Non-Linear Theory of Particle Acceleration at Astrophysical Shocks

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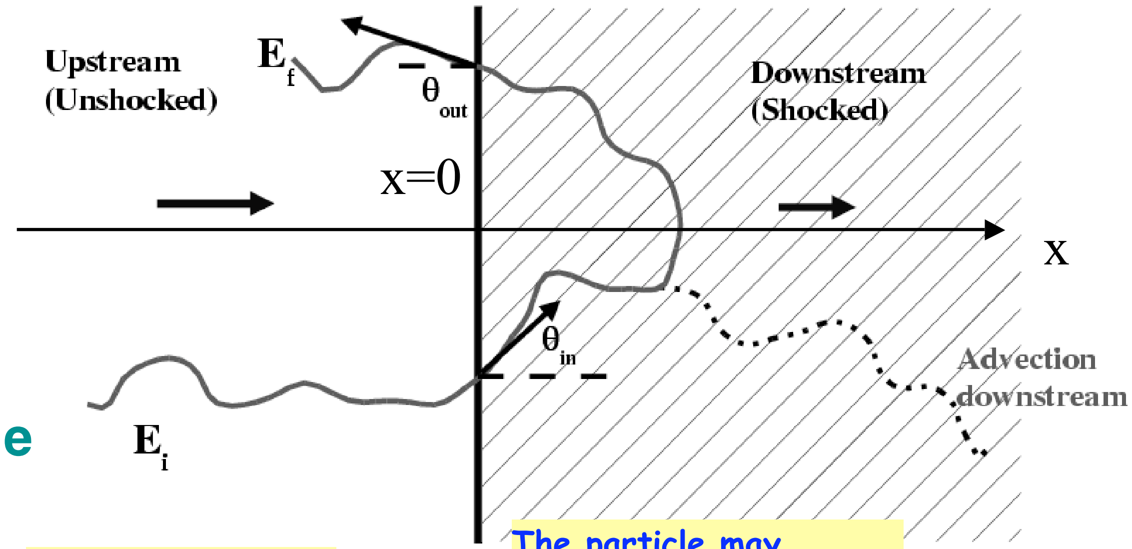
First Order Fermi Acceleration: a Primer



Test Particle

The particle is always advected back to the shock

The particle may either diffuse back to the shock or be advected downstream

$$\int d\mu \; P_u(\mu_0, \mu)\, d\mu = 1$$

[redacted]

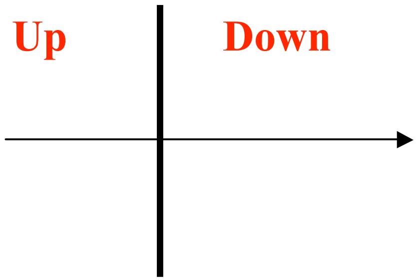
Return Probability from UP=1

Return Probability from DOWN<1

$$N(E) = N_0 \left(\frac{p}{p_0}\right)^{-\gamma-1} \qquad \gamma = -\frac{\log P}{\log G} \approx \frac{3}{r-1}$$

P	Total Return Probability from DOWN
G	Fractional Energy Gain per cycle
r	Compression factor at the shock

The Return Probability and Energy Gain for Non-Relativistic Shocks



At zero order the distribution of (relativistic) particles downstream is isotropic: $f(\mu)=f_0$

Return Probability = Escaping Flux/Entering Flux

$$\Phi_{out} = -\int f_0 (u_2+\mu)\, d\mu = \frac{1}{2}(1-u_2)^2$$

$$\Phi_{i} = \int f_0 (u_2+\mu_0)\, d\mu_0 = \frac{1}{2}(1+u_2)^2$$

$$P_{return} = \frac{(1-u_2)^2}{(1+u_2)^2} \approx 1-4u_2$$

Newtonian Limit

Close to unity for $u_2 \ll 1$!

The extrapolation of this equation to the relativistic case would give a return probability tending to zero!
The problem is that in the relativistic case the assumption of isotropy of the function f loses its validity.

$$G = \frac{\Delta E}{E} = \frac{4}{3}(u_1-u_2)$$

SPECTRAL SLOPE

$$\gamma = -\frac{\log P_{return}}{\log G} = \frac{\log(1-4u_2)}{\frac{4}{3}(u_1-u_2)} \approx \frac{3}{r-1}$$

The Diffusion-Convection Equation: A more formal approach

ADIABATIC COMPRESSION
OR SHOCK COMPRESSION

$$\frac{\partial f}{\partial t} = \frac{\partial}{\partial x}\left[D\frac{\partial f}{\partial x}\right] - u\frac{\partial f}{\partial x} + \frac{1}{3}\frac{du}{dx}\,p\frac{\partial f}{\partial p} + Q(x,p,t)$$

INJECTION
TERM

ADVECTION
WITH THE
FLUID

SPATIAL DIFFUSION

$$f_0(p) = \frac{3u_1}{u_1 - u_2}\frac{N_{inj}}{4\pi p_{inj}^2}\left(\frac{p}{p_{inj}}\right)^{-\frac{3u_1}{u_1 - u_2}}$$

The solution is a power law in momentum

The slope depends ONLY on the compression ratio (not on the diffusion coef.)

Injection momentum and efficiency are free param.

The Need for a Non-Linear Theory

- The relatively Large Efficiency may break the Test Particle Approximation…What happens then? Cosmic Ray Modified Shock Waves
- Non Linear effects must be invoked to enhance the acceleration efficiency (problem with Emax)

Self-Generation of Magnetic Field and Magnetic Scattering

Going Non Linear: Part I

Particle Acceleration in the

Non Linear Regime: Shock Modification

Why Did We think About This?

- Divergent Energy Spectrum

$$E_{CR} = \int dE\,E\,N(E) \quad \ln\left(E_{max}/E_{min}\right)$$

At Fixed energy crossing the shock front _u^2 tand at fixed efficiency of acceleration there are values of Pmax for which the integral exceeds _u^2 (absurd!)

- If the few highest energy particles that escape from upstream carry enough energy, the shock becomes dissipative, therefore more compressive

- If Enough Energy is channelled to CRs then the adiabatic index changes from 5/3 to 4/3. Again this enhances the Shock Compressibility and thereby the Modification

Approaches to Particle Acceleration at Modified Shocks

- **Two-Fluid Models**

  The background plasma and the CRs are treated as two separate fluids. These thermodynamical Models do not provide any info on the Particle Spectrum

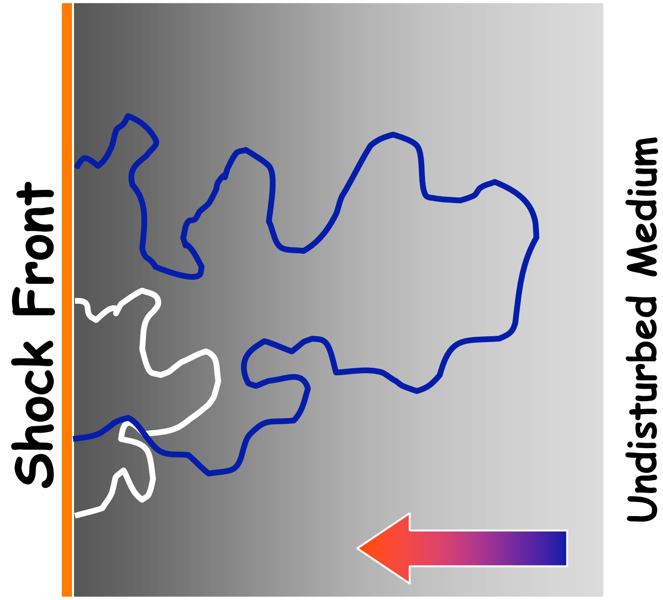
- **Kinetic Approaches**

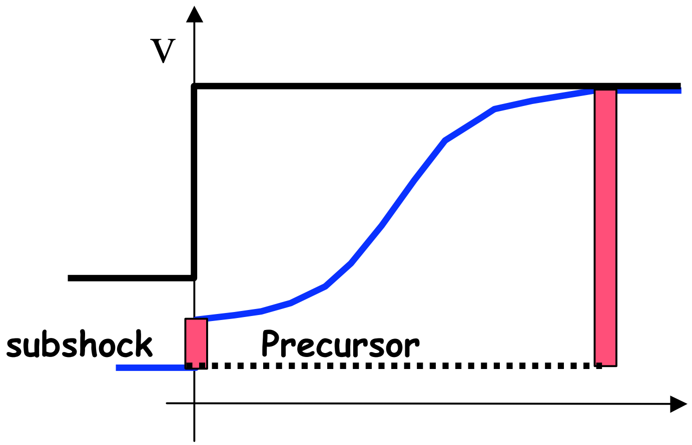
  The exact transport equation for CRs and the coservation eqs for the plasma are solved. These Models provide everything and contain the Two-fluid models

- **Numerical and Monte Carlo Approaches**

  Equations are solved with numerical integrators. Particles are shot at the shock and followed while they diffuse and modify the shock

The Basic Physics of Modified Shocks





$$\tilde{n}(x)u(x) = \tilde{n}_0 \, u_0$$

Conservation of Mass

$$\tilde{n}_0 u_0^2 + P_{g,0} = \tilde{n}(x)u(x)^2 + P_g(x) + P_{CR}(x)$$

Conservation of Momentum

[redacted]

Equation of Diffusion Convection for the Accelerated Particles

Main Predictions of Particle Acceleration at Cosmic Ray Modified Shocks

- Formation of a Precursor in the Upstream plasma
- The Total Compression Factor may well exceed 4. The Compression factor at the subshock is <4
- Energy Conservation implies that the Shock is less efficient in heating the gas downstream
- The Precursor, together with Diffusion Coefficient increasing with p-> NON POWER LAW SPECTRA!!! Softer at low energy and harder at high energy

Exact Solution for Particle Acceleration at Modified Shocks for Arbitrary Diffusion Coefficients

Amato & Blasi (2005)

Basic Equations

$$\rho(x)u(x) = \rho_0 u_0$$

$$\rho(x)u(x)^2 + P_g(x) + P_c(x) = \rho_0 u_0^2 + P_{g,0}$$

$$\frac{\partial f(p,x)}{\partial t} = \frac{\partial}{\partial x}\left[D(p,x)\frac{\partial f(p,x)}{\partial x}\right] - u(x)\frac{\partial f(p,x)}{\partial x} + \frac{1}{3}\frac{du}{dx}p\frac{\partial f(p,x)}{\partial p} + Q(x,p,t)$$

$$P_c(x) = \frac{4\pi}{3}\int dp\, p^3 v(p) f(p,x)$$

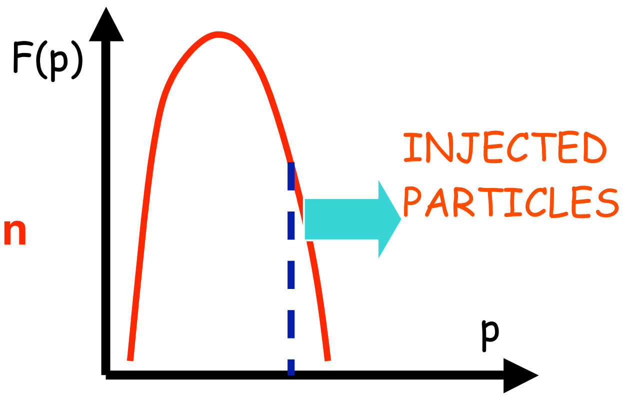
$$f(x,p) = f_0(p)\exp\left[-\frac{q(p)}{3}\int dx' \frac{u(x')}{D(x',p)}\right] \qquad q(p) = -\frac{d\ln f_0}{d\ln p}$$

$$f_0(p) = \left(\frac{3R_{tot}}{R_{tot}U(p)-1}\right)\left(\frac{\eta n_0}{4\pi p_{inj}^3}\right)\exp\left\{-\int \frac{dp'}{p'}\frac{3R_{tot}U(p')}{R_{tot}U(p')-1}\right\}$$

DISTRIBUTION FUNCTION AT THE SHOCK

$$\eta = \frac{4}{3\pi^{1/2}}\left(R_{sb}-1\right)\xi^{3}e^{-\xi^{2}}$$

Recipe for Injection



It is useful to introduce the equations in dimensionless form:

$$\xi_c(x) = 1 + \frac{1}{\gamma M_0^2} - U(x) - \frac{1}{\gamma M_0^2}U(x)^{-\gamma} \qquad \xi_c(x) = \frac{P_c(x)}{\rho_0 u_0^2}$$

$$\xi_c(x) = \frac{4\pi}{3\rho_0 u_0^2}\int dp\, p^3 v(p) f_0(p) \exp\left[-\int dx' \frac{U(x')}{x_p(x')}\right] \qquad x_p(x) = \frac{3D(p,x)}{q(p)u_0}$$

One should not forget that the solution still depends on f_0 which in turn depends on the function

$$U(p) = \int dx\, U(x)^2 \frac{1}{x_p(x)} \exp\left[-\int dx' \frac{U(x')}{x_p(x')}\right]$$

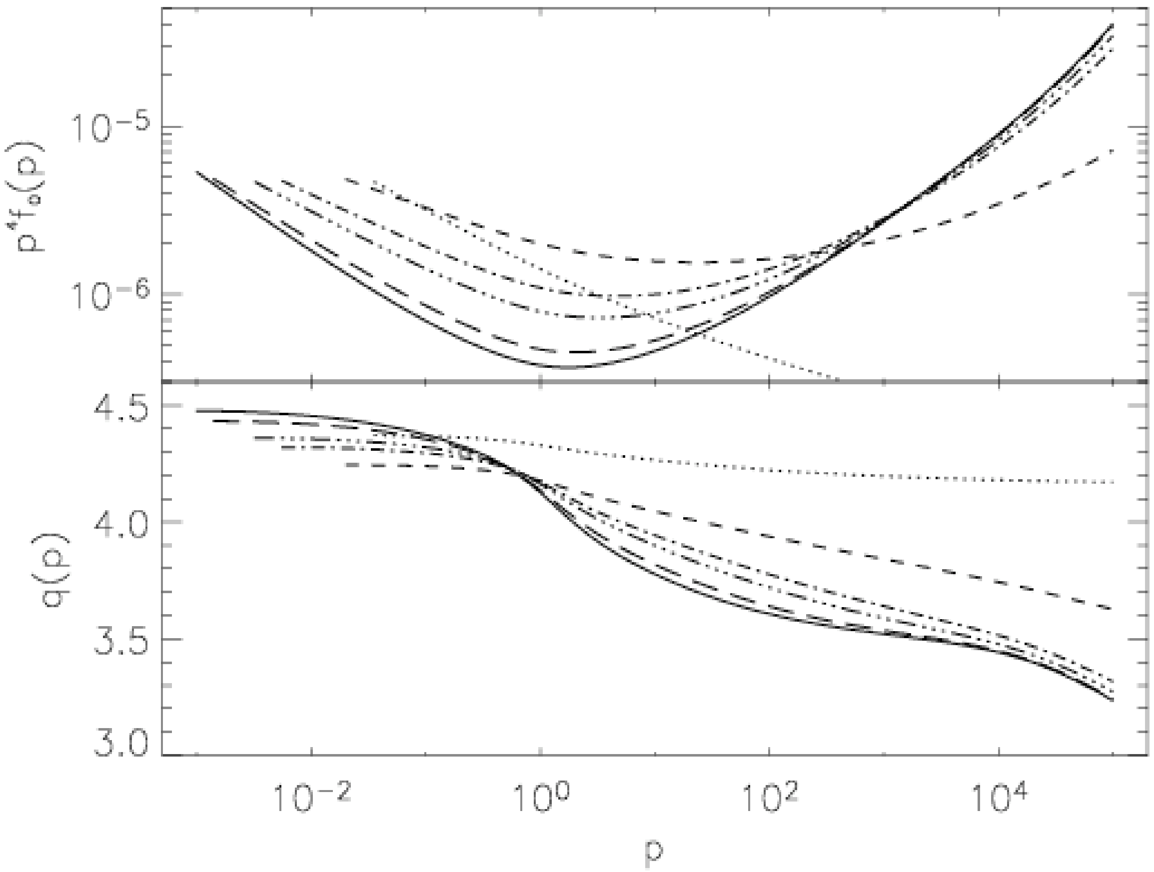
Differentiating with respect to x we get ...

$$\frac{d\xi_c}{dx} = -\lambda(x)\,\xi_c(x)\,U(x)$$

$$\lambda(x) = \frac{\int dp \quad p^3 \frac{1}{x_p(x)} v(p) f_0(p) \exp\left[-\int dx' \frac{U(x')}{x_p(x')}\right]}{\int dp \quad p^3 v(p) f_0(p) \exp\left[-\int dx' \frac{U(x')}{x_p(x')}\right]},$$

The function $\xi_c(x)$ has always the right boundary conditions at the shock and at infinity but ONLY for the right solution the pressure at the shock is that obtained with the f_0 that is obtained iteratively (NON LINEARITY)

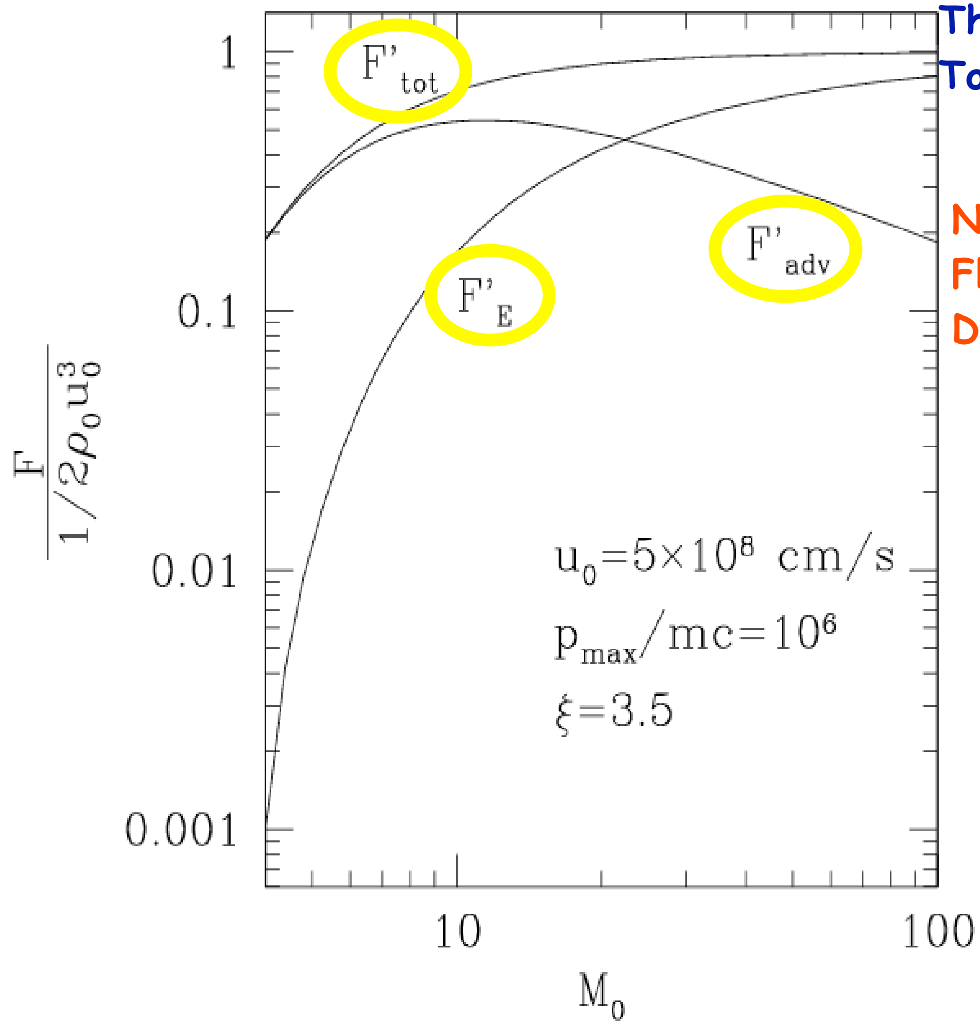
Spectra at Modified Shocks



Very Flat Spectra at high energy

Amato and PB (2005)

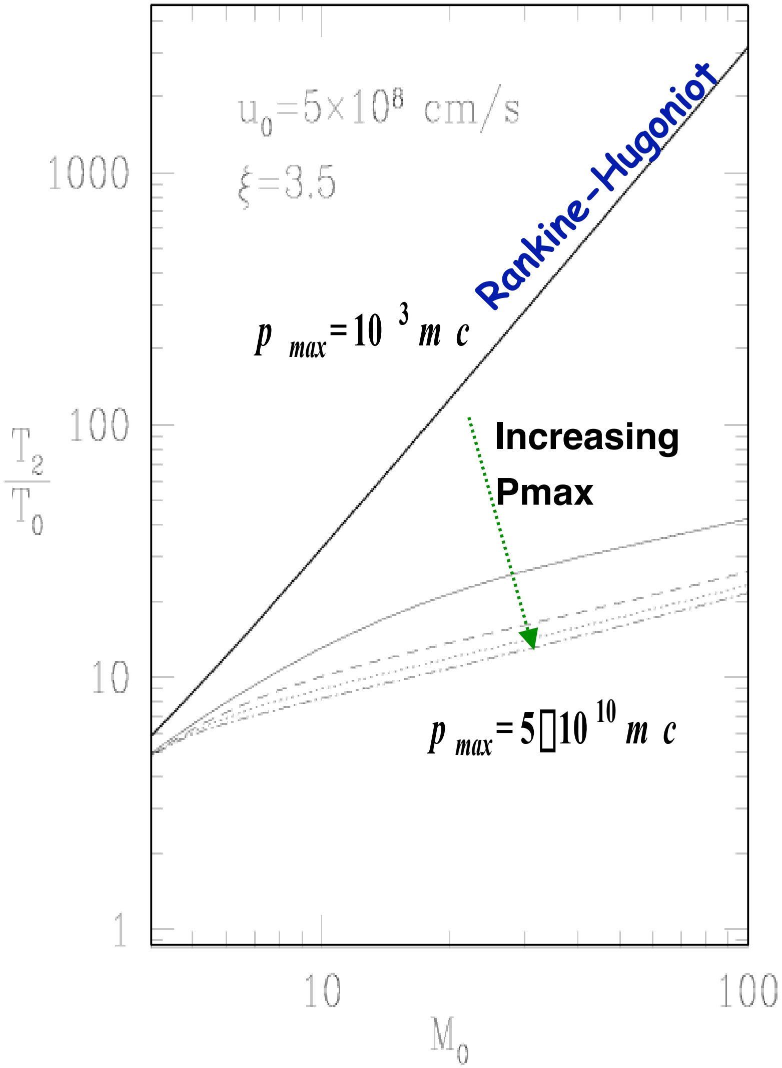
Efficiency of Acceleration (PB, Gabici & Vannoni (2005))

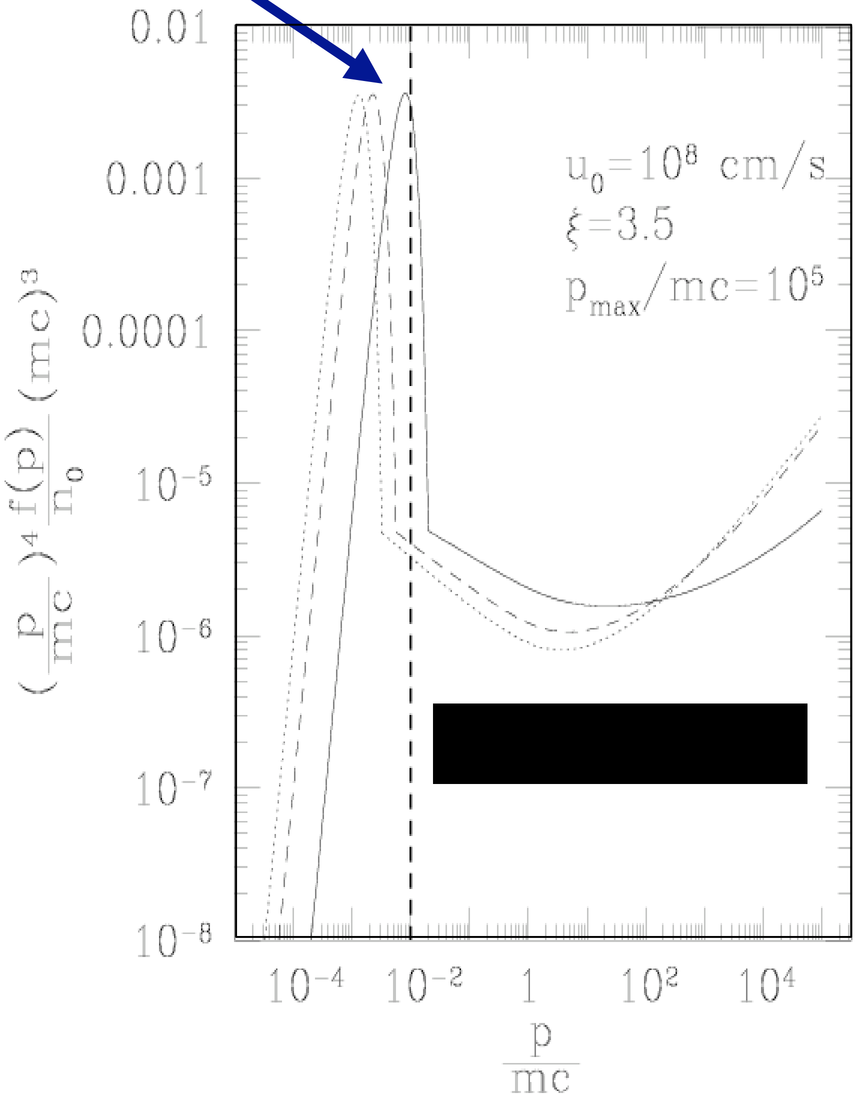


This escapes out To UPSTREAM

Note that only this Flux ends up DOWNSTREAM!!!

Suppression of Gas Heating





PB, Gabici & Vannoni (2005)

The suppressed heating might have already been detected (Hughes, Rakowski & Decourchelle (2000))

Summary of Results on Efficiency of Non Linear Shock Acceleration

Mach number M_0	R_{sub}	R_{tot}	CR frac	P_{inj}/mc	[illegible]
4	3.19	3.57	0.1	0.035	$3.4\ 10^{-4}$
10	3.413	6.57	0.47	0.02	$3.7\ 10^{-4}$
50	3.27	23.18	0.85	0.005	$3.5\ 10^{-4}$
100	3.21	39.76	0.91	0.0032	$3.4\ 10^{-4}$
300	3.19	91.06	0.96	0.0014	$3.4\ 10^{-4}$
500	3.29	129.57	0.97	0.001	$3.4\ 10^{-4}$

Amato & PB (2005)

Be Aware that the Damping
of waves to the thermal plasma
was neglected (on purpose) here!
When this effect is introduced
the efficiencies clearly decrease BUT still remain quite large

Going Non Linear: Part II

Coping with the Self-Generation of Magnetic field by the Accelerated Particles

The Classical Bell (1978) - Lagage-Cesarsky (1983) Approach

Basic Assumptions:

1. The Spectrum is a power law
2. The pressure contributed by CR's is relatively small
3. All Accelerated particles are protons

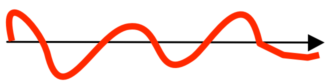
The basic physics is in the so-called streaming instability:

of particles that propagates in a plasma is forced to move at speed smaller or equal to the Alfven speed, due to the excition of Alfven waves in the medium.

Pitch angle scattering and Spatial Diffusion

The Alfven waves can be imagined as small perturbations on top of a background B-field

The equation of motion of a particle in this field is



$$\vec{B} = \vec{B}_0 + \vec{B}_1$$

$$\frac{d\vec{p}}{dt} = \frac{Ze}{c}\vec{v} \times \left(\vec{B}_0 + \vec{B}_1\right)$$

In the reference frame of the waves, the momentum of the particle remains unchanged in module but changes in direction due to the perturbation:

$$\frac{d\mu}{dt} = \frac{Zev\left(1-\mu^2\right)^{1/2} B_1}{pc}\cos\left[\left(\Omega - kv\mu\right)t + \psi\right] \qquad \Omega = ZeB_0 / mc\gamma$$

$$D(p) = \frac{1}{3}v\lambda \approx \frac{1}{3}\frac{c r_L(p)}{F(p)}$$

$$F(p(k)) = k(\delta B / B)^2$$

The Diffusion coeff reduces
To the Bohm Diffusion for
Strong Turbulence F(p)~1

Maximum Energy *a la* Lagage-Cesarsky

In the LC approach the lowest diffusion coefficient, namely the highest energy, can be achieved when F(p)~1 and the diffusion coefficient is Bohm-like.

$$D(p) = \frac{1}{3} v \lambda \approx \frac{1}{3} \frac{c r_L(p)}{F(p)}$$

$$\tau_{acc} \approx \frac{D(E)}{u_{shock}^2} = 3.3\,10^6 E_{GeV} B_{\mu}^{-1} u_{1000}^{-2} \quad sec$$

For a life-time of the source of the order of 1000 yr, we easily get

E_{max} ~ 10^{4-5} GeV

We recall that the knee in the CR spectrum is at **10^6 GeV** and the ankle at **~3 10^9 GeV**. The problem of accelerating CR's to *useful* energies remains...

BUT what generates the necessary turbulence anyway?

Wave growth **HERE IS THE CRUCIAL PART!**

$$\frac{\partial F}{\partial t} + u \frac{\partial F}{\partial x} = \sigma F - \Gamma F$$

Wave damping

Bell 1978

Standard calculation of the Streaming Instability (Achterberg 1983)

$$\frac{c^2 k^2}{\omega^2} = 1 + \chi^{R,L}$$

$$\chi^{R,L} = \frac{4\pi^2 e^2}{\omega} \int dp \int d\mu \, \frac{p^2 \left(1 - \mu^2\right) v(p)}{\omega \pm \Omega' - v \mu k} \left[\frac{\partial f}{\partial p} + \frac{1}{p} \left(\frac{kv}{\omega} - \mu \right) \frac{\partial f}{\partial \mu} \right]$$

There is a mode with an imaginary part of the frequency: CR's excite Alfven Waves resonantly and the growth rate is found to be:

$$\Gamma = \mathrm{v_A} \frac{\partial P_{CR}}{\partial z}$$

Maximum Level of Turbulent Self- Generated Field

$$\frac{\partial F}{\partial t} + u\frac{\partial F}{\partial x} = \acute{o}F$$

Stationarity

$$\mathbf{u}\frac{\partial \mathbf{F}}{\partial \mathbf{z}} = \mathbf{v}_{\mathbf{A}}\frac{\partial \mathbf{P}_{\mathbf{CR}}}{\partial \mathbf{z}}$$

Integrating

$$\frac{\Box B^2}{B_0^2} = 2M_A\frac{P_{CR}}{\Box u^2} >> 1$$

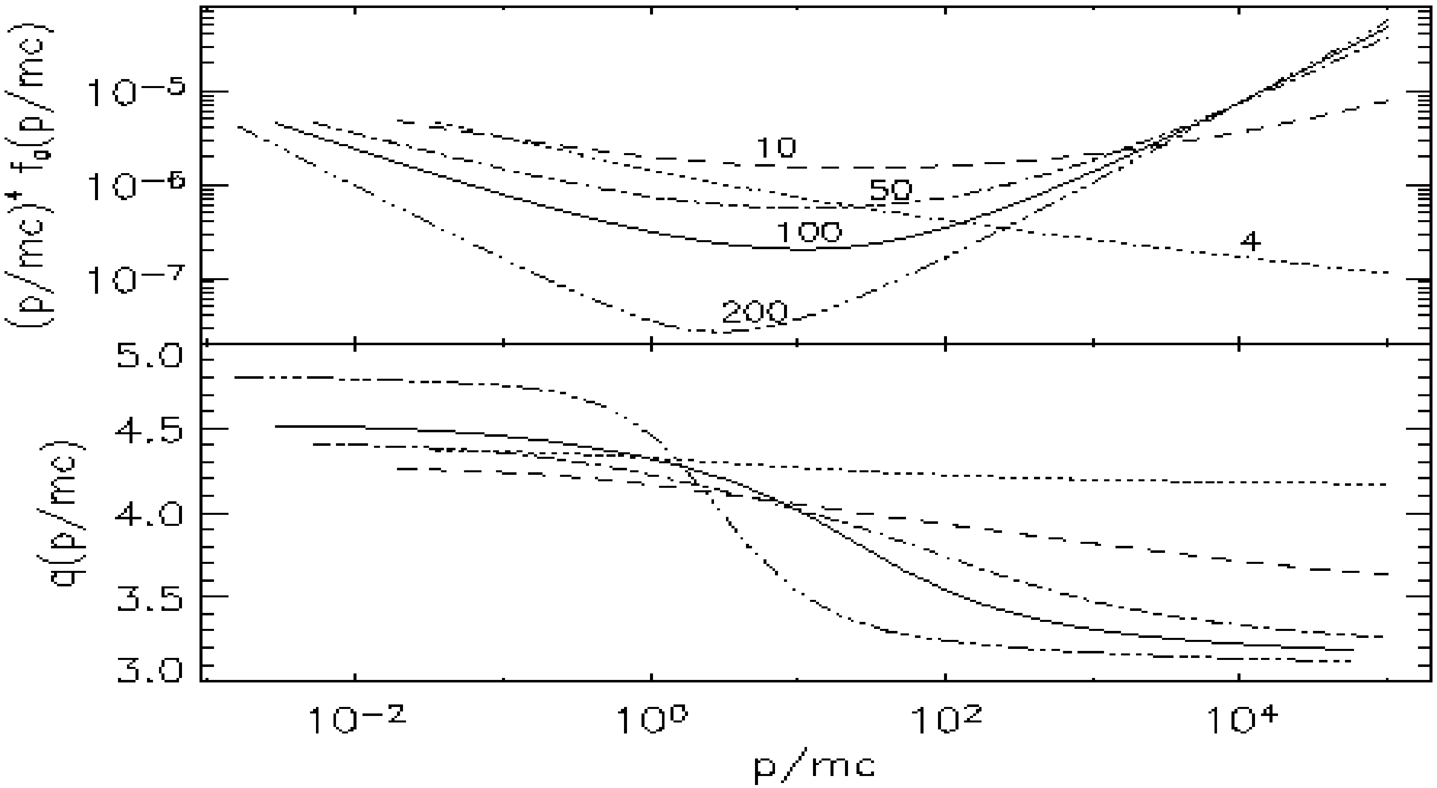
Breaking of Linear Theory…

For typical parameters of a SNR one has _B/B~20.

Non Linear DSA with Self-Generated Alfvenic turbulence (Amato & PB 2006)

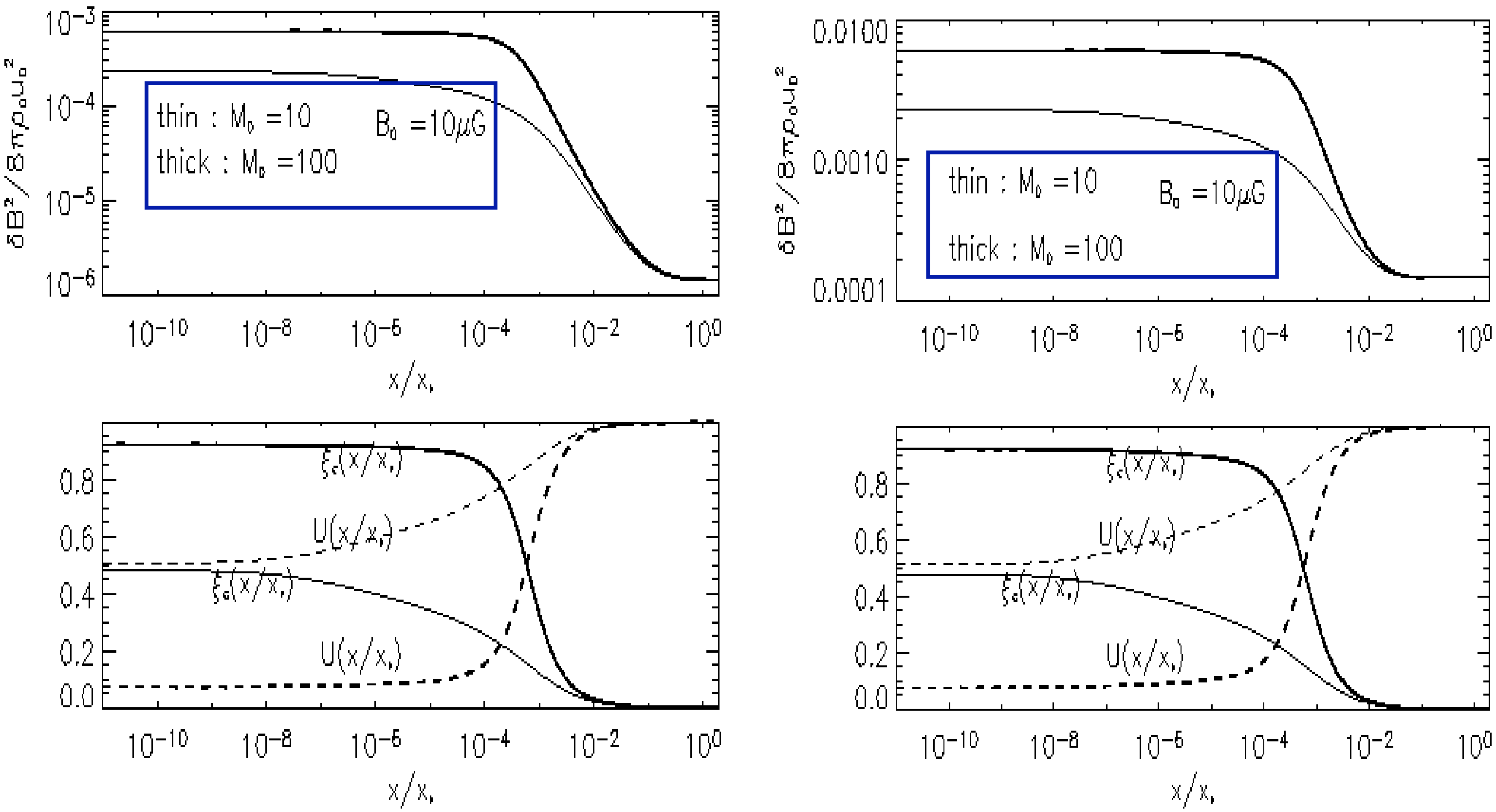
- We Generalized the previous formalism to include the Precursor!
- We Solved the Equations for a CR Modified Shock together with the eq. for the self-generated Waves
- We have for the first time a Diffusion Coefficient as an output of the calculation

Spectra of Accelerated Particles and Slopes as functions of momentum



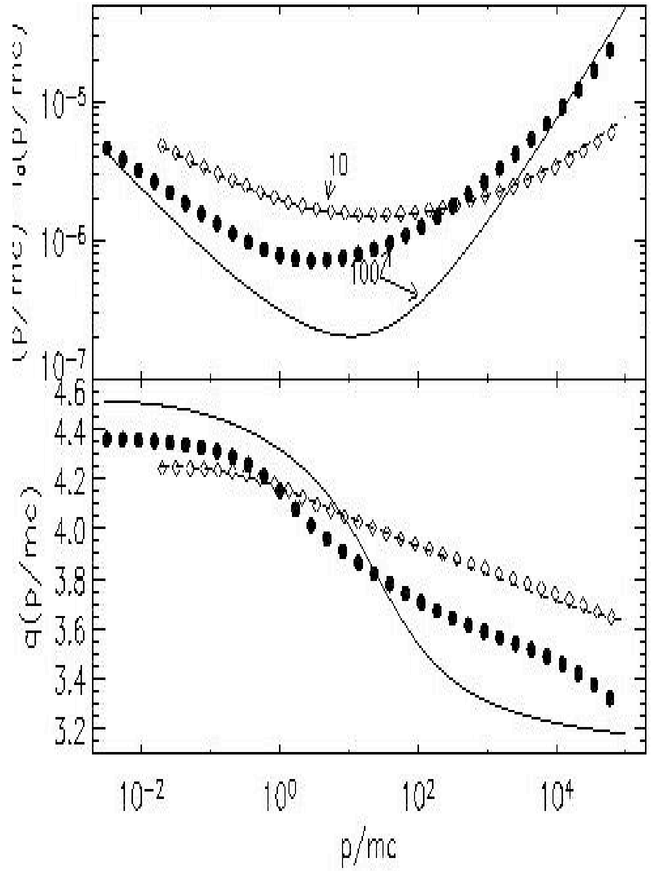
Magnetic and CR Energy as functions of the Distance from the Shock Front

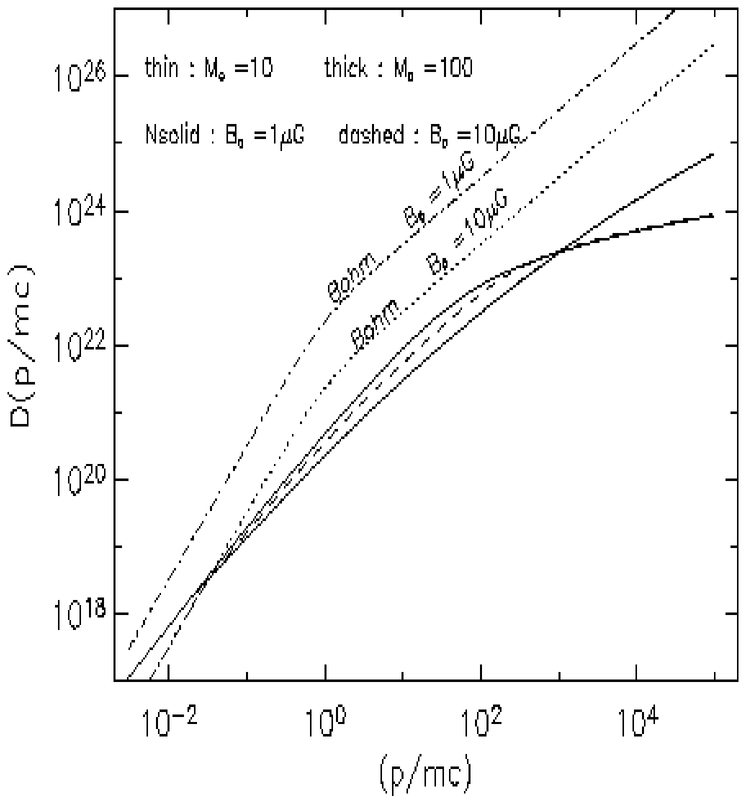
Amato & PB 2006



Super-Bohm Diffusion

Amato & PB 2006





Diffusion Coefficient

The role of turbulent heating

i) Alfven heating

ii) Acoustic instability

Non linear particle acceleration at shock waves 15

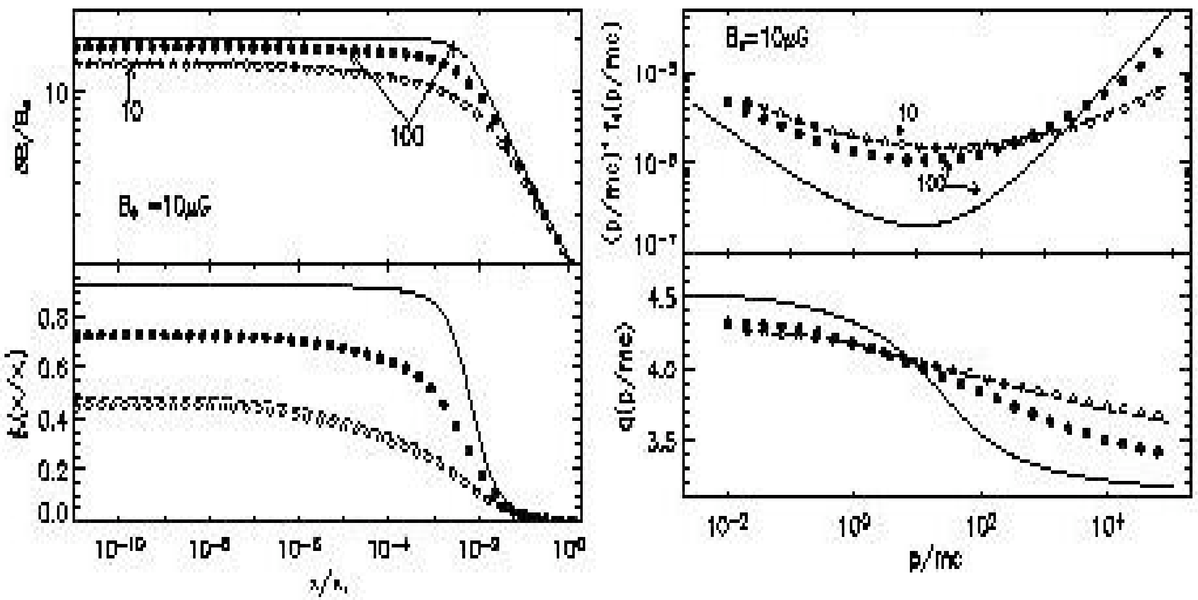


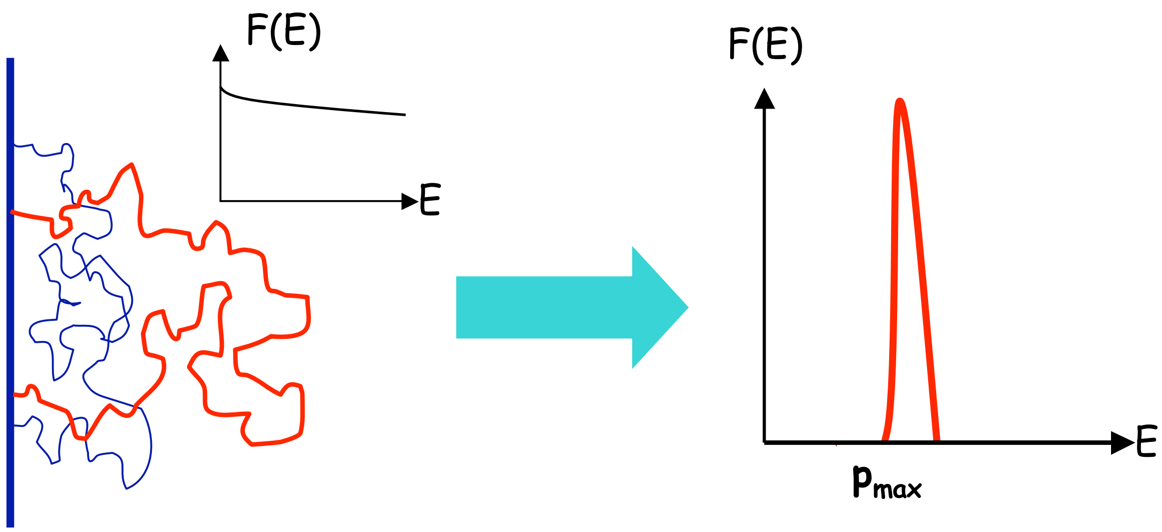
Figure 8. The plots on the left show: in the upper panel, the ratio between the turbulent and background magnetic field as a function of space for two different values of the Mach number (10 and 100), with and without inclusion of the turbulent heating; in the lower panel the corresponding normalized cosmic ray pressure. The plots on the right show the particles' spectrum and slope in the same cases. The continuous curves correspond to cases when the turbulent heating is not taken into account: dashed for $M_0 = 10$ and solid for $M_0 = 100$. The symbols correspond to cases including the turbulen heating: diamonds for $M_0 = 10$ and filled circles for $M_0 = 100$.

How do we look for NL Effects in DSA?

- Curvature in the radiation spectra (electrons in the field of protons)
  - (indications of this in the IR-radio spectra of SNRs by Reynolds)
- Amplification in the magnetic field at the shock (seen in Chandra observations of the rims of SNRs shocks)
- Heat Suppression downstream (detection claimed by Hughes, Rakowski & Decourchelle 2000)
- All these elements are suggestive of very efficient CR acceleration in SNRs shocks. BUT similar effects may be expected in all accelerators with shock fronts

A few notes on NLDSA

- The spectrum "observed" at the source through non-thermal radiation may not be the spectrum of CR's
- The spectrum at the source is most likely concave though a convolution with $p_{max}(t)$ has never been carried out
- The spectrum seen outside (upstream infinity) is peculiar of NL-DSA:



CONCLUSIONS

- Particle Acceleration at shocks occurs in the non-linear regime: these are NOT just corrections, but rather the reason why the mechanism works in the first place

- Concave spectra, heating suppression and amplified magnetic fields are the main symptoms of NL-DSA

- More work is needed to relate more strictly these complex calculations to the phenomenology of CR accelerators

- The future developments will have to deal with the crucial issue of magnetic field amplification in the fully non-linear regime