Equivalence of χ RMT and $\epsilon\chi$ PT at non zero chemical potential

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The question

...suppose we have two "different" theories describing the same objects and seeming to give the same results...

...but do they give the very same results? for any possible observable?

Which theories

ε QCD

a strange but "not-extreme" QCD...
Finite volume QCD ($1/\Lambda \ll L \ll 1/m_\pi$), when $V \to \infty$, $m\Sigma V$ and $\mu^2 F_\pi^2 V$ stay finite. Altough it cannot describe full QCD it has many applications (extrapolations of low energy constants from small lattices, sign problem, finite volume corrections...)

Dirac Operator properties in low energy regions may be computed analytically using effective theories

Which theories

$\varepsilon\chi$ PT Gasser,Leutwyler,Smilga

ε QCD

$$\int_{U(N_f)} dU \, e^{-\frac{V\Sigma_0}{2} Tr\left[MU+MU^\dagger\right]} \mathcal{D}et\,[U]^\nu$$

Which theories

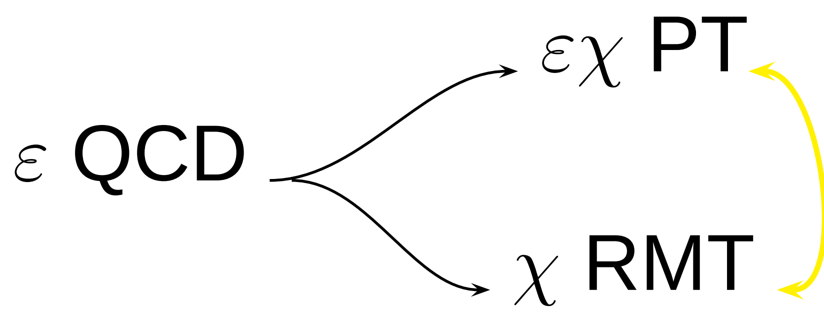
ε QCD → $\varepsilon\chi$ PT

ε QCD → χ RMT Shuryak,Verbaarschot

$$\int dA\, e^{-\alpha N Tr\left[A^\dagger A\right]} \prod_{f}^{N_f} \mathcal{D}et \begin{bmatrix} m_f \mathbf{1}_{N_+} & iA \\ iA^\dagger & m_f \mathbf{1}_{N_-} \end{bmatrix}$$

$N_+ \times N_- = (N_- + \nu) \times N_-$ Random Matrix A

Which theories



Does exist a direct link?

Which theories

$\varepsilon\chi$ PT

ε QCD

χ RMT

Does exist a direct link?

and for $\mu \neq 0$?

$\varepsilon\chi$PT $\overset{??}{\leftrightsquigarrow}$ χRMT

Do they give the very same predictions?

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- $\mu = 0$ $\quad Z_{\varepsilon\chi PT}(M) = Z_{\chi RMT}(M)$

Shuryak,Verbaarschot '93

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- $\mu = 0$ $\quad Z_{\varepsilon\chi PT}(M) = Z_{\chi RMT}(M)$
- $\mu = 0$ $\quad \rho_{\varepsilon\chi PT}(z, M=0) = \rho_{\chi RMT}(z, M=0)$

Damgaard,Osborn,Toublan,Verbaarschot ’99 + Verbaarschot,Zahed ’93

$\varepsilon\chi$PT $\overset{??}{\longleftrightarrow}$ χRMT

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- $\mu = 0$ $\; N_f = 1$, $\rho_{\varepsilon\chi PT}(z, m) = \rho_{\chi RMT}(z, m)$

Damgaard,Osborn,Toublan,Verbaarschot ’99,

Damgaard,Nishigaki,Wilke,Guhr,Wettig,Seif ’98

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Osborn '06

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- $\mu \neq 0$ $Z_{\varepsilon\chi PT}(M) = Z_{\chi RMT}(M)$
- $\mu \neq 0$ quenched $\rho_{\varepsilon\chi PT}(z, z^*) = \rho_{\chi RMT}(z, z^*)$

Splittorff,Verbaarschot '02, Osborn '04

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Why are we interested in this question?

- different universality arguments leading to these theories
- $\varepsilon\chi$PT is more physical, χRMT is solved
- (math) exact map of RMT to the underlying microscopical theory
- this equivalence is accepted, but a proof is still lacking

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The two theories have the very same spectral properties. This result holds for all N_f, masses and chemical potentials.

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D.O. in χ RMT $\quad \stackrel{?}{=} \quad$ D.O. in $\varepsilon\chi$ PT

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$$\begin{array}{ccc} \text{D.O. in } \chi \text{ RMT} & \stackrel{?}{=} & \text{D.O. in } \varepsilon\chi \text{ PT} \\ \uparrow & & \uparrow \\ \text{Z in pq-}\chi\text{RMT} & & \text{Z in pq-}\varepsilon\chi \text{ PT} \end{array}$$

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Resolvent method

k-point correlation function of the e.v. λ_j of an operator D is generate by the expectation value of product of δ function

$$\rho_k(z_1, \ldots, z_k) \sim \left\langle \prod_{\tilde{k}}^{k} \sum_{\lambda_j \in \text{e.v.}} \delta(z_{\tilde{k}} - \lambda_j) \right\rangle$$

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Resolvent method

Hermitian $\mathcal{D}$, $\mu = 0$

use the resolvent method to generate the k-point correlation function

$\Rightarrow$ partially quenched QCD, pq-QCD:

theory with N_f fermions $\rightarrow N_f + k$ fermions, k bosons

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Consider the $(n_b|n_f)$ theory with n_f fermions and n_b bosons, with generic m_i and μ_i

Outline of the proof

Outline: pq-χRMT $\rightarrow$ pq-$\varepsilon\chi$PT

- Ratio of determinants as gaussian integral of two sets of supervectors
- Explicit integration of the RM
- Explicit integration of one set of supervectors
- Apply a theorem (super-bosonisation) to write the remanining supervectors integrations as integral on $\hat{G}L(n_b|n_f)$

Outline of the proof

$$Z^{pq}_{\chi RMT} = \int dA e^{-\alpha N Tr\left[A^\dagger A\right]} \frac{\prod_f^{n_f} \mathcal{D}et \begin{bmatrix} m_f \mathbf{1}_{N_+} & iA \\ iA^\dagger & m_f \mathbf{1}_{N_-} \end{bmatrix}}{\prod_b^{n_b} \mathcal{D}et \begin{bmatrix} m_b \mathbf{1}_{N_+} & iA \\ iA^\dagger & m_b \mathbf{1}_{N_-} \end{bmatrix}}$$

$\mathcal{D}et\left[\cdot\right]$ =gaussian superintegral

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$$\int d(\psi\psi^*\phi\phi^*) e^{-\sum_{g=-n_f}^{n_b} \begin{pmatrix} \psi^*_{g,\alpha} \\ \phi^*_{g,\beta} \end{pmatrix} \begin{pmatrix} m_g \mathbf{1}_{\alpha,\alpha'} & iA_{\alpha,\beta'} \\ iA^\dagger_{\beta,\alpha'} & m_g \mathbf{1}_{\beta,\beta'} \end{pmatrix} \begin{pmatrix} \psi_{g,\alpha'} \\ \phi_{g,\beta'} \end{pmatrix}}$$

Explicit integration of the RM

$$Z^{pq}_{\chi RMT} = \int dA e^{-\alpha N Tr[A^\dagger A]} \frac{\prod_f^{n_f} \mathcal{D}et \begin{bmatrix} m_f \mathbf{1}_{N_+} & iA \\ iA^\dagger & m_f \mathbf{1}_{N_-} \end{bmatrix}}{\prod_b^{n_b} \mathcal{D}et \begin{bmatrix} m_b \mathbf{1}_{N_+} & iA \\ iA^\dagger & m_b \mathbf{1}_{N_-} \end{bmatrix}}$$

$$\int d(\psi \psi^* \phi \phi^*) e^{-\sum_{g=-n_f}^{n_b} \begin{pmatrix} \psi^*_{g,\alpha} \\ \phi^*_{g,\beta} \end{pmatrix} \begin{pmatrix} m_g \mathbf{1}_{\alpha,\alpha'} & iA_{\alpha,\beta'} \\ iA^\dagger_{\beta,\alpha'} & m_g \mathbf{1}_{\beta,\beta'} \end{pmatrix} \begin{pmatrix} \psi_{g,\alpha'} \\ \phi_{g,\beta'} \end{pmatrix}}$$

$$\int dA \, e^{-\alpha N \sum_{\alpha,\beta} A_{\alpha,\beta} A^*_{\alpha,\beta} - iA_{\alpha,\beta} \left(\sum_g \phi^*_{g,\beta} \psi_{g\alpha} \right)^* - iA^*_{\alpha,\beta} \left(\sum_g \phi^*_{g,\beta} \psi_{g\alpha} \right)}$$

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$$= e^{-\frac{1}{\alpha N} Str\left[s_g \cdot \sum_\beta \phi_{g,\beta} \otimes \phi^\dagger_{h,\beta} \cdot s_h \cdot \sum_\alpha \psi_{h,\alpha} \otimes \psi^\dagger_{g,\alpha} \right]}$$

$$Z_{pq} \propto \int d(\psi\psi^*\phi\phi^*)\, e^{-Str\left[m_g\cdot\sum_\alpha \psi_{g,\alpha}\otimes\psi^\dagger_{g,\alpha}+m_g\cdot\sum_\beta \phi_{g,\beta}\otimes\phi^\dagger_{g,\beta}\right]}$$

$$e^{-\frac{1}{\alpha N}Str\left[\sum_\beta \phi_{g,\beta}\otimes\phi^\dagger_{h,\beta}\cdot\sum_\alpha \psi_{h,\alpha}\otimes\psi^\dagger_{g,\alpha}\right]}$$

Integrate of 1 set of supervectors

$$Z_{pq} \propto \int d(\psi\psi^*\phi\phi^*)\, e^{-Str\left[m_g \cdot \sum_\alpha \psi_{g,\alpha}\otimes\psi^\dagger_{g,\alpha} + m_g \cdot \sum_\beta \phi_{g,\beta}\otimes\phi^\dagger_{g,\beta}\right]}$$

$$e^{-\frac{1}{\alpha N} Str\left[\sum_\beta \phi_{g,\beta}\otimes\phi^\dagger_{h,\beta} \cdot \sum_\alpha \psi_{h,\alpha}\otimes\psi^\dagger_{g,\alpha}\right]}$$

use that:

$$\int d(\gamma^*\gamma)\, e^{-Str\left[A \cdot \gamma\otimes\gamma^\dagger\right]} = \frac{1}{\mathcal{S}det\,[A]}$$

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$$e^{-\frac{1}{\alpha N}Str\left[\sum_\beta \phi_{g,\beta}\otimes\phi^\dagger_{h,\beta}\cdot\sum_\alpha \psi_{h,\alpha}\otimes\psi^\dagger_{g,\alpha}\right]}$$

$$Z_{pq} \propto \int d(\psi\psi^*)\, e^{-Str\left[m_g\cdot\sum_\alpha \psi_{g,\alpha}\otimes\psi^\dagger_{g,\alpha}\right]}$$

$$\mathcal{S}det\left[\frac{1}{N}M_{gh}+\frac{1}{\alpha N}\sum_\alpha \psi_{g,\alpha}\otimes\psi^\dagger_{h,\alpha}\right]^{-N_-}$$

A theorem

Super-bosonisation theorem:

$$\int d^N\psi\psi^\dagger f\left(\sum_k \psi_k \otimes \psi_k^\dagger\right) = \int_{\hat{G}L(n_b|n_f)} \mu_H(U)\, \mathcal{S}det\left[U\right]^N \; f(U)$$

$\mu_H(U)$ is the Haar measure on $\hat{G}L(n_b|n_f)$

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Independent proof to recent result by Zirnbauer et al.

Our idea:embed $\sum_k \psi_k \otimes \psi_k^\dagger$ in the manifold of $H = H^\dagger$,define a δ function and perform the $\psi, \psi^\dagger$ integration; find a relation between $H = H^\dagger$ and $\hat{G}L(n_b|n_f)$

Outline of the proof (a bit technical)

Apply the theorem

$$Z_{pq} \propto \int d(\psi\psi^*)\, e^{-Str\left[M_{gg}\cdot\frac{1}{N}\sum_\alpha \psi_{g,\alpha}\otimes\psi^\dagger_{g,\alpha}\right]}$$

$$\mathcal{S}det\left[\frac{1}{N}M_{gh} + \frac{1}{\alpha N}\sum_\alpha \psi_{g,\alpha}\otimes\psi^\dagger_{h,\alpha}\right]^{-N_-}$$

Apply the theorem

$$Z_{pq} \propto \int_{\hat{G}L(n_b|n_f)} \mu_H(U)\mathcal{S}det\,[U]^{N_-+\nu}\, e^{-Str[M_{gg}\cdot U]}$$

$$\mathcal{S}det\left[\frac{1}{N}M_{gh}+\frac{1}{\alpha}U\right]^{-N_-}$$

$$Z_{pq} \propto \int_{\hat{GL}(n_b|n_f)} \mu_H(U) \mathcal{S}det\,[U]^{N_- + \nu}\, e^{-Str[M_{gg} \cdot U]}$$

$$\mathcal{S}det\left[\frac{1}{N} M_{gh} + \frac{1}{\alpha} U\right]^{-N_-}$$

$$\propto \int_{\hat{GL}(n_b|n_f)} \mu_H(U) \mathcal{S}det\,[U]^{\nu}\, e^{-Str[M \cdot U]}$$

$$\mathcal{S}det\left[1 + \frac{\alpha}{N} M \cdot U^{-1}\right]^{-N_-}$$

perform the $N \to \infty$ limit

$$Z_{pq} \propto \int_{\hat{GL}(n_b|n_f)} \mu_H(U) \mathcal{S}det\,[U]^{N_-+\nu}\, e^{-Str[M_{gg}\cdot U]}$$

$$\mathcal{S}det \left[\frac{1}{N} M_{gh} + \frac{1}{\alpha} U \right]^{-N_-}$$

$$\propto \int_{\hat{GL}(n_b|n_f)} \mu_H(U) \mathcal{S}det\,[U]^{\nu}\, e^{-Str[M\cdot U]}$$

$$e^{-\frac{\alpha}{2} Str\left[M\cdot U^{-1}\right]}$$

The desired result for $\mu = 0$

$$Z_{pq} \propto \int_{\hat{G}L(n_b|n_f)} \mu_H(U) \mathcal{S}det\,[U]^{N_- + \nu}\, e^{-Str[M_{gg} \cdot U]}$$

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$$e^{-\frac{\alpha}{2} Str\left[M \cdot U^{-1}\right]}$$

$$Z_{pq} \propto \int_{\hat{G}L(n_b|n_f)} \mu_H(U) \mathcal{S}det\,[U]^{\nu}\, e^{-\sqrt{\frac{\alpha}{2}} Str\left[M \cdot U + M \cdot U^{-1}\right]}$$

The proof at $\mu \neq 0$

$$Z^{pq}_{\chi RMT} = \int dAdBe^{-\alpha NTr\left[A^\dagger A + B^\dagger B\right]} \frac{\prod_f^{n_f} \mathcal{D}et \begin{bmatrix} m_f \mathbf{1}_{N_+} & iA + \mu_f B \\ iA^\dagger + \mu_f B^\dagger & m_f \mathbf{1}_{N_-} \end{bmatrix}}{\prod_b^{n_b} \mathcal{D}et \begin{bmatrix} m_b \mathbf{1}_{N_+} & iA + \mu_f B \\ iA^\dagger + \mu_f B^\dagger & m_b \mathbf{1}_{N_-} \end{bmatrix}}$$

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the same proof: determinant as a gaussian superintegration, integration of RMTs A and B, integration of $\phi, \phi^\dagger$, use the theorem, $N \to \infty$ limit

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the same proof: determinant as a gaussian superintegration, integration of RMTs A and B, integration of $\phi, \phi^\dagger$, use the theorem, $N \to \infty$ limit

$$Z_{pq,\mu\neq 0} \propto \int_{\hat{GL}(n_b|n_f)} \mu_H(a)\, \mathcal{S}det\, [a]^\nu\, e^{-\sqrt{\frac{\alpha}{2}} Str\left[M \cdot a + M \cdot a^{-1}\right] + Str\left[BaBa^{-1}\right]}$$

Conclusion

- We have proved the equivalence of the spectral properties of the $N \to \infty$ limit of χ-RMT and the $\varepsilon - \chi$ PT. This equivalence holds for any correlation function and for arbitrary chemical potential (zero,real barionic,immaginary isospin or arbitrary complex).
- *super-bosonisation* theorem (express integrals over $U(n), Gl(n)/U(n)$ and $\hat{G}l(n_b|n_f)$ in terms of integrals of vectors or supervectors and viceversa)

Outline of the proof of the theorem

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M manifold of super-Hermitian matrices:

$$= \int_{F=F^\dagger} dF\, e^{iF\cdot\left(\sum_k \psi_k\otimes\psi_k^\dagger - a\right)} e^{-\varepsilon Str\left[\sum_k \psi_k\otimes\psi_k^\dagger\right]}$$

Outline of the proof of the theorem

$$F = \begin{pmatrix} F_1 & \Phi \\ \Phi^\dagger & F_2 \end{pmatrix}, a = \begin{pmatrix} a_1 & \alpha \\ \alpha^\dagger & a_2 \end{pmatrix}$$

$$\int_{F=F^\dagger} dF \frac{\mathcal{D}et\left[F_1 - i\varepsilon\right]^{-N}}{\mathcal{D}et\left[F_2 - i\varepsilon - \Phi^\dagger(F_1 - i\varepsilon)^{-1}\Phi\right]^{-N}} e^{-iF\cdot a}$$

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$$\int_{\hat{GL}(n_b|n_f)} \mu_H(a)\, e^{-\mu Str[a]} \mathcal{S}det\left[a\right]^N$$