BF theory for 2+1 topological states of matter

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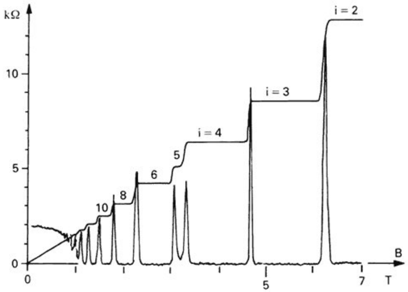
Outline

- Topological states of matter: an experimental overview
- Effective Field theory for QHE: Chern-Simons theory
- Time reversal invariant states of matter: Abelian and non-Abelian BF model
- BF model with boundary through the Symanzik's approach

Integer quantum Hall effect



- Discovered by K. Von Klitzing in 1980.
- $2d$ electron gas, $T = 4K$ and $B = 10T$.
- Plateaus observed at $R_H = \frac{h}{ie^2}$, $R_H = 25.812k\Omega, i = 1$, $R_L = 0$.

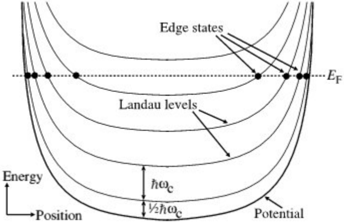
Landau levels

The Hamiltonian of a free particle in a magnetic field, $\mathbf{A} = B(-y, 0, 0)$ (Landau gauge)

$$H = \frac{[\mathbf{p} + e\mathbf{A}(\mathbf{r})]^2}{2m}$$

- Energy levels $E_n = \hbar\omega_C(n + \frac{1}{2})$, $\omega_C = \frac{eB}{mc}$.
- Degeneracy $N_s = A\frac{B}{\Phi_0}$ (Flux quantum $\Phi_0 = \frac{hc}{e}$).
- Filling fraction $\nu = \frac{N_{el}}{N_s} = \frac{nhc}{eB}$.
- When $\nu = i$, $R_H = \frac{h}{e^2 i}$

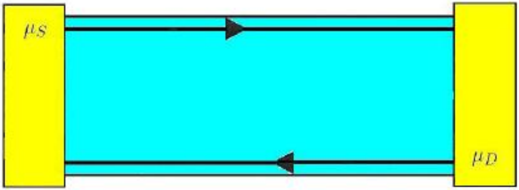
Edge states in the IQH



$$H = \frac{p_y^2}{2m} + \frac{m}{2}\omega_C(y - y_0)^2 + V_c(y)$$

$y_0 = kl_B^2$, where $l_B^2 = \sqrt{\frac{\hbar c}{eB}}$.

Edge states in the IQH, the current



The current I_n (for a given LL) is given by

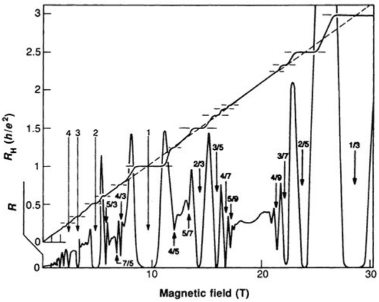
$$I_n = -\frac{e}{L}\sum_k < n, k|v_k|n, k >= -\frac{e}{L\hbar}\sum_k \frac{\partial E_{n,k}}{\partial k}$$

$$I_n = -\frac{e}{h}(\mu_S - \mu_D) = \frac{e^2}{h}V$$

Edge states in the IQH, the current

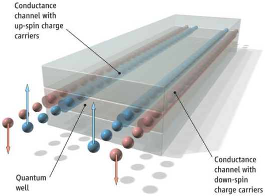
- Electrons are chiral and backscattering processes are suppressed by the sample dimension
- QH system is a very unusual electron liquid: bulk insulator with perfectly conducting edges

Fractional quantum Hall effect



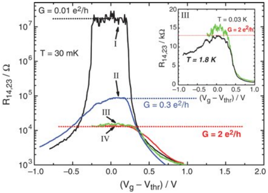
- Discovered in 1981 by D. Tsui and H. Stormer.
- Laughlin wave function for $\nu = \frac{1}{2q+1}$, q integer
- Fractional charge excitations and statistics.

Quantum Spin Hall Effect



- no magnetic field
- strong spin-orbit coupling
- two edges with opposite chirality and spin
- time reversal invariance

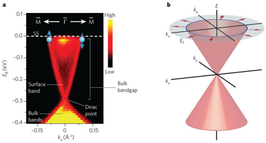
Quantum Spin Hall Effect



Konig et al., Science 314, 2006

- observation of QSHE in HgTe quantum wells
- transition from insulator to topological insulator varying thickness of quantum well

3+1 D topological insulator



J. Moore, Nature 464, 2010, experiment by Hasan group

- observation of 3D topological insulator in Bi_2Se_3
- surface helical states
- domain wall fermions in lattice gauge theory

Effective field theory for QHE

- Electron in a flatland (2+1 D)
- Fundamental current symmetry of the gapped phase

$$\sigma_{xy} = \nu \frac{e^2}{h} \qquad \sigma_{xx} = 0$$

- Topological effective theory describe low energy physics
- Parity (P) and time-reversal (T) symmetries are broken due to external magnetic field
- Chern-Simons successfully describe it

Chern-Simons theory

Given the current expressed in terms of gauge field a_μ

$$J^\mu = -\frac{e}{2\pi}\epsilon^{\mu\nu\rho}\partial_\nu a_\rho \quad (1)$$

the Lagrangian density satisfying the constrains on the current is

$$\mathcal{L}_{CS} = \frac{k}{2\pi}\epsilon^{\mu\nu\rho}a_\mu\partial_\nu a_\rho \quad (2)$$

that is naturally the Chern-Simons theory (k integer)

Froehlich & Zee, NPB 91; Wen, Adv.Phys. 95,....

Fundamental properties of Chern-Simons theory

$$S = \int dx^{\alpha} \Big(\underbrace{\frac{k}{4\pi}\epsilon^{\mu\nu\rho} a_{\mu}\partial_{\nu} a_{\rho}}_{Chern-Simons} - \underbrace{\frac{e}{2\pi}\epsilon^{\mu\nu\rho} a_{\mu}\partial_{\nu} A_{\rho}}_{Coupling\ \ with\ \ EM} - \underbrace{a_{\mu} j^{\mu}}_{Coupling\ \ with\ \ qp.} \Big)$$

A_{μ} external electromagnetic (EM) field
j^{μ} gauge field sources are the localized qp. Integrating out the a_{μ} fields we have

$$S_A = \int dx^{\alpha} \Big(\underbrace{-\frac{e^2}{4\pi k}\epsilon^{\mu\nu\rho} A_{\mu}\partial_{\nu} A_{\rho}}_{Bulk\ \ conductance} + \underbrace{\frac{e}{k} A_{\mu} j^{\mu}}_{qp.charge} - \underbrace{\frac{\pi}{k} j^{\mu}\Delta_{\mu\nu} j^{\nu}}_{qp.statistics} \Big)$$

For odd k we describe Laughlin states. Extension with many gauge field and non-Abelian statistics were proposed Wen, Adv.Phys. 95,...

Chern Simons with boundary: edge states

Imposing a confinement to the bulk theory (real sample) breaks gauge invariance. Requiring the full gauge invariance of the bulk+boundary theory leads naturally to require a chiral Kac-Moody algebra on the boundary

$$[A(z), A(z')] = \frac{2\pi}{k}\partial\delta(z - z')$$

where the chiral coordinate on the boundary is $z = (x + t)/\sqrt{2}$. We have 1+1 chiral boson $a_\mu = \partial_\mu\varphi$ corresponding to gapless modes, the edge states with action

$$S_{edge} = \frac{k}{4\pi}\int dxdt\ \partial_x\varphi(\partial_t - \partial_x)\varphi$$

Wen, Adv.Phys. 95, Floreanini and Jackiw, PRL 59, 1987

BF model in condensed matter

- Topological superconductors
  Diamantini, Sodano and Trugenberger,Phys. Rev. B82 2010.....
- Topological superconductors, Z_2 lattice gauge theory, $U(1)$ lattice gauge theory with charge 2 Higgs, RVB state.
  Hansson, Oganesyan and Sondhi, Ann. of Phys. 2004
- Topological insulators
  Cho and Moore, Ann. Phys. 2011

BF theory: Abelian case

BF theory is naturally T invariant in any dimension
For 2+1 D case

$$S_{BF}^{(2+1D)} = \frac{k}{2\pi} \int dx^{\alpha} \epsilon^{\mu\nu\rho} F_{\mu\nu} B_{\rho}$$

with $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ with A_ν and B_μ gauge fields.
For 3+1 D case

$$S_{BF}^{(3+1D)} = \frac{k}{2\pi} \int dx^{\alpha} \epsilon^{\mu\nu\rho\eta} F_{\mu\nu} B_{\rho\eta}$$

with $B_{\rho\eta}$ two-form gauge field

topological insulator Cho and Moore, Ann. Phys. 11

BF theory simmetries

In 2+1D for light-cone coordinates $z = \frac{t+x}{\sqrt{2}} \quad \bar{z} = \frac{t-x}{\sqrt{2}} \quad u = y$

$$S_{BF} = \frac{k}{\pi} \int dud^2z \left[B \left(\bar{\partial} A_u - \partial_u \bar{A} \right) + \bar{B} \left(\partial_u A - \partial A_u \right) + B_u \left(\partial \bar{A} - \bar{\partial} A \right) \right]$$

The BF action satisfy the two discrete symmetries
Parity (P) $\quad (u, z, A, A_u, B, B_u) \leftrightarrow (-u, \bar{z}, \bar{A}, -A_u, \bar{B}, -B_u)$
Time rev. (T) $(u, z, A, A_u, B, B_u) \leftrightarrow (u, -\bar{z}, -\bar{A}, A_u, \bar{B}, -B_u)$

So the gauge field can be interpreted
A_μ *charge density*
B_μ *spin density*

BF theory with boundary

The aim of this work is to present a detailed analysis of the 2+1 D BF Abelian and non-Abelian model in the presence of a boundary in the framework of Topological Quantum Field theories

We consider the axial gauge where ghost sector decouple

Boundary term must to be added to the action *to restore the total gauge invariance*. They have to satisfy the requirements:

- Locality
- Power counting
- Symanzik separability condition (i.e. propagators on the opposite sides of the boundary must vanish) Symanzik, NPB 81

Boundary terms

$$S_{BD} = \frac{k}{\pi} \int dud^2 z\delta(u) \left(\alpha_1 A\bar{A} + \alpha_2 A\bar{B} + \alpha_3 \bar{A}B + \alpha_4 B\bar{B}\right)$$

Boundary terms lead to to the *boundary conditions* on the gauge fields on one side

$$\begin{aligned} (1-\alpha_2)A - \alpha_4 B &= 0, & (3) \\ \alpha_1 A - (1-\alpha_3)B &= 0 & (4) \\ (1+\alpha_3)\bar{A} + \alpha_4 \bar{B} &= 0 & (5) \\ \alpha_1 \bar{A} + (1+\alpha_2)\bar{B} &= 0. & (6) \end{aligned}$$

non trivial solution are obtained for $\alpha_3 = -\alpha_2$ (T symmetry) and

$$\alpha_1\alpha_4 - (1-\alpha_2^2) = 0 \qquad (7)$$

Residual Ward identities

Axial gauge is not a complete gauge fixing, there is a residual gauge invariance expressed by a WI

$$\frac{\pi}{k}\int du\, H[Z_c] \equiv$$
$$-\frac{\pi}{k}\int du\left(\bar{\partial} j_{\bar{A}} + \partial j_A\right) = -\left[\alpha_1\left(\bar{\partial} A_+ + \partial \bar{A}_+\right) + \alpha_2\left(\partial \bar{B}_+ - \bar{\partial} B_+\right)\right]$$
$$\frac{\pi}{k}\int du\, N[Z_c] \equiv$$
$$-\frac{\pi}{k}\int du\left(\bar{\partial} j_{\bar{B}} + \partial j_B\right) = -\left[\alpha_2\left(\bar{\partial} A_+ - \partial \bar{A}_+\right) + \alpha_4\left(\bar{\partial} B_+ + \partial \bar{B}_+\right)\right].$$

Conserved Kac-Moody chiral current

Introducing the fields

$$\begin{aligned} R &\equiv (1-\alpha_2)\,A+\alpha_4 B \\ S &\equiv (\alpha_2-1)A_+ +\alpha_4 B \end{aligned}$$

such that $\bar{R}=S=\bar{\partial}R=\partial\bar{S}=0$ one can show they satisfy Kac-Moody algebra ($\left[R(z),\bar{S}(\bar{z}')\right]=0$)

$$\begin{aligned} \left[R(z),R(z')\right] &= \frac{2\pi\alpha_4(1-\alpha_2)}{k}\partial\delta(z-z') \\ \left[\bar{S}(\bar{z}),\bar{S}(\bar{z}')\right] &= \frac{2\pi\alpha_4(1-\alpha_2)}{k}\bar{\partial}\delta(\bar{z}-\bar{z}') \end{aligned}$$

where R and $\bar{S}$ have opposite chirality and they are connected by T symmetry

Non-Abelian BF theory

$$S_{BF} = \frac{k}{2\pi}\int dx^{\alpha}\ \varepsilon^{\mu\nu\rho}\Big\{F^{a}_{\mu\nu}B^{a}_{\rho} + \underbrace{\frac{\lambda}{3}f^{abc}B^{a}_{\mu}B^{b}_{\nu}B^{c}_{\rho}}_{\textit{Cosmological term}}\Big\}$$

$$F^{a}_{\mu\nu} = \partial_{\mu}A^{a}_{\nu} - \partial_{\nu}A^{a}_{\mu} + f^{abc}A^{b}_{\mu}A^{c}_{\nu} \tag{8}$$

We have included the *Cosmological term* consistent with gauge invariance only in 2+1 D and one can limit the analysis to $\lambda = 1$ because by rescaling

$$B^{a}_{\mu} \to \frac{B^{a}_{\mu}}{\sqrt{\lambda}} \quad \Longrightarrow \quad S_{BF}[k,\lambda] \to S_{BF}[\frac{k}{\sqrt{\lambda}}, \lambda = 1]$$

The boundary terms are analogous to the Abelian case.

Kac-Moody algebras in the non-Abelian case

Proceeding in analogy with the Abelian case if we introduce the two linear combinations

$$R^a \equiv A^a + B^a \qquad S^a \equiv -A^a + B^a$$

that satisfy $S^a(z,\bar{z}) = \bar{R}^a(z,\bar{z}) = \bar{\partial} R^a(z,\bar{z}) = \partial \bar{S}^a(z,\bar{z}) = 0$ we recover two Kac-Moody algebra

$$\left[R^a(z), R^b(z')\right] = f^{abc}\delta(z-z')R^c(z) + \frac{2\pi}{k}\delta^{ab}\delta'(z-z')$$

$$\left[\bar{S}^a(\bar{z}), \bar{S}^b(\bar{z}')\right] = f^{abc}\delta(\bar{z}-\bar{z}')\bar{S}^c(\bar{z}) + \frac{2\pi}{k}\delta^{ab}\delta'(\bar{z}-\bar{z}')$$

which represent the non-Abelian generalization for the boundary currents

Maggiore & Provero, Helv. Phys. Acta 1992

Chiral currents with opposite chirality

- In conclusion also for the non-Abelian case, at the boundary, there are two currents R^a and $\bar{S}^a$ propagating with opposite chirality and connected by T symmetry.
- The central charge of these currents are determined uniquely by the winding number k (and equivalently on the cosmological constant λ with the substitution $k \to k/\sqrt{\lambda}$ of the theory.
- In this perspective it is useful to compare the Abelian result discussed before with the non-Abelian result presented taking the limit $f^{abc} \to 0$. A proper way is to develop the Abelian theory as a limit of the non-Abelian one.
- If the T symmetry is not required at the boundary other solutions are possible where only one chiral current is obtained

Conclusion

- Abelian and non-Abelian theories appear good candidates as effective theories for topological insulators
- In general at the boundary we find edge states with opposite chiralities and Kac-moody algebras connected with T symmetry, consistently with the experimental observation of QSH.
- Fractional QSH state?
- Perspective: higher dimension 3+1 D topological insulators, spontaneous breaking of T symmetry at the boundary