Thermodynamics of the $O(N)$ sigma model in the $1/N$ expansion

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QCD in Extreme Conditions 2007
Outline

1. \( O(N) \) sigma model and 1PI-1/\( N \) expansion
   - \( O(N) \) sigma model
   - Auxiliary field technique
   - 1PI-1/\( N \) expansion

2. Leading order
   - Sigma and pion condensates
   - Phase diagram

3. Next-to-leading order
   - Renormalization
   - Sigma and pion condensates
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Introduction

- The general framework: study of the QCD phase diagram.
- Low temperature and chemical potential: only LE DOFs excited.
- Simplest possible approximation: include only the light pseudoscalar mesons, i.e., the Goldstone bosons of the spontaneously broken $SU_L(N_f) \times SU_R(N_f)$ chiral symmetry.
- In particular for $N_f = 2$ make use of the isomorphism $SU(2) \times SU(2) \simeq SO(4)$ and study the $O(4)$ model.
- A special case of the general $O(N)$ model; applications besides low-energy QCD also e.g. in the electroweak interactions (Higgs mechanism), or in condensed matter physics (spin models).
- At large $N$, a $1/N$ expansion possible as a substitute for the conventional perturbation theory.
- Nonperturbative technique which preserves, even at the leading order, “much more of the nonlinear structure of the exact theory than does ordinary lowest-order perturbation theory”.

Coleman, Jackiw, and Politzer, PRD10 (1974) 2491
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1/$N$ expansion

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  Coleman, Jackiw, and Politzer, PRD10 (1974) 2491 (LO at $T = 0$)
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- Here: 1PI-1/N expansion to NLO including the solution of the gap equation, at nonzero temperature and chemical potential.
Generalize the $O(4)$ model to arbitrary $N$ by a suitable redefinition of the couplings.

Perform the (nonperturbative) resummation to a given order in $1/N$ and set $N = 4$ afterwards. Hope that 4 is large enough.

Some leading-order, $O(N)$, contributions to the pressure:

Some next-to-leading-order, $O(1)$, contributions to the pressure:
Auxiliary field technique

- A real scalar $N$-vector, $\phi = (\pi_1, \pi_2, \ldots, \pi_{N-1}, \sigma)$.
- Include isospin chemical potential for the pair $(\pi_1, \pi_2) \sim (\pi^+, \pi^-)$.
- Include explicit chiral symmetry breaking (quark mass) in terms of a source for $\phi_N \sim \sigma$.
- Define all the couplings so that the action scales naturally with $N$.

$$
\mathcal{L} = \frac{1}{2} (\partial_\mu \phi_i)^2 + \frac{\lambda_b}{8N} (\phi_i \phi_i - \mathcal{N}f_{\pi, b}^2)^2
$$

- Introduce a new auxiliary field $\alpha$ and add pure Gaussian integral over $\alpha$.

$$
\Delta \mathcal{L} = \frac{N}{2\lambda_b} \left[ \alpha - \frac{i\lambda_b}{2N} (\phi_i \phi_i - \mathcal{N}f_{\pi, b}^2) \right]^2
$$

- Integrate over the non-condensing fields $\pi_3, \ldots, \pi_{N-1}$.

$$
\mathcal{L} = \frac{1}{2} (N - 3) \text{Tr} \log (-\partial^2 - i\alpha) + \frac{1}{2} \sum_{i=1,2,N} [(\partial_\mu \phi_i)^2 - i\alpha \phi_i^2] +
$$

$$
\frac{i}{2} \mathcal{N}f_{\pi, b}^2 \alpha \left( \alpha^2 - \sqrt{N}H\phi_N - i\mu \delta_{\mu 0} (\phi_1 \partial_\mu \phi_2 - \phi_2 \partial_\mu \phi_1) - \frac{1}{2} \mu^2 (\phi_1^2 + \phi_2^2) \right)
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Introduce the condensates \( \bar{\rho} = \frac{1}{\sqrt{N}} \langle \phi_1 \rangle \), \( \bar{\sigma} = \frac{1}{\sqrt{N}} \langle \phi_N \rangle \), \( m^2 = -i \langle \alpha \rangle \), and shift the fields.

\[
\phi_1 = \sqrt{N} \bar{\rho} + \tilde{\pi}_1, \quad \phi_N = \sqrt{N} \bar{\sigma} + \tilde{\sigma}, \quad \alpha = im^2 + \frac{\tilde{\alpha}}{\sqrt{N}}
\]

Expand the action (thermodynamic potential) in powers of \( 1/N \).

At NLO, the path integral over the field fluctuations is Gaussian and hence exactly calculable.

\[
\frac{S_{\text{NLO}}^{\text{eff}}}{\beta V} = \frac{1}{2} (N - 3) \int_P \log(P^2 + m^2) - NH\bar{\sigma} - \frac{Nm^4}{2\lambda_b} + \\
+ \frac{1}{2} Nm^2(\bar{\sigma}^2 - f_{\pi,b}^2) + \frac{1}{2} (m^2 - \mu^2) N\bar{\rho}^2 + \frac{1}{2} \int_P \chi^T D^{-1} \chi^*
\]

\[
D^{-1} = \begin{pmatrix}
\frac{1}{2} \Pi(P, m) + \frac{1}{\lambda_b} & -i\bar{\sigma} & -i\bar{\rho} & 0 \\
-i\bar{\sigma} & P^2 + m^2 & 0 & 0 \\
-i\bar{\rho} & 0 & P^2 + m^2 - \mu^2 & -2\mu\omega \\
0 & 0 & +2\mu\omega & P^2 + m^2 - \mu^2
\end{pmatrix}, \quad \chi = \begin{pmatrix}
\tilde{\alpha} \\
\bar{\sigma} \\
\tilde{\pi}_1 \\
\pi_2
\end{pmatrix}
\]

\[
\Pi(P, m) = \int_Q \frac{1}{Q^2 + m^2} \frac{1}{(P + Q)^2 + m^2}
\]
Inverse propagators

For simplicity, only consider the case $\mu = 0 \Rightarrow$ mixing only in the $(\tilde{\alpha}, \tilde{\sigma})$ block, separate propagation of degenerate pions.

- Inverse propagator of $\tilde{\alpha}$.

$$D_{\tilde{\alpha}}^{-1}(P, m) = \frac{1}{2} \Pi(P, m) + \frac{1}{\lambda_b} + \frac{\tilde{\sigma}^2}{P^2 + m^2} \xrightarrow{\lambda_b \to 0} \frac{1}{\lambda_b}$$

- Inverse propagator of $\tilde{\sigma}$.

$$D_{\tilde{\sigma}}^{-1}(P, m) = P^2 + m^2 + \frac{\tilde{\sigma}^2}{\frac{1}{2} \Pi(P, m) + \frac{1}{\lambda_b}} \xrightarrow{\lambda_b \to 0} P^2 + m^2 + \lambda_b \tilde{\sigma}^2$$

- Both $\tilde{\sigma}$ and $\tilde{\alpha}$ describe the propagation of the same mode.

$$D_{\tilde{\alpha}}^{-1} = D_{\tilde{\sigma}}^{-1} \frac{\frac{1}{2} \Pi + \frac{1}{\lambda_b}}{P^2 + m^2}$$

Those are all results of the resummation of the NLO in the $1/N$ expansion. In the $\lambda_b \to 0$ limit, we recover perturbative results.
Inverse propagators

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Those are all results of the resummation of the NLO in the $1/N$ expansion. In the $\lambda_b \to 0$ limit, we recover perturbative results.
At LO, the only quantum contribution to the thermodynamic potential comes from the $N - 3$ “pions” $\pi_3, \cdots, \pi_{N-1}$.

\[
V_{LO} = \frac{m^2}{2} (f_{\pi,b}^2 - \bar{\sigma}^2 - \bar{\rho}^2) + \frac{m^4}{2 \lambda_b} + \frac{1}{2} \mu^2 \bar{\rho}^2 + H\bar{\sigma} - \frac{1}{2} \int \log(P^2 + m^2)
\]

\[
= \frac{m^2}{2} (f_{\pi,b}^2 - \bar{\sigma}^2 - \bar{\rho}^2) + \frac{T^4}{64\pi^2} J_0(\beta m) + \frac{m^4}{64\pi^2} \left( \frac{32\pi^2}{\lambda} + \log \frac{Q^2}{m^2} + \frac{1}{2} \right) + \frac{1}{2} \mu^2 \bar{\rho}^2 + H\bar{\sigma}
\]

where $J_0(\beta m) = \frac{32}{3 T^4} \int_0^\infty dp \frac{p^4}{\omega_p} n(\omega_p)$ and $\omega_p = \sqrt{p^2 + m^2}$

LO renormalization by redefinition of the parameters.

\[
f_{\pi}^2 = f_{\pi,b}^2 - \frac{\Lambda^2}{16\pi^2}, \quad \frac{32\pi^2}{\lambda} = \frac{32\pi^2}{\lambda_b} + \log \frac{\Lambda^2}{Q^2}
\]

LO $\beta$-function: $\beta(\lambda) = \frac{\lambda^2}{16\pi^2}$. 

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LO gap equations

- Solve the LO gap equations.

\[
\begin{align*}
G &= 16\pi^2 f_\pi^2, \quad H = m^2 \bar{\sigma}, \quad (m^2 - \mu^2) \bar{\rho} = 0 \\
G &= 16\pi^2 (\bar{\sigma}^2 + \bar{\rho}^2) + T^2 J_1(\beta m) - m^2 \log \frac{Q^2}{m^2} - \frac{32\pi^2 m^2}{\lambda} \\
J_1(\beta m) &= \frac{8}{T^2} \int_0^\infty dp \frac{p^2}{\omega_p} n(\omega_p)
\end{align*}
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- Chiral limit: \( H = 0 \).
  - Chiral-symmetry-breaking phase, just the \( \bar{\sigma} \) condensate.
  - Pion-condensation phase, just the \( \bar{\rho} \) condensate.
  - Normal phase

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The chiral-symmetry-breaking phase is never energetically favorable, except at $\mu = 0$ where it is degenerate with pion condensation.

- **Normal phase:** $\bar{\sigma}$ is degenerate with the pions. Their common mass is given implicitly by
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  16 \pi^2 f_{\pi}^2 = T^2 J_1(\beta m) - m^2 \log \frac{Q^2}{m^2} - \frac{32 \pi^2 m^2}{\lambda}
  \]
  - Nonzero $m^2$ only for $T \geq T_{c}^{(m)} = \sqrt{12} f_{\pi}$.
  - Above $T_{c}^{(m)}$, $m^2$ increases with $T$ until it reaches the value $m^2 = \mu^2$ at the critical temperature for pion condensation.

- **Pion-condensation phase:**
  $m^2 = \mu^2$, pion condensate $\bar{\rho}$ is given explicitly by
  \[
  16 \pi^2 f_{\pi}^2 = 16 \pi^2 \bar{\rho}^2 + T^2 J_1(\beta \mu) - \mu^2 \log \frac{Q^2}{\mu^2} - \frac{32 \pi^2 \mu^2}{\lambda}
  \]
  - Pion condensate grows with chemical potential and decreases with temperature.
  - Critical temperature for pion condensation increases with chemical potential.
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- Above $T_c^{(m)}$, $m^2$ increases with $T$ until it reaches the value $m^2 = \mu^2$ at the critical temperature for pion condensation.

- **Pion-condensation phase**: $m^2 = \mu^2$, pion condensate $\bar{\rho}$ is given explicitly by

$$16\pi^2 f_\pi^2 = 16\pi^2 \bar{\rho}^2 + T^2 J_1(\beta \mu) - \mu^2 \log \frac{Q^2}{\mu^2} - \frac{32\pi^2 \mu^2}{\lambda}$$

- Pion condensate grows with chemical potential and decreases with temperature.
- Critical temperature for pion condensation increases with chemical potential.
The chiral-symmetry-breaking phase is never energetically favorable, except at $\mu = 0$ where it is degenerate with pion condensation.

- **Normal phase**: $\tilde{\sigma}$ is degenerate with the pions. Their common mass is given implicitly by

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- **Pion condensate grows with chemical potential and decreases with temperature.**

- **Critical temperature for pion condensation increases with chemical potential.**
Analytical expression for the chiral condensate $\bar{\sigma}$ (or $\bar{\rho}$) at $\mu = 0$:

$$\bar{\sigma}^2 = f_\pi^2 - \frac{T^2}{12}$$
Physical point

- $\bar{\sigma}$ (as well as $m^2$) is always nonzero $\Rightarrow$ it is not an order parameter for spontaneous symmetry breaking.
- In the normal phase, $m^2$ solves the implicit equation
  \[ 16\pi^2 f_\pi^2 = 16\pi^2 \left( \frac{H}{m^2} \right)^2 + T^2 J_1(\beta m) - m^2 \log \frac{Q^2}{m^2} - \frac{32\pi^2 m^2}{\lambda} \]
- In the pion-condensation phase, $\bar{\rho}$ solves the implicit equation
  \[ 16\pi^2 f_\pi^2 = 16\pi^2 \left[ \left( \frac{H}{\mu^2} \right)^2 + \bar{\rho}^2 \right] + T^2 J_1(\beta \mu) - \mu^2 \log \frac{Q^2}{\mu^2} - \frac{32\pi^2 \mu^2}{\lambda} \]
- At critical chemical potential for pion condensation, $m^2 = \mu^2$ and $\bar{\rho} = 0$.
- At fixed temperature, pion condensation occurs at chemical potential equal to the renormalized pion mass at $\mu = 0$. 

\[ \text{J. O. Andersen and T. Brauner} \]
Parameter set:

$$\lambda(Q = 100 \text{ MeV}) = 30, \quad f_\pi = 47 \text{ MeV}, \quad H = (104 \text{ MeV})^3$$

Chiral-symmetry-breaking parameter $H$ chosen in order to reproduce the (LO) renormalized pion mass $m_\pi = 138 \text{ MeV}$.

Warringa, hep-ph/0604105 (PhD thesis)
The phase boundary is given by the solution of the implicit equation

\[ 16\pi^2 f_\pi^2 = \frac{16\pi^2 H^2}{\mu^4} + T^2 J_1(\beta \mu) - \mu^2 \log \frac{Q^2}{\mu^2} - \frac{32\pi^2 \mu^2}{\lambda} \]

In the weak-coupling limit, we get an analytic expression for the critical temperature.

\[ T_c^2 = 12 \left( f_\pi^2 + \frac{2\mu^2}{\lambda} \right) \]

At large chemical potential, the effects of explicit chiral symmetry breaking are negligible.
Outline

1. $O(N)$ sigma model and 1PI-1/$N$ expansion
   - $O(N)$ sigma model
   - Auxiliary field technique
   - 1PI-1/$N$ expansion

2. Leading order
   - Sigma and pion condensates
   - Phase diagram

3. Next-to-leading order
   - Renormalization
   - Sigma and pion condensates
   - Phase diagram
NLO effective potential

- NLO expression for the effective potential (pressure):
  \[ V_{NLO} = \int_P \log(P^2 + m^2) - \frac{1}{2} \int_P \log[(P^2 + M^2)^2 + (2\mu^2)^2] - \frac{1}{2} \int_P \log J(P,m) \]

- \( \frac{1}{32\pi^2} J(P,m) = \frac{1}{2} \Pi(P,m) + \frac{1}{\lambda_b} + \frac{\bar{\sigma}^2}{P^2 + m^2} + \frac{\bar{\rho}^2(P^2 + M^2)}{(P^2 + M^2)^2 + (2\mu^2)^2} \)

- The correction due to the finite chemical potential of the free pion gas is “analytically” calculable.

\[ \frac{T^4}{64\pi^2} \left[ K_0^+(\beta m, \beta \mu) + K_0^-(\beta m, \beta \mu) - 2J_0(\beta m) \right], \quad K_0^\pm(\beta m, \beta \mu) = \frac{32}{3T^4} \int_0^\infty dp \frac{p^4}{\omega_p} n(\omega_p \pm \mu) \]

- The correction due to the dynamics of \( \bar{\sigma} \) and \( \bar{\alpha} \) must be evaluated numerically.

- However, at least the divergences may be extracted analytically, which is essential for the renormalization.
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Extraction of divergences

- Expand the log in $V_{\text{NLO}}$ in powers of momentum.

$$
\log J = \log \left( \alpha + \log \frac{\Lambda^2}{p^2} \right) + \frac{2}{p^2} \left( m^2 + \frac{G - 2m^2}{\alpha + \log \frac{\Lambda^2}{p^2}} \right) - \frac{2}{p^4} \left[ 2m^4 + \frac{3m^2(G - \frac{3}{2}m^2)}{\alpha + \log \frac{\Lambda^2}{p^2}} + \frac{(G - 2m^2)^2}{\left( \alpha + \log \frac{\Lambda^2}{p^2} \right)^2} \right] + \text{UV-finite terms}
$$

$$
G = 16\pi^2(\bar{\sigma}^2 + \bar{\rho}^2) + T^2 J_1(\beta m) - m^2 \log \frac{Q^2}{m^2} - \frac{32\pi^2m^2}{\lambda}, \quad \alpha = 1 + \frac{32\pi^2}{\lambda_b}
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- Quartic UV-divergence, independent of $m, \bar{\sigma}, \bar{\rho}$.
- Quadratic UV-divergence.
- Logarithmic UV-divergence. Also IR divergent: This IR divergence is absent in the full expression for $J(P, m)$, but for sake of numerical computation must be conveniently regulated.
Divergent part of the effective potential after integration using momentum cutoff $\Lambda$:

$$V_{NLO}^{div} = \frac{1}{16\pi^2} \left[ (G - 2m^2)\Lambda^2 e^{\alpha \text{li}} \left( \frac{1}{e^\alpha} \right) - m^2 \Lambda^2 + 2m^4 \log \frac{\Lambda^2}{m^2} - 3m^2 (G - \frac{3}{2} m^2) \log \alpha + \frac{(G - 2m^2)^2}{\alpha} \right]$$

The divergences only $T$-independent after using the LO gap equation $G = 16\pi^2 f_\pi^2$! $\Rightarrow$ it was reported that the effective potential is only renormalizable at the minimum.

Andersen, Boer, and Warringa, PRD70 (2004) 116007

The NLO renormalization prescription is then

$$f_\pi^2 = f_{\pi,b}^2 + \frac{\Lambda^2}{16\pi^2} \left\{ 1 + \frac{2}{N} \left[ 1 + 2e^{\alpha \text{li}} \left( \frac{1}{e^\alpha} \right) \right] \right\}$$

$$\frac{32\pi^2}{\lambda} = \frac{32\pi^2}{\lambda_b} + \left( 1 + \frac{8}{N} \right) \log \frac{\Lambda^2}{Q^2}, \quad \beta(\lambda) = \frac{\lambda^2}{16\pi^2} \left( 1 + \frac{8}{N} \right)$$
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Only the LO gap equation for $\langle \alpha \rangle = im^2$ was used!

When this is treated as merely a constraint to eliminate $m^2$ in favor of $\bar{\sigma}, \bar{\rho}$, we get an effective potential of $\bar{\sigma}, \bar{\rho}$ solely, which is renormalizable for any value of the classical fields.

Coleman, Jackiw, and Politzer, PRD10 (1974) 2491

Systematic expansion of the effective potential as a function of $\bar{\sigma}, \bar{\rho}$ then starts with

$$V_{LO}(m_{LO}(\bar{\sigma}, \bar{\rho}), \bar{\sigma}, \bar{\rho}) + \frac{1}{N} V_{NLO}(m_{LO}(\bar{\sigma}, \bar{\rho}), \bar{\sigma}, \bar{\rho}) + \cdots$$

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The chiral condensate increases significantly. Also, the critical temperature for symmetry restoration increases by about 25%.

The parameter $m$ acquires nonzero value even in the symmetry-broken phase. This is no contradiction with the Goldstone theorem, since at NLO $m$ has no longer the interpretation as the pion mass.
Both the chiral condensate and the mass parameter change rather decently $\Rightarrow 1/N$ is a reasonable expansion parameter even for $N = 4$.

- Chiral condensate at $T = 0$ slightly increases $\Rightarrow$ the physical pion mass slightly reduces.
Physical point: chemical potential axis

- The change in the critical chemical potential for pion condensation is in accord with the correction of the physical pion mass.

- Minimum of the NLO effective potential was only found for not-too-large chemical potential. For larger $\mu$ the effective potential seems to acquire nonzero imaginary part.
NLO phase diagram

- So far, only the NLO correction to the phase diagram in the chiral limit.
- The NLO critical temperature stays almost constant up to chemical potential about 75 MeV, where it merges with the LO curve.

Apparently a strong-coupling effect, since the weak-coupling analysis predicts a decrease of critical temperature at NLO.

\[ T_c = \sqrt{\frac{12}{1 + \frac{2}{N}}} f_\pi \quad \text{at } \mu = 0 \]
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\[ T_c = \sqrt{\frac{12}{1 + \frac{2}{N}}} f_\pi \] at \( \mu = 0 \)
To determine the NLO contribution to the pressure, it is only necessary to solve the gap equation at the leading order. We have extended previous calculations to nonzero chemical potential.

We have shown that the NLO effective action in the $1/N$ expansion can be consistently renormalized for all values of the classical field.

We have renormalized and solved the NLO gap equation, and determined the NLO phase diagram (so far, in the chiral limit).

The NLO corrections to the pressure are decent, less than $1/N \Rightarrow 1/N$ is a good expansion parameter even for $N = 4$.

However, the chiral and pion condensates may display rather large deviations from the LO values, especially in the chiral limit.
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