Quantum phase transitions in the 3d Thirring model

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Research Training Group
Quantum and Gravitational Fields

seit 1558

Outline

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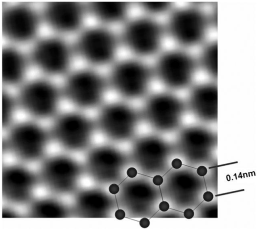
Introduction

3d N_f-flavor Thirring model:

$$\mathcal{L} = \bar{\Psi}^i \mathrm{i}\partial\!\!\!/ \Psi^i + \frac{\bar{g}}{2N_\mathrm{f}} (\bar{\Psi}^i \gamma_\mu \Psi^i)^2, \quad i = 1, 2, \ldots, N_\mathrm{f}$$

$N_\mathrm{f} = 2$: effective model for

- high-T_c cuprate superconductors
  [Herbut, PRL 94 (2005) 237001]
- graphene
  [Herbut, Juričić, Roy, PRB 79 (2009) 146401]



TEM image of graphene

But also ...

... 3d Thirring model per se very interesting field theory!

Previous results

"Chiral" symmetry and/or parity symmetry spontaneously broken?

- $1/N_\mathrm{f}$ expansion: $\langle\bar{\Psi}\Psi\rangle = 0$ ✗
- Dyson-Schwinger equation: [Kondo, NPB 450 (1995) 251] [Sugiura, PTP 97 (1997) 311] [Hong, Park, PRD 49 (1994) 5507] ...

$$0 \neq \langle\bar{\Psi}\Psi\rangle \sim \mathrm{e}^{-f(N_\mathrm{f})} \Leftrightarrow N_\mathrm{f} < N_\mathrm{f}^\mathrm{cr} \simeq 2, 3.24, 4.32, \infty \ \textbf{?}$$

- Monte-Carlo simulations: [Christofi, Hands, Strouthos, PRD 75 (2007) 101701]

$$N_\mathrm{f}^\mathrm{cr} \simeq 6.6(1) \ \textbf{?}$$

Universal value for N_f^cr?

- Power counting: (superficially) non-renormalizable
- $1/N_\mathrm{f}$ expansion: renormalizable at $\mathcal{O}(1/N_\mathrm{f})$, appears to hold to all orders [Parisi, NPB 100 (1975) 368] [Hands, PRD 51 (1995) 5816]
- Beyond $1/N_\mathrm{f}$ expansion?

"Chirality" in 3d: 4-component formalism

$$\mathcal{L} = \bar{\Psi}^i \mathrm{i}\partial\!\!\!/ \Psi^i + \frac{\bar{g}}{2N_\mathrm{f}} (\bar{\Psi}^i \gamma_\mu \Psi^i)^2, \quad i = 1, 2, \ldots, N_\mathrm{f}$$

[Pisarski, PRD 29 (1984) 2423]
[Appelquist *et al.*, PRD 33 (1986) 3704]

- $\bar{\Psi}, \Psi$: 4-spinors $\Leftrightarrow$ γ_μ: 4×4 matrices
- 2 "fifth-γ" matrices γ_4 & γ_5: $\{\gamma_4, \gamma_\mu\} = 0, \quad \{\gamma_5, \gamma_\mu\} = 0$
- For each flavor $i = 1, \ldots, N_\mathrm{f}$: U(2) "chiral" symmetry
  - 2 possible $\mathrm{U_A}(1)$ generated by γ_4, γ_5: $\quad \Psi \mapsto e^{\mathrm{i}\alpha\gamma_4}\, \Psi$, etc.
  - 2 possible $\mathrm{U_V}(1)$ generated by $\mathbb{1}, \gamma_{45} := \mathrm{i}\gamma_4\gamma_5$: $\quad \Psi \mapsto e^{\mathrm{i}\beta\gamma_{45}}\, \Psi$, etc.
- $\mathrm{U}(N_\mathrm{f})$ flavor rotations: $\Psi^i \mapsto U^{ij}\Psi^j$, etc.

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Symmetry of 3d massless Thirring model:

$\mathrm{U}(2N_\mathrm{f})$ generated by $\lambda_i \otimes \{\mathbb{1}, \gamma_4, \gamma_5, \gamma_{45}\}$

$\mathbb{1}, \gamma_4, \gamma_5, \gamma_{45}$: generators of U(2)

$\lambda_1, \lambda_2, \ldots, \lambda_{N_\mathrm{f}^2}$: generators of $\mathrm{U}(N_\mathrm{f})$ ($N_\mathrm{f}\times N_\mathrm{f}$ Gell-Mann matrices)

"Chiral" symmetry as part of flavor symmetry

$U(2N_f)$ in 2-component formalism:

$$\Psi^i \equiv \begin{pmatrix} \psi^i \\ \psi^{i+N_f} \end{pmatrix} \quad \Rightarrow \quad \mathcal{L} = \mathrm{i}\bar{\psi}^a \sigma_\mu \partial_\mu \psi^a + \frac{\bar{g}}{2N_f}(\bar{\psi}^a \sigma_\mu \psi^a)^2$$

where $i = 1, \ldots, N_f$ and $a = 1, \ldots, 2N_f$

- $U(2N_f)$ symmetry manifest: $\psi^a \mapsto U^{ab}\psi^b$, etc.
  $\Rightarrow$ "Chiral" symmetry $\subsetneq$ flavor symmetry

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Dynamical mass generation?

- 4 possible mass terms: $\bar{\Psi}\Psi$, $\bar{\Psi}\gamma_4\Psi$, $\bar{\Psi}\gamma_5\Psi$, $\bar{\Psi}\gamma_{45}\Psi$, $\quad \gamma_{45} := \mathrm{i}\gamma_4\gamma_5$
- E.g. $\bar{\Psi}\gamma_4\Psi \xrightarrow{\mathrm{U}(2N_\mathrm{f})} \bar{\Psi}\Psi \quad \Rightarrow \quad$ generic mass term: $\bar{m}_\chi \bar{\Psi}\Psi + \bar{m}_\mathcal{P} \bar{\Psi}\gamma_{45}\Psi$

"Chiral" symmetry as part of flavor symmetry

$\mathrm{U}(2N_\mathrm{f})$ in 2-component formalism:

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Dynamical mass generation: χSB vs. $\mathcal{P}$SB

$$\bar{m}_\chi \propto \langle \bar{\Psi}^i \Psi^i \rangle = \langle \bar{\psi}^i \psi^i - \bar{\psi}^{i+N_\mathrm{f}} \psi^{i+N_\mathrm{f}} \rangle$$

$$\bar{m}_\mathcal{P} \propto \langle \bar{\Psi}^i \gamma_{45} \Psi^i \rangle = \langle \bar{\psi}^i \psi^i + \bar{\psi}^{i+N_\mathrm{f}} \psi^{i+N_\mathrm{f}} \rangle = \langle \bar{\psi}^a \psi^a \rangle$$

- $\bar{m}_\chi \neq 0 \Leftrightarrow \mathrm{U}(\cancel{2}N_\mathrm{f}) \to \mathrm{U}(N_\mathrm{f}) \otimes \mathrm{U}(N_\mathrm{f}), \quad \mathcal{P}$ ✔
- $\bar{m}_\mathcal{P} \neq 0 \Leftrightarrow \mathrm{U}(2N_\mathrm{f})$ ✔, $\quad \cancel{\mathcal{P}}$

[Pisarski, PRD 29 (1984) 2423]
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Outline

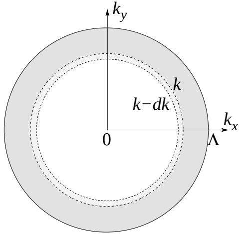
Functional Renormalization Group (FRG)

Effective action Γ

$$\Gamma[\Psi,\bar{\Psi}] = \mathcal{L}\left(W[\eta,\bar{\eta}]\right), \quad W[\eta,\bar{\eta}] = \log \int_\Lambda \mathcal{D}\hat{\Psi}\mathcal{D}\hat{\bar{\Psi}}\, \mathrm{e}^{-S[\hat{\Psi},\hat{\bar{\Psi}}]+\int \bar{\eta}\hat{\Psi}+\int \hat{\bar{\Psi}}\eta}$$

Idea (Wilson):

Integrate momentum shell by momentum shell!

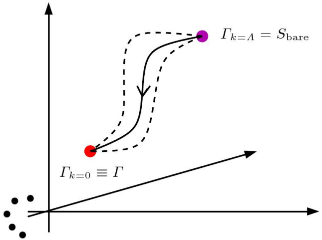


Effective average action:

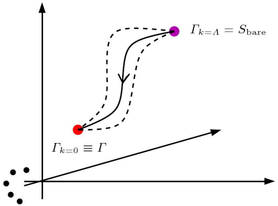
$$\Gamma[\bar{\Psi},\Psi] \mapsto \Gamma_k[\bar{\Psi},\Psi]$$

such that

$$\begin{aligned} \text{UV}: \quad & \Gamma_{k=\Lambda} = S_{\text{bare}} \\ \text{IR}: \quad & \Gamma_{k=0} = \Gamma \end{aligned}$$



RG flow of effective average action Γ_k



Evolution for Γ_k:

[Wetterich, PLB 301 (1993) 90]

$$\partial_k \Gamma_k[\Psi, \bar{\Psi}] = -\frac{1}{2} \operatorname{Tr} \frac{\partial_k R_k}{\Gamma_k^{(2)}[\Psi, \bar{\Psi}] + R_k}$$

with $\Gamma_k^{(2)} = \begin{pmatrix} \frac{\delta^2 \Gamma}{\delta \hat{\bar{\Psi}} \delta \hat{\Psi}} & \frac{\delta^2 \Gamma}{\delta \hat{\bar{\Psi}} \delta \hat{\bar{\Psi}}} \\ \frac{\delta^2 \Gamma}{\delta \hat{\Psi} \delta \hat{\Psi}} & \frac{\delta^2 \Gamma}{\delta \hat{\Psi} \delta \hat{\bar{\Psi}}} \end{pmatrix} \quad \Rightarrow \quad \left(\Gamma_k^{(2)} + R_k\right)^{-1}$ full propagator

Non-perturbative expansion schemes:

$$\Gamma_k = \int_x \left[V_k(\Psi, \bar{\Psi}) + Z_k(\Psi, \bar{\Psi})\, \bar{\Psi}(x) \mathrm{i} \partial\!\!\!/ \Psi(x) + \mathcal{O}(\partial^2) \right]$$

$$\Gamma_k = \Gamma^{(0)} + \sum_{\mathcal{O}_X} \int_{x_1, x_2} \Gamma^{(2)}(x_1, x_2)\, \bar{\Psi}(x_1) \mathcal{O}_X \Psi(x_2)$$
$$+ \sum_{\mathcal{O}_X, \mathcal{O}_Y} \int_{x_1, \ldots, x_4} \Gamma^{(4)}(x_1, \ldots, x_4)\, \bar{\Psi}(x_1) \mathcal{O}_X \Psi(x_2) \bar{\Psi}(x_3) \mathcal{O}_Y \Psi(x_4) + \mathcal{O}(\Psi^6)$$

Computation procedure

FRG Feynman rules

$$\beta_i = k\partial_k g_i \sim \sum \text{(non-perturbative) 1-loop diagrams}$$

with

- vertices: full vertices $\Gamma_k^{(n)} = \frac{\delta^n \Gamma_k}{\delta\hat{\bar{\Psi}}\ldots\delta\hat{\Psi}} \sim g_j$
- inner lines: full propagators $\left(\Gamma_k^{(2)} + R_k\right)^{-1}$

Remarks:

- Fully non-perturbative ("exact")
- Perturbation theory: $\Gamma_k = S + \hbar\Gamma_k^{\text{1-loop}} + \mathcal{O}(\hbar^2)$

$$\Rightarrow \qquad \partial_k \Gamma_k^{\text{1-loop}} = \frac{1}{2}\,\mathrm{Tr}\left[\partial_k R_k \left(S^{(2)} + R_k\right)^{-1}\right] = \frac{1}{2}\,\mathrm{Tr}\log\left(S^{(2)} + R_k\right)$$

$$\Rightarrow \qquad \Gamma^{\text{1-loop}} = S + \frac{1}{2}\,\mathrm{Tr}\log S^{(2)} + \text{const.}$$

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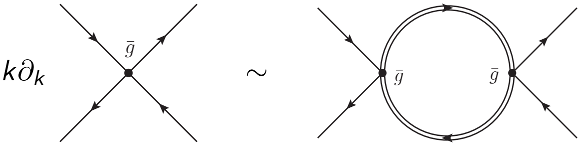
Example: 4-fermion β function (point-like limit, $\cancel{\mathcal{O}(\Psi^6)}$)



$\Rightarrow$ recover RG-improved 1-loop β function

Outline

Classification of bilinear/4-fermi terms w.r.t. $\mathrm{U}(2N_{\mathrm{f}}) \otimes \mathcal{P}$

- Bilinears:
  - No mass terms: $\bar{\Psi}\Psi$ ✗, $\bar{\Psi}\gamma_{45}\Psi$ ✗
  - First order in derivative: $\bar{\Psi}i\partial\!\!\!/\Psi$ ✓

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- 4-fermi terms:

$$(\bar{\Psi}^i\gamma_\mu\Psi^i)^2$$

$$(\bar{\Psi}^i\gamma_{45}\Psi^i)^2$$

$$(\bar{\Psi}^i\Psi^j)^2 - (\bar{\Psi}^i\gamma_4\Psi^j)^2 - (\bar{\Psi}^i\gamma_5\Psi^j)^2 + (\bar{\Psi}^i\gamma_{45}\Psi^j)^2$$

$$(\bar{\Psi}^i\gamma_\mu\Psi^j)^2 + (\bar{\Psi}^i\frac{\sigma_{\mu\nu}}{\sqrt{2}}\Psi^j)^2 - (\bar{\Psi}^i\mathrm{i}\gamma_\mu\gamma_4\Psi^j)^2 - (\bar{\Psi}^i\mathrm{i}\gamma_\mu\gamma_5\Psi^j)^2$$

where $(\bar{\Psi}^i\Psi^j)^2 \equiv \bar{\Psi}^i\Psi^j\bar{\Psi}^j\Psi^i$, etc.

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- 4-fermi terms:

$$(\bar{\Psi}^i\gamma_\mu\Psi^i)^2 = (\bar{\psi}^a\sigma_\mu\psi^a)^2 \checkmark$$

$$(\bar{\Psi}^i\gamma_{45}\Psi^i)^2 = (\bar{\psi}^a\psi^a)^2 \checkmark$$

$$(\bar{\Psi}^i\Psi^j)^2 - (\bar{\Psi}^i\gamma_4\Psi^j)^2 - (\bar{\Psi}^i\gamma_5\Psi^j)^2 + (\bar{\Psi}^i\gamma_{45}\Psi^j)^2 = (\bar{\psi}^a\psi^b)^2 \checkmark$$

$$(\bar{\Psi}^i\gamma_\mu\Psi^j)^2 + (\bar{\Psi}^i\frac{\sigma_{\mu\nu}}{\sqrt{2}}\Psi^j)^2 - (\bar{\Psi}^i\mathrm{i}\gamma_\mu\gamma_4\Psi^j)^2 - (\bar{\Psi}^i\mathrm{i}\gamma_\mu\gamma_5\Psi^j)^2 = (\bar{\psi}^a\sigma_\mu\psi^b)^2 \checkmark$$

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  - First order in derivative: $\bar{\Psi}\mathrm{i}\not{\partial}\Psi$ ✓
- 4-fermi terms:

$$(\bar{\Psi}^i\gamma_\mu\Psi^i)^2 = (\bar{\psi}^a\sigma_\mu\psi^a)^2 \checkmark$$

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  where $(\bar{\Psi}^i\Psi^j)^2 \equiv \bar{\Psi}^i\Psi^j\bar{\Psi}^j\Psi^i$, etc.
- Fierz transform:

$$(\bar{\psi}^a\psi^b)^2 = -(\bar{\psi}^a\sigma_\mu\psi^a)^2 - (\bar{\psi}^a\psi^a)^2$$

$$(\bar{\psi}^a\sigma_\mu\psi^b)^2 = (\bar{\psi}^a\sigma_\mu\psi^a)^2 - 3(\bar{\psi}^a\psi^a)^2$$

Result:

$N_f = 1$: [Herbut *et al.*, PRB 79 (2009) 146401]

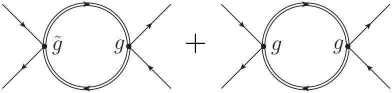
2 independent couplings: e.g. $(\bar{\Psi}^i\gamma_\mu\Psi^i)^2$, $(\bar{\Psi}^i\gamma_{45}\Psi^i)^2$

Simple truncation of effective average action

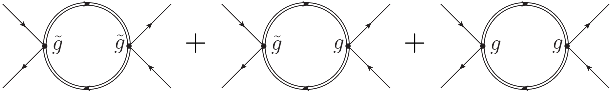
$$\Gamma_k = \int_x \left[Z_k \bar{\Psi}^i \mathrm{i}\partial\!\!\!/ \Psi^i + \frac{\bar{g}_k}{2N_\mathrm{f}} (\bar{\Psi}^i \gamma_\mu \Psi^i)^2 + \frac{\bar{\tilde{g}}_k}{2N_\mathrm{f}} (\bar{\Psi}^i \gamma_{45} \Psi^i)^2 \right]$$

[Gies, LJ, PRD 82 (2010) 085018]
$N_\mathrm{f} = 1$: [Herbut *et al.*, PRB 79 (2009) 146401]

$k\partial_k g = (d-2)g +$



$k\partial_k \tilde{g} = (d-2)\tilde{g} +$



$\eta_\psi \equiv -k\partial_k \log Z_k = 0$

$g, \tilde{g}$: dim'less couplings

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$N_\mathrm{f} = 1$: [Herbut *et al.*, PRB 79 (2009) 146401]

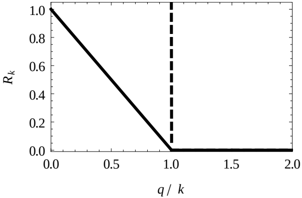
$$k\partial_k g = (d-2)g + \frac{4\ell_1^{(\mathrm{F})}}{\pi^2} \left[\frac{1}{2N_\mathrm{f}} \tilde g g + \frac{2N_\mathrm{f}+1}{6N_\mathrm{f}} g^2 \right]$$

$$k\partial_k \tilde g = (d-2)\tilde g - \frac{4\ell_1^{(\mathrm{F})}}{\pi^2} \left[\frac{2N_\mathrm{f}-1}{2N_\mathrm{f}} \tilde g^2 - \frac{3}{2N_\mathrm{f}} \tilde g g - \frac{1}{N_\mathrm{f}} g^2 \right]$$

$$\eta_\psi \equiv -k\partial_k \log Z_k = 0$$

$g, \tilde g$: dim'less couplings

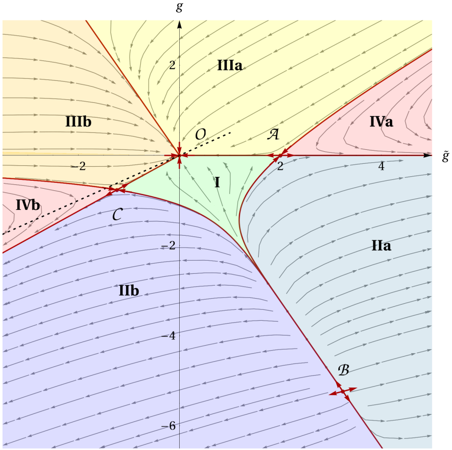
$$\ell_1^{(\mathrm{F})} = \begin{cases} 2/3 & \text{linear cut-off} \\ 1 & \text{sharp cut-off} \end{cases}$$



Fermionic RG flow for $N_\mathrm{f} = 1$

Fixed-point (FP) condition:

$$\beta_i(g^*) = k\partial_k g_i\big|_{g=g^*} = 0$$



c.f. [Herbut, Juričić, Roy, PRB 79 (2009) 146401]

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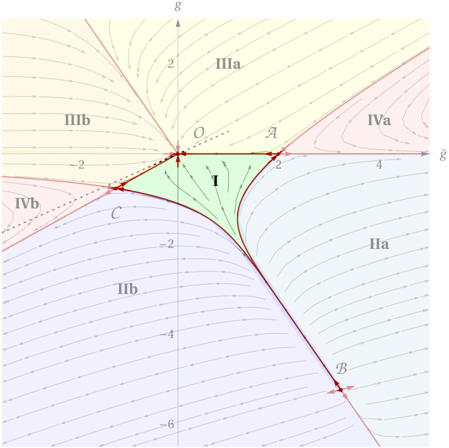
I: $(\tilde{g}, g) \overset{\mathrm{IR}}{\rightarrow} 0$: free theories



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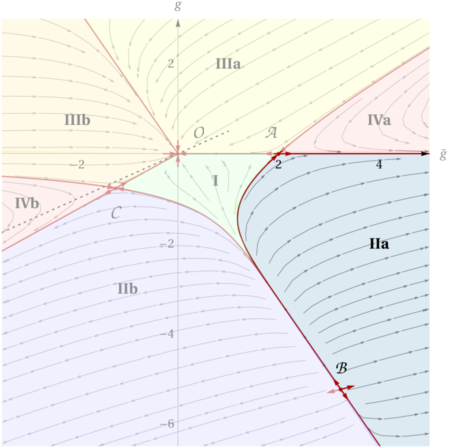
I: $(\tilde{g}, g) \xrightarrow{\text{IR}} 0$: free theories

IIa: $(\tilde{g}, g) \xrightarrow{\text{IR}} (+\infty, 0)$

- $\Gamma_k^{\text{int}} \xrightarrow{\text{IR}} \tilde{g}(\bar{\Psi}\gamma_{45}\Psi)^2$
- $\langle\bar{\Psi}\gamma_{45}\Psi\rangle \neq 0 \;\Leftrightarrow\; \mathcal{P}$SB!



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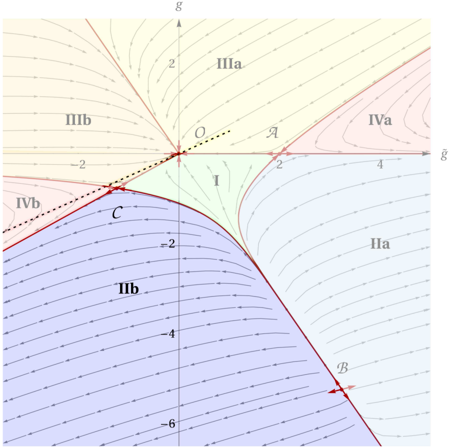

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IIa: $(\tilde{g}, g) \xrightarrow{\text{IR}} (+\infty, 0)$

- $\Gamma_k^{\text{int}} \xrightarrow{\text{IR}} \tilde{g}(\bar{\Psi}\gamma_{45}\Psi)^2$
- $\langle \bar{\Psi}\gamma_{45}\Psi \rangle \neq 0 \;\Leftrightarrow\; \mathcal{P}$SB!

IIb: $(\tilde{g}, g) \xrightarrow{\text{IR}} \mathcal{C} \cdot \infty$

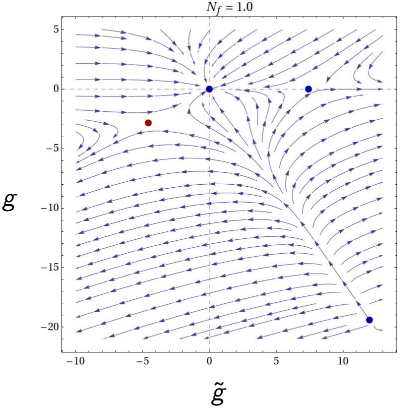
- $\Gamma_k^{\text{int}} \overset{\text{Fierz}}{\propto} (2g - \tilde{g})(\bar{\Psi}\gamma_\mu\Psi)^2 - \tilde{g}[(\bar{\Psi}\Psi)^2 - (\bar{\Psi}\gamma_4\Psi)^2 - (\bar{\Psi}\gamma_5\Psi)^2]$
- IR: $|\tilde{g}| \gtrsim 4 \times |2g - \tilde{g}| \;\rightsquigarrow\; \langle \bar{\Psi}\Psi \rangle \neq 0 \;\Leftrightarrow\; \chi$SB!

c.f. [Herbut, Juričić, Roy, PRB 79 (2009) 146401]

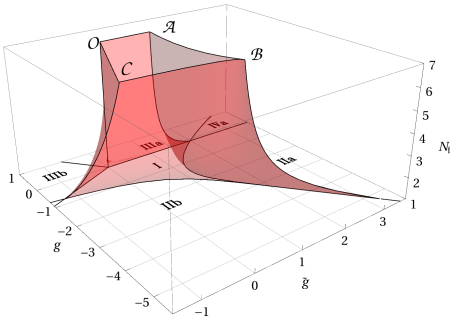
Fermionic RG flow for $N_f \geq 1$

[Gies, LJ, PRD 82 (2010) 085018]

(a) FPs for $N_f = 1, \ldots, 10$:



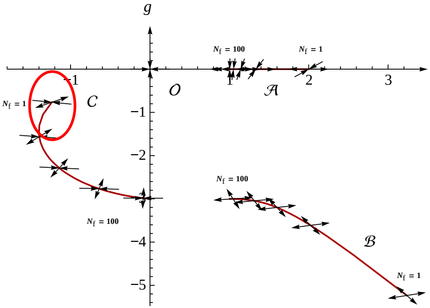
(b) . . . and separatrices:

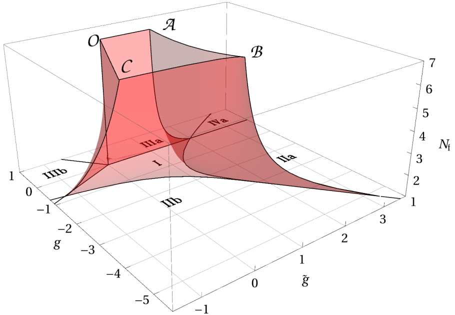


Fermionic RG flow for $N_f \geq 1$

[Gies, LJ, PRD 82 (2010) 085018]

(a) FPs for $N_f = 1, 2, 4, 10, 100$: (b) . . . and separatrices:





$$\Gamma_k^{\text{int}} \sim (2g - \tilde{g})(\bar{\Psi}^i \gamma_\mu \Psi^i)^2 - \tilde{g}[(\bar{\Psi}^i \Psi^j)^2 - (\bar{\Psi}^i \gamma_4 \Psi^j)^2 - (\bar{\Psi}^i \gamma_5 \Psi^j)^2]$$

Small N_f:

$|\tilde{g}| \gg |2g - \tilde{g}|$:
$\langle \bar{\Psi}^i \Psi^j \rangle \neq 0 \Leftrightarrow \chi$SB

Fermionic RG flow for $N_\mathrm{f} \geq 1$

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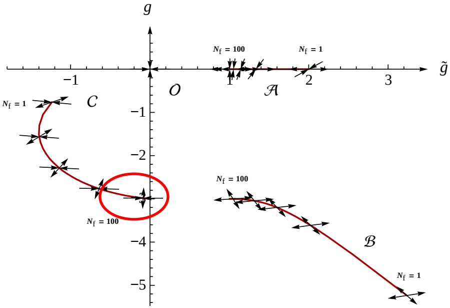

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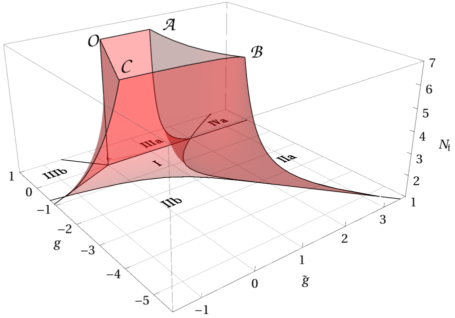

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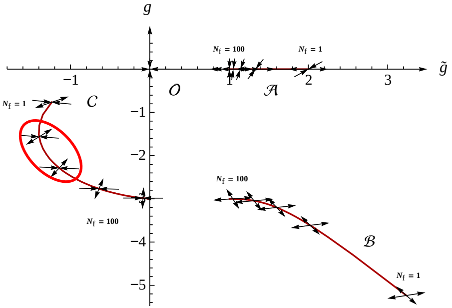
Large N_f:

$|\tilde{g}| \ll |2g - \tilde{g}|$:
$V_\mu \propto \bar{\Psi}^i \gamma_\mu \Psi^i \Rightarrow \cancel{\chi}\mathrm{SB}$

Fermionic RG flow for $N_\mathrm{f} \geq 1$

[Gies, LJ, PRD 82 (2010) 085018]

(a) FPs for $N_\mathrm{f} = 1, 2, 4, 10, 100$:



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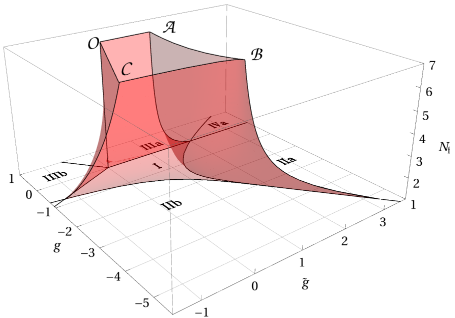

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Small N_f:

$|\tilde{g}| \gg |2g - \tilde{g}|$:
$\langle \bar{\Psi}^i \Psi^j \rangle \neq 0 \Leftrightarrow \chi$SB

$N_\mathrm{f} \sim \mathcal{O}(7/4)$:

$|\tilde{g}| \approx |2g - \tilde{g}|$:
$\bar{\Psi}^i \Psi^j$? $\leftrightsquigarrow$ $\bar{\Psi}^i \gamma_\mu \Psi^i$?

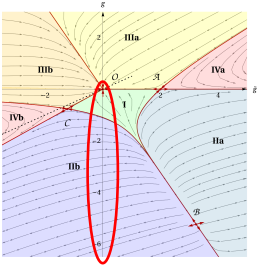
Large N_f:

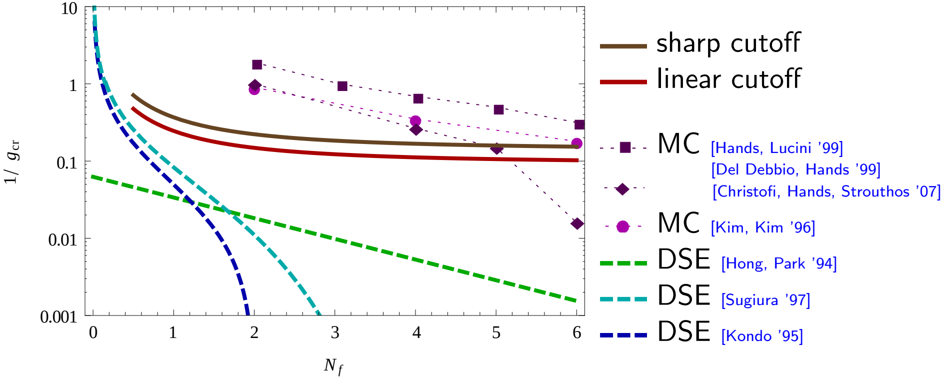
$|\tilde{g}| \ll |2g - \tilde{g}|$:
$V_\mu \propto \bar{\Psi}^i \gamma_\mu \Psi^i \Rightarrow \cancel{\chi}$SB

Comparison with MC/DSE studies

- $S_{\text{bare}} \sim (\bar{\Psi}\gamma_\mu\Psi)^2$ renormalizable?
  - $N_f \to \infty$: FP $\mathcal{C} \to$ "Thirring" axis $\tilde{g} = 0$
  - $N_f < \infty$: $(\bar{\Psi}\gamma_{45}\Psi)^2$ generated by RG flow
  - IIb in attractive domain of $\mathcal{C}$: 1 relevant coupl.
- $g_{\text{cr}} \sim$ intersection separatrix $\mathcal{BC}$ / Thirring axis

⚠ FP values not universal!





Outline

Thirring model in 2-component formalism

Investigate competition between ...

scalar channel:

$$(\bar\Psi^i\Psi^j)^2 - (\bar\Psi^i\gamma_4\Psi^j)^2$$
$$-(\bar\Psi^i\gamma_5\Psi^j)^2 + (\bar\Psi^i\gamma_{45}\Psi^j)^2$$

vs. $\leftrightsquigarrow$

vector channel:

$$(\bar\Psi^i\gamma_\mu\Psi^i)^2$$

Using the 2-component formalism:

$$\Psi^i \equiv \begin{pmatrix} \psi^i \\ \psi^{i+N_\mathrm{f}} \end{pmatrix}, \quad \bar\Psi^i \equiv \left(\bar\psi^i, -\bar\psi^{i+N_\mathrm{f}}\right)$$

scalar channel:

$$\bar\psi^a\psi^b\bar\psi^b\psi^a$$

vs. $\leftrightsquigarrow$

vector channel:

$$(\bar\psi^a\sigma_\mu\psi^a)^2$$

with $a, b = 1, 2, \ldots, 2N_\mathrm{f}$

Hubbard-Stratonovich-Transformation

Partition function

$$Z[0] = \int \mathcal{D}\psi\mathcal{D}\bar{\psi}\,\exp\left[-\left(\bar{\psi}^a \mathrm{i}\partial\!\!\!/\psi^a + \frac{\bar{g}_\phi}{2N_\mathrm{f}}\bar{\psi}^a\psi^b\bar{\psi}^b\psi^a - \frac{\bar{g}_V}{2N_\mathrm{f}}\left(\bar{\psi}^a\sigma_\mu\psi^a\right)^2\right)\right]$$

Fierz transform:

$$\bar{g}_\phi = -2\tilde{\bar{g}} > 0, \qquad \bar{g}_V = \tilde{\bar{g}} - \bar{g} > 0$$

Hubbard-Stratonovich-Transformation

Partition function

$$Z[0] = \int \mathcal{D}\psi\mathcal{D}\bar{\psi}\,\exp\left[-\left(\bar{\psi}^a \mathrm{i}\partial\!\!\!/\psi^a + \frac{\bar{g}_\phi}{2N_\mathrm{f}}\bar{\psi}^a\psi^b\bar{\psi}^b\psi^a - \frac{\bar{g}_V}{2N_\mathrm{f}}\left(\bar{\psi}^a\sigma_\mu\psi^a\right)^2\right)\right]$$

Multiply by:

$$1 = \mathcal{N}\int \mathcal{D}\phi\ \exp\left[-\frac{1}{2}\left(\bar{m}_\phi\phi^{ab} + \mathrm{i}\frac{\bar{h}_\phi}{\bar{m}_\phi}\bar{\psi}^b\psi^a\right)\left(\bar{m}_\phi\phi^{ba} + \mathrm{i}\frac{\bar{h}_\phi}{\bar{m}_\phi}\bar{\psi}^a\psi^b\right)\right]$$

$$1 = \mathcal{N}\int \mathcal{D}V\ \exp\left[-\frac{1}{2}\left(\bar{m}_V V_\mu - \frac{\bar{h}_V}{\bar{m}_V}\bar{\psi}^a\sigma_\mu\psi^a\right)^2\right]$$

$$Z[0] = \mathcal{N}\int \mathcal{D}\psi\mathcal{D}\bar{\psi}\mathcal{D}\phi\,\exp\left[-\left(\bar{\psi}^a\mathrm{i}\partial\!\!\!/\psi^a + \frac{1}{2}\bar{m}_\phi^2\phi^{ab}\phi^{ba} + \frac{1}{2}\bar{m}_V^2 V_\mu^2 - \bar{h}_V V_\mu\bar{\psi}^a\sigma_\mu\psi^a + \mathrm{i}\bar{h}_\phi\bar{\psi}^a\phi^{ab}\psi^b\right)\right]$$

if $\frac{\bar{h}_\phi^2}{2\bar{m}_\phi^2} = \frac{\bar{g}_\phi}{2N_\mathrm{f}}$ and $\frac{\bar{h}_V^2}{2\bar{m}_V^2} = \frac{\bar{g}_V}{2N_\mathrm{f}}$

Effective action with collective fields

$$\Gamma_k = \int_x \Big[Z_{\psi,k}\bar{\psi}^a \mathrm{i}\partial\!\!\!/ \psi^a + \frac{Z_{\phi,k}}{2}\partial_\mu\phi^{ab}\partial_\mu\phi^{ba} + \frac{Z_{V,k}}{4}V_{\mu\nu}V_{\mu\nu} + \frac{Z_{V,k}}{2\xi}(\partial_\mu V_\mu)^2$$
$$+ U_k(\phi) + \frac{\bar{m}^2_{V,k}}{2}V_\mu V_\mu + \frac{\bar{\mu}_k}{8}(V_\mu V_\mu)^2 + \frac{\bar{\nu}_k}{4}V_\mu V_\mu \phi^{ab}\phi^{ba}$$
$$- \bar{h}_{V,k}V_\mu \bar{\psi}^a\gamma_\mu\psi^a + \mathrm{i}\bar{h}_{\phi,k}\bar{\psi}^a\phi^{ab}\psi^b \Big]$$

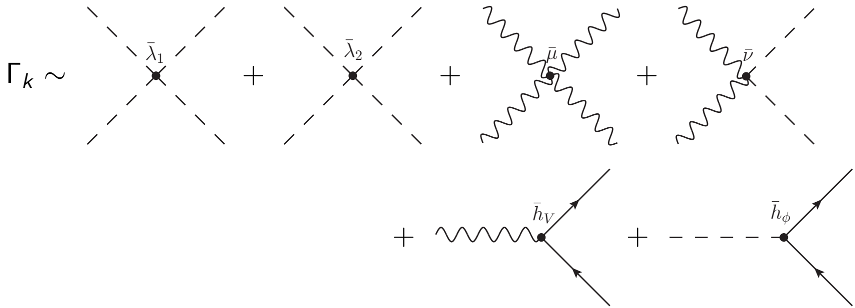
where $V_{\mu\nu} = \partial_\mu V_\nu - \partial_\nu V_\mu$ and the scalar potential

$$U_k(\phi) = \begin{cases} \bar{m}^2_{\phi,k}\rho + \frac{\bar{\lambda}_{1,k}}{2}\rho^2 + \bar{\lambda}_{2,k}\tau, & \text{SYM regime} \\ \frac{\bar{\lambda}_{1,k}}{2}(\rho - \rho_{0,k})^2 + \bar{\lambda}_{2,k}\tau, & \text{SSB regime} \end{cases}$$

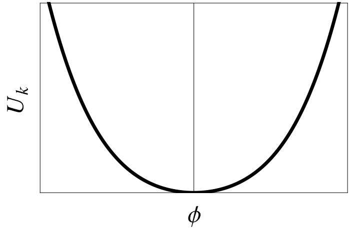
with the $U(2N_\mathrm{f})$ invariants

$$\rho \equiv \frac{1}{2}\,\mathrm{tr}\,\phi^2 \quad \text{and} \quad \tau \equiv \frac{1}{2}\,\mathrm{tr}\left(\frac{1}{2}\phi^2 - \frac{1}{n}\rho\right)^2$$

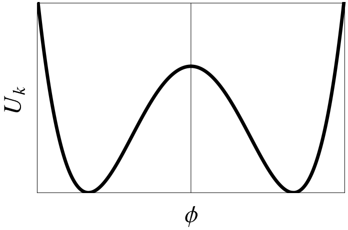
Effective action with collective fields: diagrammatically



Scalar potential:



SYM regime



SSB regime

Spontaneous symmetry breaking

Consider configurations with vev

$$\langle\hat{\phi}\rangle = \sqrt{\frac{\rho_{0,k}}{N_\mathrm{f}}}\begin{pmatrix} \mathbb{1}_{N_\mathrm{f}} & 0 \\ 0 & -\mathbb{1}_{N_\mathrm{f}} \end{pmatrix}$$

Order parameters:

$$\chi\mathrm{SB}: \quad \left\langle \sum_{i=1}^{N_\mathrm{f}} \left(\hat{\bar{\psi}}^i\hat{\psi}^i - \hat{\bar{\psi}}^{i+N_\mathrm{f}}\hat{\psi}^{i+N_\mathrm{f}}\right)\right\rangle \propto \lim_{k\to 0}\sqrt{\rho_{0,k}}\,\bar{h}_{\phi,k}$$

$$\mathcal{P}\mathrm{SB}: \quad \left\langle \sum_{a=1}^{2N_\mathrm{f}} \hat{\bar{\psi}}^a\hat{\psi}^a\right\rangle = 0$$

Procedure:

- Initial couplings: close to “Thirring” FP with $\rho_{0,\mathrm{UV}} = 0$
- Succesively integrate out fluctuations by lowering k
- Switch to SSB regime once $U_k''(\phi) \sim \bar{m}_\phi^2 \to 0$
- Continue RG flow until IR FP is reached / massive modes decouple

Benefits & drawbacks of our bosonic formulation

✔ Benefits

- Momentum-dependent 4-fermi couplings

$$g(q) \sim \frac{h^2}{Zq^2 + m^2}$$

- SSB may be explicitly studied via $\langle\hat{\phi}\rangle \neq 0$

✗ Drawbacks

- Only 2 (out of 4) possible condensation channels
- Fierz ambiguity: results depend on distribution on channels

[Jäckel, Wetterich, PRD 68 (2003) 025020]

Scalar mass spectrum

Mass matrix

$$\left(\frac{\partial^2 U_k}{\partial\phi^{cd}\partial\phi^{ab}}\right)\Bigg|_{\phi=\langle\hat{\phi}\rangle}$$

eigenvalue	degeneracy
$\partial_\rho U_k + 2\rho_{0,k}\partial_\rho^2 U_k$	1
$\partial_\rho U_k + \frac{2\rho_{0,k}}{n}\partial_\tau U_k$	$2N_\mathrm{f}{}^2 - 1$
$\partial_\rho U_k$	$2N_\mathrm{f}{}^2$

In SSB regime:

$$\partial_\rho U_k\Big|_{(\rho,\tau)=(\rho_{0,k},0)} = 0$$

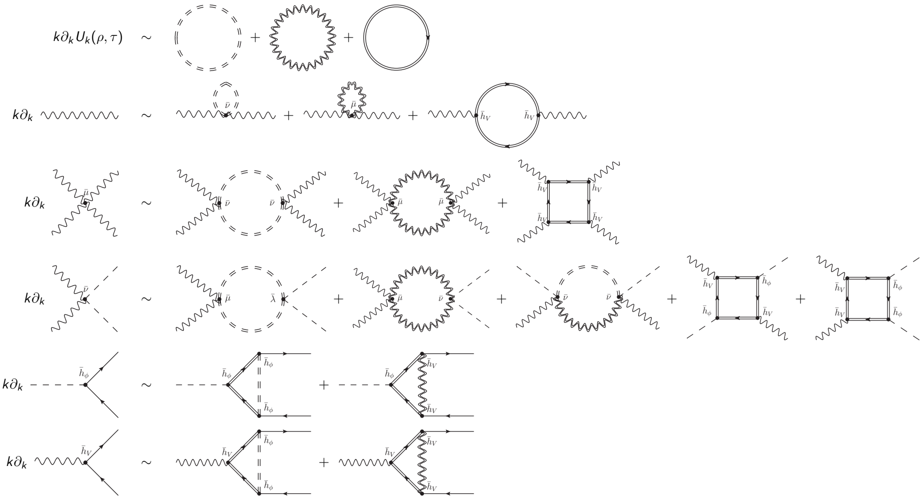
Goldstone's theorem

$\#$ massless modes $=$ $\#$ broken symmetry generators

breaking pattern:	$\mathrm{U}(2N_\mathrm{f}) \to$	$\mathrm{U}(N_\mathrm{f}) \otimes$	$\mathrm{U}(N_\mathrm{f})$
$\#$ generators:	$(2N_\mathrm{f})^2$	$N_\mathrm{f}{}^2$	$N_\mathrm{f}{}^2$

$2N_\mathrm{f}{}^2$ broken generators ✔

Beta functions (SYM regime)



& flow of wave function renormalizations $Z_{\phi,k}$, $Z_{\psi,k}$, $Z_{V,k}$

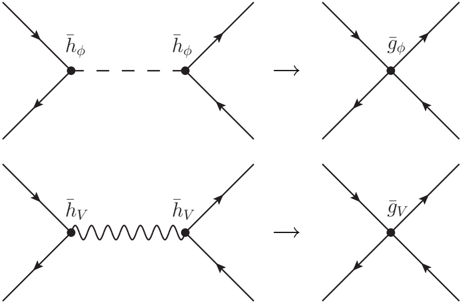
$\Rightarrow$ Possibility to automatize: DoFun [Braun, Huber, arXiv:1102.5307]

Preliminary results: Recover point-like limit

Point-like limit: $Z_{\phi/V} \to 0$

$$m_{\phi/V} \propto Z_{\phi/V}^{-1} \to \infty, \quad h_{\phi/V} \propto Z_{\phi/V}^{-1} \to \infty \quad \text{with} \quad \frac{h_{\phi/V}^2}{2m_{\phi/V}^2} = \frac{g_{\phi/V}}{2N_\mathrm{f}} = \text{const.}$$

Diagrammatically:

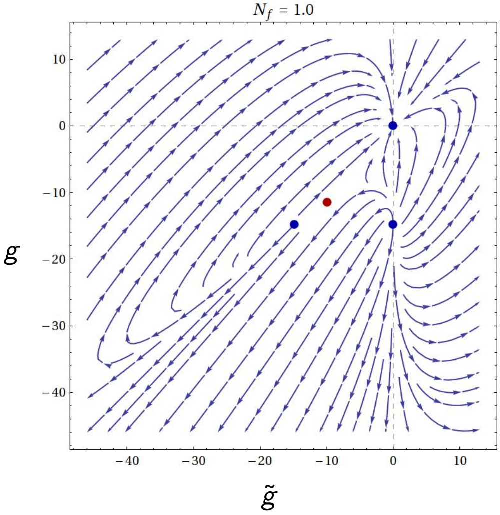


$$\Gamma_k = \int_x \Big[Z_\psi \bar{\psi}^a \mathrm{i}\partial\!\!\!/ \psi^a + \frac{1}{2}\bar{m}_\phi^2 \phi^{ab}\phi^{ba} + \frac{1}{2}\bar{m}_V^2 V_\mu^2 - \bar{h}_V V_\mu \bar{\psi}^a \gamma_\mu \psi^a + \mathrm{i}\bar{h}_\phi \bar{\psi}^a \phi^{ab} \psi^b \Big]$$

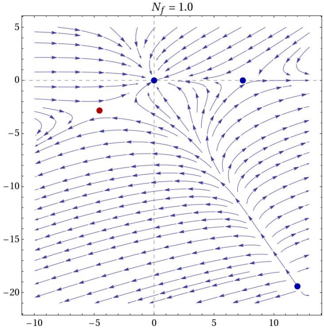
Preliminary results: RG flow in point-like limit

[Gies, LJ, work in progress]

Flow of ratios $(\tilde{g}, g) = -\frac{N_\mathrm{f}}{2}\left(\frac{h_\phi^2}{m_\phi^2}, 2\frac{h_V^2}{m_V^2} + \frac{h_\phi^2}{m_\phi^2}\right)$:

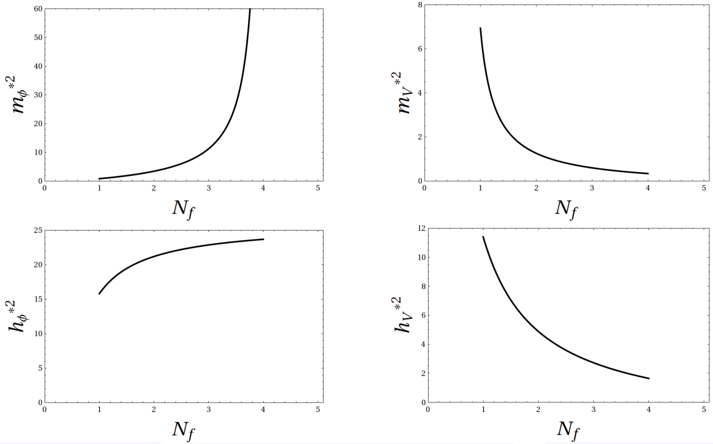


... not equivalent to fermionic formulation (Fierz ambiguity):



Preliminary results: Thirring fixed point in point-like limit

[Gies, LJ, work in progress]



$N_f \sim 1 : m_\phi^2 \ll m_V^2$

V loops freeze out

$\Rightarrow \langle \bar{\Psi}\Psi \rangle \neq 0$

$1 \ll N_f \ll 4 : m_\phi^2 \sim m_V^2$

competition: V vs. ϕ

$N_f \sim 4 : m_\phi^2 \gg m_V^2$

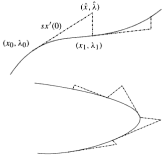
ϕ loops freeze out

$\Rightarrow \langle \bar{\Psi}\Psi \rangle = 0$

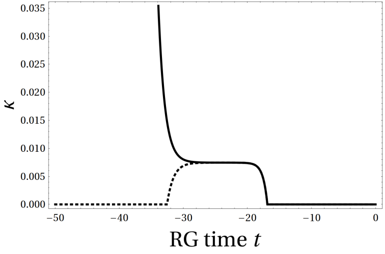
To-do list

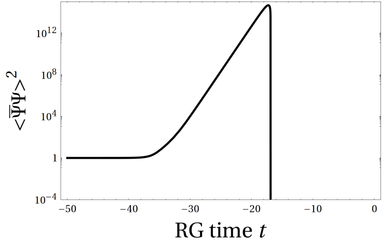
...in order to make a statement about N_f^{cr}:

1. Find "Thirring" fixed point beyond point-like limit:
   - Embed $\beta_i(g_j) = 0$ into class $\beta_{i,\alpha}(g_j) = 0$ such that
     - ★ $\alpha \to 0$: point-like limit
     - ★ $\alpha \to 1$: $\beta_{i,\alpha=1}(g_j) \equiv \beta_i(g_j)$
   - Continue point-like solution $g^*_{j,\alpha=0}$ to full solution $g^*_{j,\alpha=1}$
2. Integrate out RG flow
   - Start in the vicinity of "Thirring" fixed point
   - Compute chiral condensate $\langle\bar{\Psi}\Psi\rangle \propto \sqrt{\rho_{0,\mathrm{IR}}}$ as function of N_f

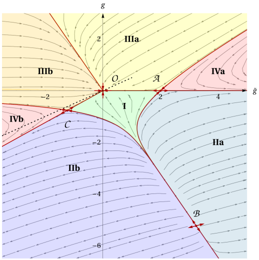


Example (FP in scalar-fermion sector):





Conclusions

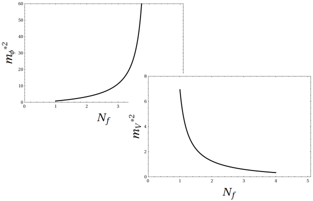


RG flow in point-like limit:

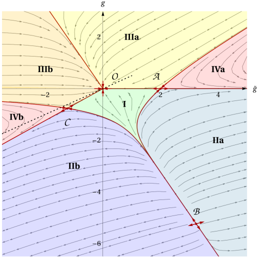
- Full basis of fermionic 4-point functions
- Non-perturbative renormalizability?
  - 2d coupling plane $\sim$ e.g. $(\bar{\Psi}\gamma_\mu\Psi)^2,\ (\bar{\Psi}\gamma_{45}\Psi)^2$
  - Justification for MC/DSE studies with $S_{\text{bare}} \sim g(\bar{\Psi}\gamma_\mu\Psi)^2$
- Mechanism for N_f^{cr}: competing condensation channels

RG flow for collective fields:

- Point-like limit shows competition: ϕ vs. V
- "Full" Thirring FP to be found



Conclusions



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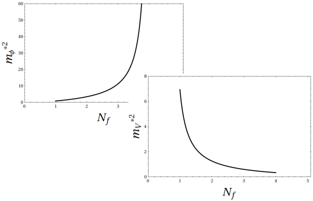

- Point-like limit shows competition: ϕ vs. V
- "Full" Thirring FP to be found

Outlook:

- Remove Fierz ambiguity: rebosonization
- Implications for graphene/cuprates ($N_{\text{f}} = 2$)?