Algebra of planar zero-mode operators

R. Jackiw and S.-Y. Pi
(MIT and BU)

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Bosons: particle $\neq \overline{\text{particle}}$ $\Rightarrow$ complex field

particle $= \overline{\text{particle}}$ $\Rightarrow$ real field

Fermions: particle $\neq \overline{\text{particle}}$ $\Rightarrow$ complex field

particle $= \overline{\text{particle}}$ $\Rightarrow$ real field (Majorana)

Dirac equation:

$$(\boldsymbol{\alpha}\cdot\mathbf{p} + \beta m)\,\Psi = i\;\partial_t\;\Psi \quad \left(\mathbf{p} = \frac{1}{i}\nabla\right)$$

Majorana: find matrices such that

$\boldsymbol{\alpha}$ is real: $\boldsymbol{\alpha}^* = \boldsymbol{\alpha}$, β is imaginary: $\beta^* = -\beta$

$\Rightarrow \Psi$ is real: $\Psi = \Psi^*$(Majorana representation)

More generally: $\boldsymbol{\alpha}^* = C^{-1}\,\boldsymbol{\alpha}\,C, \quad \beta^* = -C^{-1}\,\beta\,C, \quad \Psi = C\,\Psi^*$

e.g. Weyl representation

$$\boldsymbol{\alpha} = \begin{pmatrix} \boldsymbol{\sigma} & 0 \\ 0 & -\boldsymbol{\sigma} \end{pmatrix}, \quad \beta = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$$

$$C = C^\dagger = C^{-1} = \tilde{C} = C^* = \begin{pmatrix} 0 & -i\sigma_2 \\ i\sigma_2 & 0 \end{pmatrix}$$

Majorana constraint: $\Psi = C\Psi^*$

($C = I$ in Majorana representation)

Majorana Equation (2 component)

$$\begin{bmatrix} \boldsymbol{\sigma}\cdot\mathbf{p} & m \\ m & -\boldsymbol{\sigma}\cdot\mathbf{p} \end{bmatrix} \begin{bmatrix} \psi \\ \varphi \end{bmatrix} = i\partial_t \begin{bmatrix} \psi \\ \varphi \end{bmatrix}$$

but Majorana constraint $\Rightarrow$

$$\begin{pmatrix} \psi \\ \varphi \end{pmatrix} = \begin{pmatrix} 0 & -i\sigma_2 \\ i\sigma_2 & 0 \end{pmatrix} \begin{pmatrix} \psi^* \\ \varphi^* \end{pmatrix} \Rightarrow \varphi = i\sigma^2\psi^*$$

$$\boldsymbol{\sigma}\cdot\mathbf{p}\,\psi + m\,i\,\sigma^2\psi^* = i\partial_t\,\psi$$

NB. Dirac mass term: preserves quantum numbers (charge, particle number)

Majorana mass term: does not preserve any quantum numbers
$\Rightarrow$ no distinction between particle and anti-particle
since there are no conserved quantities to tell
them apart, particle is its own anti-particle

Dirac field operator

$$\Psi = \sum_E \left(a_E\, e^{-iEt}\,\Psi_E + b_E^\dagger\, e^{iEt}\, C\,\Psi_E^* \right)$$

Majorana field operator

$$\Psi = \sum_E \left(a_E\, e^{-iEt}\,\Psi_E + a_E^\dagger\, e^{iEt}\, C\,\Psi_E^* \right)$$

Are there Majorana fermions in Nature?

neutrinos?

– recent development in neutrino physics

experimental observation of neutrino oscillations $\Rightarrow$

- neutrinos have mass (< 0.1eV)
- lepton number is not conserved separately!

$\Rightarrow$ they could be Majorana fermions

Hypothetical Majorana fermions:

- supersymmetry – supersymmetric partners of photon, neutral Higgs boson, etc. are necessarily Majorana fermions
- cosmology – dark matter candidates

Majorana fermions in superconductor in contact with a topological insulator

superconductor

proximity effects ⇒ Cooper pairs

tunnel through to the surface of TI

topological insulator

Hamiltonian density for the model:

$$H = \psi^{*T}\ (i\,\boldsymbol{\sigma}\cdot\boldsymbol{\partial} - \mu)\ \psi + \frac{1}{2}\ (\triangle\psi^{*T}\,i\,\sigma_2\,\psi^* + h.c.)$$

$\psi = \begin{pmatrix} \psi_\uparrow \\ \psi_\downarrow \end{pmatrix}, \boldsymbol{\sigma} = (\sigma_1, \sigma_2), \mu$ is chemical potential and $\triangle$ is the order parameter that may be constant or takes vortex profile, $\triangle(r) = v(r)e^{i\theta}$.

Equation of motion: $$i\,\partial_t\,\psi = (\boldsymbol{\sigma}\cdot\mathbf{p} - \mu)\ \psi + \triangle\,i\,\sigma_2\,\psi^*$$

In the absence of μ, and with constant $\triangle$, the above system is a (2+1)-dimensional version of the (3+1)-dimensional, two component Majorana equation!

⇒ governs chargeless spin $\frac{1}{2}$ fermions with Majorana mass $|\triangle|$.

In the presence of a single vortex order parameter $\Delta(\mathbf{r}) = v(r)e^{i\theta}$ there exists a zero-energy (static) isolated mode
$\big($L. Fu and C. Kane, (2008); P. Rossi and RJ (1981)$\big)$

$$\psi_0 = N \begin{pmatrix} J_0(\mu r)\,\mathrm{exp}\,\{-i\pi/4 - V(r)\} \\ J_1(\mu r)\,\mathrm{exp}\,\{i(\theta + \pi/4) - V(r)\} \end{pmatrix}$$

N real constant, $V'(r) = v(r)$

Majorana field expansion:

$$\Psi = \underset{E\neq 0 \text{ modes}}{\ldots\ldots\ldots\ldots\ldots\ldots} \quad + a\,\Psi_0$$

where zero mode operator satisfies

$$\{a, a^\dagger\} = 1, a^\dagger = a \Rightarrow a^2 = 1/2$$

How to realize a on states?

Realizing a on states $\Rightarrow$ 2 possibilities
(C. Chamon, Y. Nishida, S.-Y. Pi, L. Santos, RJ; (2010))

(i) Two 1-dimensional realizations: take vacuum state to be eigenstate of a, with possible eigenvalue $\pm 1/\sqrt{2}$.

$$a\,|0\pm\rangle = \pm\,\frac{1}{\sqrt{2}}\;|0\pm\rangle$$

There are two ground states $|0+\rangle$ and $|0-\rangle$. Two towers of states are constructed by repeated application of $a_E^\dagger$. No operator connects the two.

Fermion parity is broken because a is a fermonic operator. Like in spontaneous breaking, a vacuum $|0+\rangle$ or $|0-\rangle$ must be chosen, and no tunneling connects to the other ground state.

(ii) One 2-dimensional realization: vacuum doubly degenerate $|1\rangle\,, |2\rangle$, and a connects the two vacua.

$$a\,|1\rangle = \tfrac{1}{\sqrt{2}}\;|2\rangle$$

$$a\,|2\rangle = \tfrac{1}{\sqrt{2}}\;|1\rangle$$

Two towers of states are constructed by repeated application of $a_E^\dagger$. a connects the towers. Fermion parity is preserved.

Which representation to choose?

Argument for 2-dimensional representation based on separated vortex/anti-vortex background: no zero modes but a state with small positive energy ε and another with equal but opposite energy $-\varepsilon$.

Field operator

$$\Psi = \underset{\text{finite E modes}}{\cdots\cdots} + A\, e^{-i\varepsilon t}\, \Psi_\varepsilon + A^\dagger\, e^{i\varepsilon t}\, C\, \Psi_\varepsilon^*$$

No ambiguity about states: two are low lying:

$$\text{vacuum } |\Omega\rangle, \text{ excited state } |\varepsilon\rangle$$

$$A\,|\Omega\rangle = 0 \quad A^\dagger\,|\Omega\rangle = |\varepsilon\rangle$$

$$A\,|\varepsilon\rangle = |\Omega\rangle \quad A^\dagger\,|\varepsilon\rangle = 0$$

As separation between vortex/anti-vortex becomes arbitrarily large:
$\varepsilon \to 0, \Psi_\varepsilon, C\,\Psi_\varepsilon^* \to \Psi_0$

$\Rightarrow$ last terms in Ψ become $(A + A^\dagger)\ \Psi_0$

$\Rightarrow a = A + A^\dagger$ acts on $|\Omega\rangle$ and on $|\varepsilon \to 0\rangle$

Two-state representation!

Matrix realization of two-state representation

$$|1\rangle = \begin{pmatrix}1\\0\end{pmatrix}, |2\rangle = \begin{pmatrix}0\\1\end{pmatrix}, \quad a = \frac{\sigma_1}{\sqrt{2}}$$

[The one-state representation would use same states for $|0\pm\rangle$
and the diagonal $\frac{\sigma_3}{\sqrt{2}}$ for a.]

How many states $\mathcal{N}$ for higher N?

Known for even N: $\mathcal{N} = 2^{N/2}$. Odd N? (unphysical ?) $(*)$

We verify even $-N$ formula and derive odd $-N$ formula by generalizing 2-dimensional, $N = 1$ case (we reject the 1-dimensional $N = 1$ realization)

$$\text{odd } N : \mathcal{N} = 2^{(N+1)/2} \ (= 2 \text{ for } N = 1) \quad (**)$$

$N = 2$: two zero mode operators: a and b

$a = a^\dagger, a^2 = \frac{1}{2}; b = b^\dagger, b^2 = \frac{1}{2}; ab + ba = 0$

use two states, as for $N = 1$, with

$a\,|1\rangle = \frac{\alpha}{\sqrt{2}}\ |2\rangle \quad a\,|2\rangle = \frac{1}{\sqrt{2}\alpha}\ |1\rangle$

$b\,|1\rangle = \frac{\beta}{\sqrt{2}}\ |2\rangle \quad b\,|1\rangle = \frac{1}{\sqrt{2}\beta}\ |1\rangle$

$\mid \alpha \mid = \mid \beta \mid = 1$ (because a, b are unitary, up to $\frac{1}{\sqrt{2}}$ factor)

anti-commutator requires

$\frac{\alpha}{\beta} + \frac{\beta}{\alpha} = 0 \Rightarrow \alpha = 1, \beta = i$

$2^{N/2} = 2$ for $N = 2$ verifies $(*)$

Matrix realization: $|1\rangle = \binom{1}{0}, |2\rangle = \binom{0}{1} \quad a = \frac{\sigma_1}{\sqrt{2}},\ b = \frac{\sigma_2}{\sqrt{2}}$

$N = 3$: three zero mode operators: a, b and c.

$$a = a^\dagger, \quad a^2 - \frac{1}{2}; \; b = b^\dagger, \quad b^2 = \frac{1}{2}; \quad c = c^\dagger, \quad c^2 = \frac{1}{2}$$

$$ab + ba = 0, \quad bc + cb = 0, \quad ca + ac = 0$$

try two states

$$a\,|1\rangle = \frac{\alpha}{\sqrt{2}} \;|2\rangle \qquad a\,|2\rangle = \frac{1}{\sqrt{2}\alpha} \;|1\rangle$$

$$b\,|1\rangle = \frac{\beta}{\sqrt{2}} \;|2\rangle \qquad b\,|2\rangle = \frac{1}{\sqrt{2}\beta} \;|1\rangle$$

$$c\,|1\rangle = \frac{\gamma}{\sqrt{2}} \;|2\rangle \qquad c\,|2\rangle = \frac{1}{\sqrt{2}\gamma} \;|1\rangle$$

norm: $\mid \alpha \mid = \mid \beta \mid = \mid \gamma \mid = 1$

anti-commutators require $\frac{\alpha}{\beta} + \frac{\beta}{\alpha} = 0, \frac{\beta}{\gamma} + \frac{\gamma}{\beta} = 0, \frac{\gamma}{\alpha} + \frac{\alpha}{\gamma} = 0 \;\Rightarrow$ inconsistent

also if we try $a \sim \sigma_1, b \sim \sigma_2, c \sim \sigma_3$ (to satisfy anti-commutators)

but σ_3 is diagonal and cannot be used if fermion parity is preserved

$\Rightarrow$ use four states and 4×4 realization

Dirac 4×4 matrices : $\boldsymbol{\alpha} = \begin{pmatrix} 0 & i\boldsymbol{\sigma} \\ -i\boldsymbol{\sigma} & 0 \end{pmatrix} \; \beta = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$

states: $|1\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \; |2\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \; |3\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \; |4\rangle = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$

Choose 3 of 4 Dirac matrices $\frac{1}{\sqrt{2}}(\boldsymbol{\alpha}, \beta)$. They square to $\frac{1}{2}$, anti-commute with each other, and act on the 4 basis vectors. Formula $(**)$ is verified for $N = 3$

$$\mathcal{N} = 2^{\frac{N+1}{2}} = 4 \text{ for } N = 3$$

Note: matrix $\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}$ also anti-commutes but cannot be used because it is diagonal, and would lead to fermion parity breaking.

$N = 4$

For $N = 4$ there are four zero modes and governing operators. We use 4-dimensional realization, with the operators represented by the four, 4×4 Dirac matrices already encountered for $N = 3 : \frac{1}{\sqrt{2}} (\boldsymbol{\alpha}, \beta)$. Similarly the states are the four Cartesian vectors introduced for $N = 3$. The formula $(*)$ is verified with $\mathcal{N} = N = 4$.

Higher N

The pattern is now clear. We use a Clifford algebra realized by $\mathcal{N} \times \mathcal{N}$ matrices and $\mathcal{N}$-component Cartesian vectors to represent N zero-mode operators acting on states.

In selecting the members of the Clifford algebra, diagonal elements (in the Cartesian basis) must not be used, because they correspond to fermion parity violating realizations.

Conclusions

We have shown that the fermion parity preserving realization leads to a number of states given by formulas $(*)$ and $(**)$ for even number and odd number vortices.

It is an interesting open question, what role, if any, should be assigned to the fermion parity violating realizations. Let me remind that fermion parity violation arises in supersymmetry (Shifman).