Physics of non-Extreme QCD at finite Isospin Potential

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Overview

- Consider finite-volume QCD with asymptotically *vanishing* imaginary isospin chemical potential: $\mu \sim 1/\sqrt{V}$.

- For very light quarks $m \sim 1/V$ this can actually be a pretty extreme isospin chemical potential: $m_\pi \sim 1/\sqrt{V}$ – same as μ.

- Both with and without chemical potential the connection between chiral Random Matrix Theory and the chiral Lagrangian is well established.

- Random Matrix Theory gives the finite-volume scaling limit of QCD deep in the ϵ-regime.

- Scaling depends to leading order on one fitting parameter: the chiral condensate Σ.

- To next orders the other low-energy constants of QCD begin to appear. Then Random Matrix Theory cannot be used.

- Goal: To find finite-volume scaling for F_π already at leading order.

- We need to introduce an appropriate source. *This is where we choose imaginary isospin chemical potential*.

- Because isospin chemical potential corresponds to having two different Dirac operators, we need a Random Two-Matrix Theory.

- A theory with (imaginary) isospin chemical has no sign problem: it can be simulated by Monte Carlo methods.

- Better: We can use lattice configurations obtained at zero isospin chemical potential ("partial quenching").

The Aim

To extract F_π from the distribution of just one single Dirac operator eigenvalue with chemical potential.

The ϵ-regime

Spontaneous breaking of chiral symmetry

$$\Sigma \;=\; \lim_{m\to 0}\lim_{V\to\infty}\langle\bar{\psi}\psi\rangle$$

Finite-volume scaling in the ϵ-regime:

$$V \to \infty$$

$$m_\pi L \ll 1$$

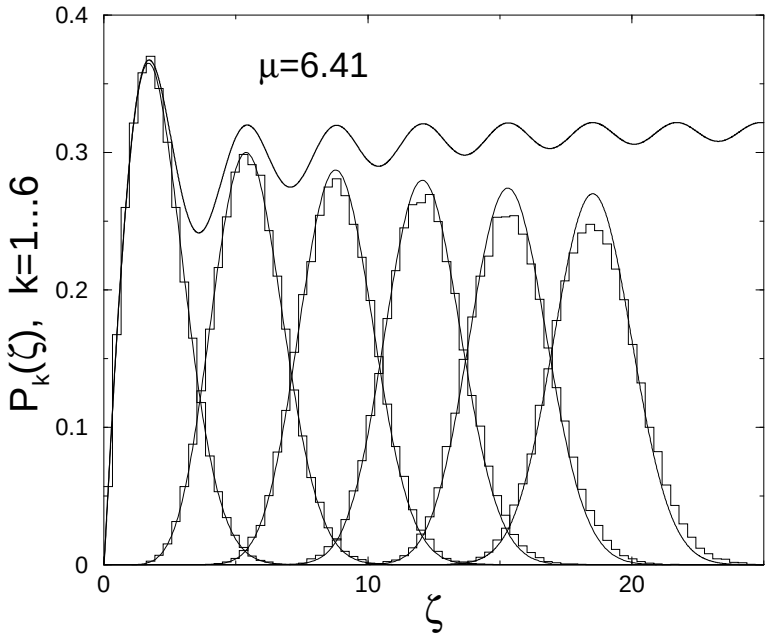
Low-energy QCD depends only on the scaling variable

$$\hat{m} \;\equiv\; m\Sigma V$$

Rescale Dirac operator eigenvalues λ

$$\xi \equiv \lambda \Sigma V$$

Universal distributions and correlation functions.



Random Matrix Theory I

Chiral Random Matrix Theory [Verbaarschot, 1994]:

$$\mathcal{Z}_\nu \;=\; \int d\Phi \prod_{f=1}^{N_f} \det[D+m_f] e^{-NV(\Phi\Phi^\dagger)}$$

with

$$\mathcal{D} = \begin{pmatrix} 0 & i\Phi \\ i\Phi^\dagger & 0 \end{pmatrix}$$

Finite-size scaling in the matrices: $N \to \infty$, fix

$$\xi = 2\pi\rho(0)\lambda N$$

Spectral correlation functions of ξ's coincide exactly with those of the **Dirac operator** in the ϵ-regime.

$$2\pi\rho(0)\lambda_{RMT}N \quad \sim \quad \Sigma\lambda_{Dirac}V$$

The approach based on the effective low-energy theory is more tedious, but systematic.

The Chiral Lagrangian

$$\mathcal{L} \;=\; \frac{F_\pi}{4}\mathrm{Tr}\left[\partial_\mu U(x)\partial_\mu U^\dagger(x)\right] - \frac{\Sigma}{2}\mathrm{Tr}\left[\mathcal{M}U^\dagger(x) + \mathcal{M}^\dagger U(x)\right] + \ldots$$

Leading term in the ϵ-expansion

$$\mathcal{L} \;=\; -\frac{\Sigma}{2}\mathrm{Tr}\left[\mathcal{M}U^\dagger(x) + \mathcal{M}^\dagger U(x)\right]$$

Partition function in sector of fixed topological charge ν

$$\mathcal{Z}_\nu = \int_{U(N_f)} dU\,(\det U)^\nu e^{\frac{1}{2}\Sigma V\mathrm{Tr}(\mathcal{M}^\dagger U + \mathcal{M}U^\dagger)}$$

Include Isospin Chemical Potential

Dirac operators

$$D_+\psi_n \equiv [\not{D}(A) + i\mu\gamma_0]\psi_n = i\lambda_n^{(+)}\psi_n$$

$$D_-\psi_n \equiv [\not{D}(A) - i\mu\gamma_0]\psi_n = i\lambda_n^{(-)}\psi_n$$

Transcription to the effective theory:

$$\partial_0 U(x) \rightarrow \nabla_0 U(x) = \partial_0 U(x) - i\mu[\sigma_3, U(x)],$$

Leading term in the ϵ-regime ($B = \sigma_3$)

$$\mathcal{Z}_\nu = \int dU \det(U)^\nu e^{\frac{1}{4}VF_\pi^2\mu^2 \mathrm{Tr}[U,B][U^\dagger,B] + \frac{1}{2}\Sigma V \mathrm{Tr}(M^\dagger U + MU^\dagger)}$$

New finite-volume scaling variable

$$\hat{\mu}^2 \equiv V F_\pi^2 \mu^2$$

Partition functions be computed analytically. Example, $N_f = 2$

$$\mathcal{Z}_\nu = e^{-2\hat{\mu}^2} \int_0^1 dt\, t e^{2\hat{\mu}^2 t^2} I_\nu(t\hat{m}_+) I_\nu(t\hat{m}_-),$$

Consider the **2-pt spectral correlation function**

$$\begin{aligned}\rho(\lambda_+, \lambda_-) \equiv & \left\langle \sum_n \delta\left(\lambda_+ - \lambda_+^{(n)}\right) \sum_m \delta\left(\lambda_- - \lambda_-^{(m)}\right) \right\rangle \\ & - \left\langle \sum_n \delta\left(\lambda_+ - \lambda_+^{(n)}\right) \right\rangle \left\langle \sum_m \delta\left(\lambda_- - \lambda_-^{(m)}\right) \right\rangle\end{aligned}$$

Example: the quenched formula

$$
\begin{aligned}
& \rho_s(\xi_+,\xi_-) \\
= \quad & \xi_+\xi_- \left[\int_0^1 dt\, t e^{2\hat{\mu}^2 t^2} J_\nu(t\xi_+) J_\nu(t\xi_-)\right] \\
& \times \left[\frac{1}{4\hat{\mu}^2} \exp\left(-\frac{(\xi_+^2+\xi_-^2)}{8\hat{\mu}^2}\right) I_\nu\left(\frac{\xi_+\xi_-}{4\hat{\mu}^2}\right) - \int_0^1 dt\, t e^{-2\hat{\mu}^2 t^2} J_\nu(t\xi_+) J_\nu(t\xi_-)\right]
\end{aligned}
$$

[PHD, U.M. Heller, K. Splittorff and B. Svetitsky, 2005]

Random Matrix Theory

The "simple" form of the final results for $N_f = 0$ and $N_f = 2$ suggests that Random Matrix Theory is at play again.

Two different Dirac operators suggest that we need a **chiral Random Two-Matrix Theory**.

Consider ($N \sim$ size of matrix)

$$\mathcal{Z}_\nu = \int d\Phi_1 d\Phi_2 \prod_{f1=1}^{N_1} \det[\mathcal{D}_1 + m_{f1}] \prod_{f2=1}^{N_2} \det[\mathcal{D}_2 + m_{f2}] \; e^{-N \mathrm{Tr} V(\Phi_1, \Phi_2)}$$

with

$$\mathcal{D}_{1,2} = \begin{pmatrix} 0 & i\Phi_{1,2} \\ i\Phi_{1,2}^\dagger & 0 \end{pmatrix} \tag{1}$$

The two random matrices are coupled

$$V(\Phi_1, \Phi_2) = c_1 \Phi_1^\dagger \Phi_1 + c_2 \Phi_2^\dagger \Phi_2 - d \left(\Phi_1^\dagger \Phi_2 + \Phi_2^\dagger \Phi_1 \right)$$

with

$$\begin{aligned} c_1 &= (1+\mu_2^2)/\delta^2 \\ c_2 &= (1+\mu_1^2)/\delta^2 \\ d &= (1+\mu_1\mu_2)/\delta^2 \\ \delta &= \mu_2 - \mu_1 \end{aligned}$$

In the scaling limit the partition function of this chiral Random Two-Matrix Theory coincides with that of the leading term of the chiral Lagrangian in the ϵ-regime.

Eigenvalue Representation

Decompose

$$\begin{aligned} \Phi_1 &= U_1 X V_1^\dagger \\ \Phi_2 &= U_2 Y V_2^\dagger \end{aligned}$$

and perform the unitary integrations (chiral analogs of Itzykson-Zuber Integral)

$$\int dU dV \exp\left[dN \mathrm{Tr}(VXUY) \ + \mathrm{h.c.}\ \right] \ = \ \frac{\det[I_\nu(2dNx_i y_j)]}{\prod (x_i y_i)^\nu \Delta_N(\{x^2\}) \Delta_N(\{y^2\})}$$

$\Delta_N(z)$ is the Vandermonde determinant.

Eigenvalue representation

$$\begin{aligned}\mathcal{Z}_\nu^{(N_f)} &= \int_0^\infty \prod_i^N \left(dx_i dy_i (x_i y_i)^{\nu+1} \prod_{f1=1}^{N_1} (x_i^2 + m_{f1}^2) \prod_{f2=1}^{N_2} (y_i^2 + m_{f2}^2) \right) \\ &\times \Delta_N(\{x^2\}) \Delta_N(\{y^2\}) \det\left[I_\nu(2dN x_i y_j) \right] e^{-N \sum_i c_1 x_i^2 + c_2 y_i^2} \end{aligned} \quad (2)$$

Now ALL eigenvalue correlation functions can be computed!

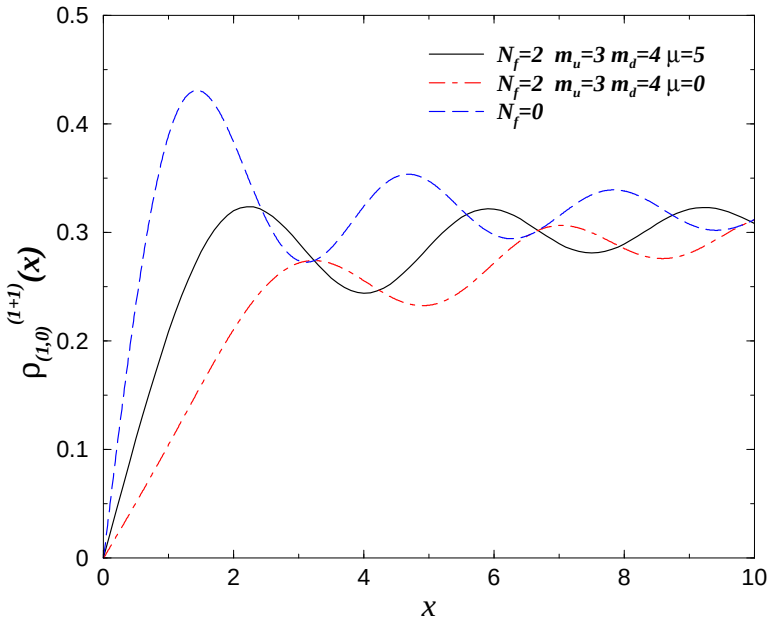
Two and More Light Flavors

$$\rho^{(1+1)}_{(n,k)}(\{\hat{x}\}_n,\{\hat{y}\}_k) = \prod_{i=1}^{n}\hat{x}_i \prod_{j=1}^{k}\hat{y}_j \left(\mathcal{I}^+(i\hat{m}_u, i\hat{m}_d)\right)^{-n-k}$$

$$\times \quad \det_{\substack{1\le i_1,i_2\le n\\ 1\le j_1,j_2\le k}} \begin{bmatrix} \mathcal{I}^+(i\hat{m}_u,i\hat{m}_d)\mathcal{I}^0(\hat{x}_{i_1},\hat{x}_{i_2}) & -\mathcal{I}^+(i\hat{m}_u,i\hat{m}_d)\tilde{\mathcal{I}}^-(\hat{y}_{j_1},\hat{x}_{i_2}) \\ -\mathcal{I}^0(\hat{x}_{i_1},i\hat{m}_u)\mathcal{I}^+(\hat{x}_{i_2},i\hat{m}_d) & -\mathcal{I}^0(\hat{y}_{j_1},i\hat{m}_d)\mathcal{I}^0(\hat{x}_{i_2},i\hat{m}_u) \\ \\ \mathcal{I}^+(i\hat{m}_u,i\hat{m}_d)\mathcal{I}^+(\hat{y}_{j_1},\hat{x}_{i_2}) & \mathcal{I}^+(i\hat{m}_u,i\hat{m}_d)\mathcal{I}^0(\hat{y}_{j_1},\hat{y}_{j_2}) \\ -\mathcal{I}^+(\hat{y}_{j_1},i\hat{m}_u)\mathcal{I}^+(\hat{x}_{i_2},i\hat{m}_d) & -\mathcal{I}^+(\hat{y}_{j_1},i\hat{m}_u)\mathcal{I}^0(\hat{y}_{j_2},i\hat{m}_d) \end{bmatrix}$$

with known integrals $\mathcal{I}$'s.

Now even the spectral density itself is $\hat{\mu}$-dependent – we have the closed analytical expression.



Moreover: $\rho_{(1,1)}(\hat{x},\hat{y})$ agrees precisely with the result from the chiral Lagrangian!

Partial Quenching

Use *ordinary configurations* without chemical potentials to extract F_π with the same method. The chemical potential enters only in the valence sector.

Forbiddingly difficult to perform the calculation from the chiral Lagrangian without new tricks or new insight.

Can be computed straightforwardly from the Random Two-Matrix Theory. We have the general (k, n)-correlation function analytically.

It is "simple", but I will not show it.

One Eigenvalue Splits into Two

Suppose we have only one ordinary Dirac eigenvalue available.

Compare the distribution of this single eigenvalue to that of a valence quark eigenvalue with chemical potential.

Work in Progress:

Let us derive the distribution of the lowest Dirac operator eigenvalue for a valence quark with chemical potential.

We already have expansions.

Let there be "ordinary", physical, Dirac eigenvalues x, and valence Dirac eigenvalues y with imaginary chemical potential. Probability distributions

$$p_{k,\ell}(x,y)$$

Example:

$$p_{(1,1)}(x,y) = \rho_{(1,1)}(x,y) - \int_0^s dx'\, \rho_{(2,1)}(x',x,y) - \int_0^t dy'\, \rho_{(1,2)}(x,y,y') + \ldots$$

– the probability distribution for the smallest "ordinary" eigenvalue to be x while that of the valence quark is y.

Similarly for the probability distribution of the smallest eigenvalue of the valence quark (*with* chemical potential)

$$p_{(0,1)} = \rho_{(0,1)}(y) - \int_0^y dy' \; \rho_{(0,2)}(y, y') + \dots$$

This function depends on Σ and F_π.

Fit Σ from the distribution of the eigenvalue with zero chemical potential.

Fit F_π from the distribution of the eigenvalue with non-zero chemical potential!

Conclusions

- A chiral Random Two-Matrix Theory and a scaling regime with an external vector source ("isospin chemical potential")
- The source acts as a probe of the low-energy constant F_π
- The lowest Dirac operator eigenvalue is distributed in a manner that depends on Σ and F_π
- So far an accurate expansion exists – work on the exact analytical expression is in progress
- A new and inexpensive way to extract F_π from lattice simulations at finite volume