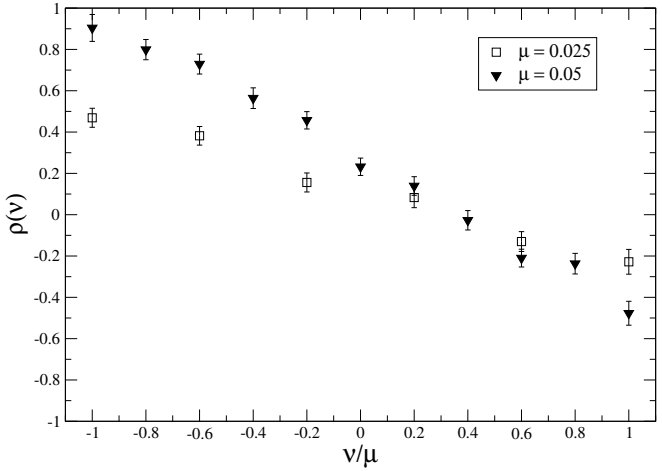
$$\langle e^{i2\theta} \rangle_{i\mu} = \exp \left( \int_{-\mu}^{\mu} \rho(\nu) d\nu \right) \tag{1}$$

**hence obtained without any determinant computation. The derivative** $\rho(\nu)$ **is computed for a discrete set of** $\nu$ **and then integrated numerically.**

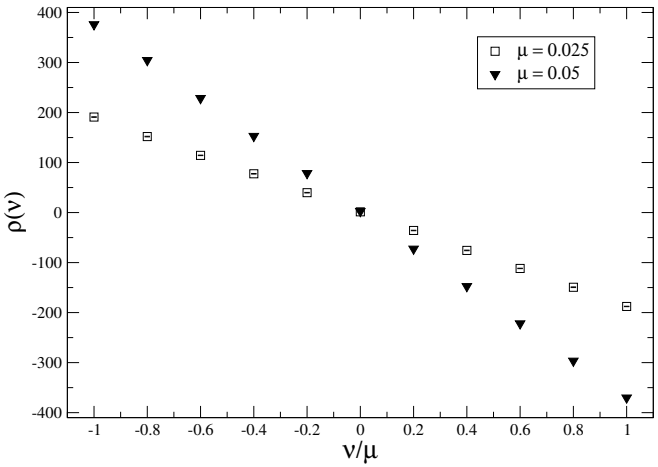
**We have tested the method for 8 staggered flavors of mass** $am = 0.1$**. A strong first order transition is present at** $\mu = 0$**: we have explored both phases.**




$4^3 \times 4$ , $\beta = 4.8$

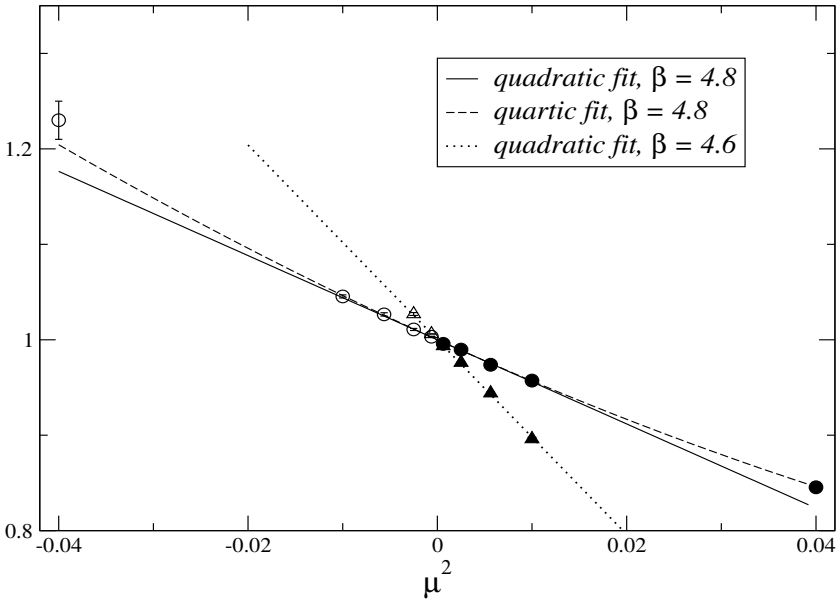



$4^3 \times 4$ , $\beta = 4.6$




$16^3 \times 4$ , $\beta = 4.8$

**$\rho(\nu)$ is always a smooth function of $\nu$ $\Longrightarrow$ systematics well under control**

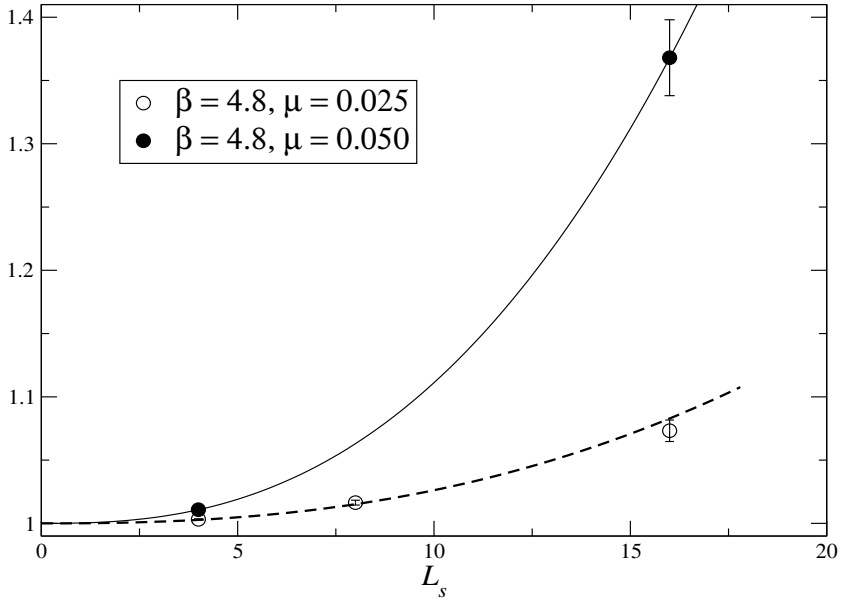



**Analyticity of the phase factor around $\mu^2 = 0$ is well verified on the smallest lattices. In our largest range of values at $\beta = 4.8$ we have:**

$$\langle e^{i2\theta} \rangle = 1 + A\mu^2 + B\mu^4 \qquad (2)$$

**with** $A = -4.48(8), B = 15.7 \pm 2.5$ **and** $\chi^2/\text{d.o.f.} \simeq 1.3$**.**

**The study of the large volume scaling of $\langle e^{i2\theta} \rangle$ is affordable at imaginary $\mu$**




**We have studied two different values of $i\mu$. In both cases the scaling with $L_s$ is well described by:**

$$\langle e^{i2\theta} \rangle = 1 + C L_s^{\gamma} \qquad (3)$$

**with $\gamma \sim 2.5$.**

$\Longrightarrow$ **The region at real chemical potentials where the sign problem is not important becomes irrelevant in the thermodynamical limit, phase factor suppressed exponentially with the volume in that limit (compare also with results presented by Ph. de Forcrand in his talk)**

# SUMMARY and OUTLOOK

⋆ **The method of analytic continuation from an imaginary chemical potential is providing us with interesting (even if not "extreme") results about the QCD phase diagram:**

- **slope and nature of the critical line for relatively small chemical potentials**
- **properties of the hadron gas below $T_c$**
- **properties of deconfined matter in the strongly interacting phase and approaching the perturbative regime**

⋆ **Exact analytic results are also being obtained in special regimes or models:**

- **yesterday talk by P.H. Damgaard, tomorrow talk by K. Splittorff about the $\varepsilon$-regime**
- **F. Karbstein and M. Thies have recently shown how to obtain the whole phase diagram of the $1+1$ large $N$ Gross-Neveu model by analytic continuation of the effective potential of the order parameter. Extensions to QCD?**

⋆ **The method is applicable to other theories with a sign problem (consider for instance an imaginary $\theta$-term, see e.g. poster by B.Alles and A.Papa at Lattice 2007).**

⋆ **Systematic studies of the validity of the method can be performed in theories where the sign problem is absent or treatable**

⋆ **Much work yet to be done: further efforts and more refined studies will surely lead to even more interesting results in the near future.**