Geometry of the Fractional Quantum Hall effect

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- A new viewpoint on the Laughlin State leads to a quantitative description of incompressibility in the FQHE
- A marriage of Chern-Simons topological field theory with “quantum geometry”

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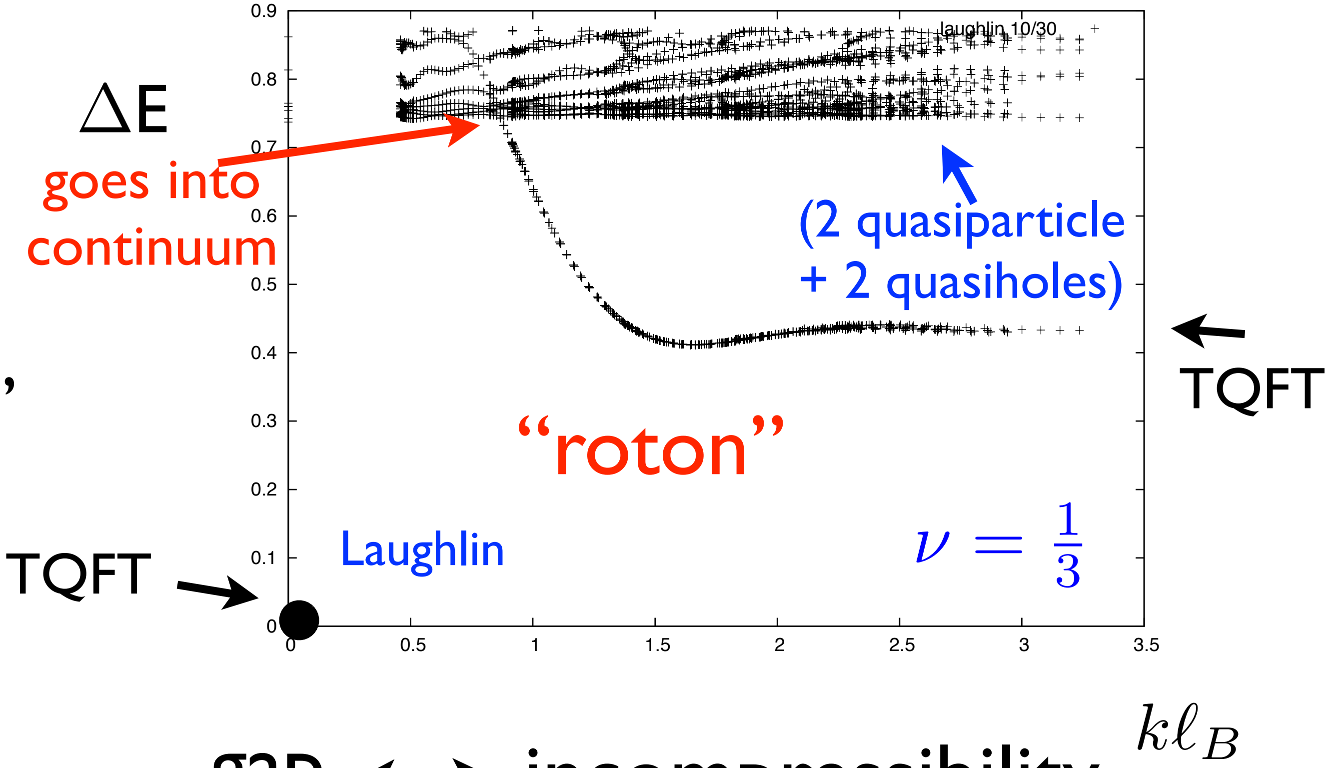
Geometry of the Fractional QHE

- For at least 20 years, most "theoretical" work (as opposed to numerical simulation) has been "topological" in character.

- Finite-size exact diagonalization (up to 20 particles) confirms that microscopic Hamiltonians exhibit incompressibility



gap ⟷ incompressibility

- Energy-scale analysis shows the gap must be of order

$$\Delta E \sim \frac{e^2}{4\pi\epsilon_0\epsilon\ell_B} \qquad \ell_B = \sqrt{\frac{\hbar}{|eB|}}$$

Topological Quantum Field Theory

- (Abelian) Chern-Simons theory

$$S = \int d^3x \, \frac{\hbar}{4\pi} K^{IJ} \epsilon^{\mu\nu\lambda} a_{I\mu} \partial_\mu a_{J\lambda} + \frac{e}{2\pi} t^I A_\mu \partial_\nu a_\lambda$$

- Choose gauge a_{I0} = 0

$$S = \int d^3x \, \tfrac{1}{2} \sigma_H \epsilon^{\mu\nu\lambda} A_\mu \partial_\nu A_\lambda + \boxed{\frac{\hbar}{4\pi} K^{IJ} \epsilon^{ab} a_{Ia} \partial_t a_{jb} + \frac{e}{2\pi} t^I \epsilon^{ab} A_a \partial_t a_{Ib}}$$

linear in time-derivative

$$\sigma_H = \frac{e^2}{2\pi\hbar} (t^I K^{-1}_{IJ} t^J)$$

K^{IJ} is an integer topological matrix, t^I is an integer vector

- Topological degeneracy on a 2-manifold of genus g

$$|\det K|^g$$

Topological quantum field theory

- The TQFT description of FQHE assumes the existence of incompressibility.
- It can classify the different types of topological excitations, and what happens when **external agents** move them around braided paths or and selection rules for fusing them.
- .TQFT models the systems with a Lagrangian that is **linear in time-derivatives.**
- The weak point of TQFT follows from this:

The Hamiltonian has the simple form:

$$H = 0$$

- TQFT is a fundamentally incomplete description of the FQHE
- It does not even know about the fundamental magnetic area $2\pi\ell_B^2 = (\Phi_0/B)$
- It does not describe the relative energies of the point-like topological excitations it classifies, just their electric charge and mutual statistics.

Origin of incompressibility?

Various "narratives" have been developed

- Ginzburg-Landau Chern-Simons theory suggests that it could be explained by a theory where CS flux is "attached" to Galilean-invariant non-relativistic particles with mass m.
- The composite fermion (CF) idea (Jain) says it arises because composite fermions fill "effective Landau levels". "Effective Hamiltonian Theory" (Shankar and Murthy) tries to implement this with an uncontrolled Ansatz.

CF method produces model wavefunctions for 2/5, 2/7, .. . series of FQHE states that can be used as successful variational states with numerically-evaluated energies, but provides no analytic insight.

The “narratives” appear to be “conforting make-believe stories” that reassure us we have some understanding, even if it is non-quantitative?

- These approaches have not been based on microscopic analysis, and attempt to solve the problem by appealing to superficially-plausible analogies
- Dimensional analysis suggests gaps must be related to

$$\Delta = \left(\frac{e^2}{4\pi\epsilon_0\epsilon}\right)\frac{1}{\ell_B}$$

We may hope for something more quantitative

- The clue to a quantitative approach is provided by seminal work of Girvin MacDonald and Platzman (1985)
- It was never properly interpreted and followed up, but turns out to be the only source of correct physics on which a microscopic picture can be based
- The new results presented here can be viewed as a translation of GMP into a "geometric field theory"

GMP:

$$\|v\| = \int \frac{d^2\boldsymbol{q}\ell_B^2}{2\pi}|v(\boldsymbol{q})|$$ Is finite

$$H = \int \frac{d^2\boldsymbol{q}\ell_B^2}{2\pi} v(\boldsymbol{q})\rho(\boldsymbol{q})\rho(-\boldsymbol{q})$$

Hamiltonian

$$\rho(\boldsymbol{q}) = \sum_{i=1}^{N} e^{i\boldsymbol{q}\cdot\boldsymbol{R}_i}$$ (needs regularization)

$$[R_i^a, R_j^b] = -i\ell_B^2\delta_{ij}$$

$$[\rho(\boldsymbol{q}), \rho(\boldsymbol{q}')] = 2i\sin(\tfrac{1}{2}\boldsymbol{q}\times\boldsymbol{q}'\ell_B^2)\rho(\boldsymbol{q}+\boldsymbol{q}')$$

Fundamental Lie algebra

Landau level filling

$$\delta\rho(\boldsymbol{q}) = \rho(\boldsymbol{q}) - 2\pi\nu\delta^2(\boldsymbol{q}\ell_B)$$ also obeys this algebra

- The one step missed by GMP (regularization)

  replace $\rho(\boldsymbol{q})$ by $\delta\rho(\boldsymbol{q})$ $\quad \frac{\mathrm{u}(\infty)}{\mathrm{u}(1)} = \mathrm{su}(\infty)$

$$\lim_{\lambda\to 0}\delta\rho(\lambda\boldsymbol{q}) = 0$$

- regularized generator of translations

$$P_a = \frac{\hbar}{\ell_B^2} \lim_{\lambda\to 0} \lambda^{-1} \epsilon_{ab} \nabla_q^b \delta\rho(\lambda \boldsymbol{q})$$

$$[P_a, P_b] = i\epsilon_{ab}\frac{\hbar^2}{\ell_B^2}\delta\rho(\mathbf{0}) = 0$$

Components commute after regularization!

(this solves a fundamental problem in taking the thermodynamic limit of the FQHE)

Translationally-invariant vacuum state

$$P_a|0\rangle = 0 \quad \langle 0|\delta\rho(\boldsymbol{q})|0\rangle = 0$$

$$\langle 0|\delta\rho(\boldsymbol{q})\delta\rho(\boldsymbol{q}')|0\rangle = 2\pi S(\boldsymbol{q})\delta^2(\boldsymbol{q}\ell_B - \boldsymbol{q}'\ell_B)$$

Guiding-center structure function: the importance of this correlation function is central to the theory of FQHE incompressibility

- correlation energy per magnetic area

$$\frac{E_0}{N_\Phi} = \int \frac{d^2\boldsymbol{q}}{2\pi} v(\boldsymbol{q}) S(\boldsymbol{q}) \qquad \frac{N}{N_\Phi} = \nu$$

- $S(\boldsymbol{q})$ is the ground-state structure factor, which describes the zero-point fluctuations of the ***R***$_i$

- make an area-preserving shear deformation of the ground state, i.e., of $S(\boldsymbol{q})$; the energy increase will be quadratic in the deformation

$$q_a \rightarrow \left(\delta_a^b + \lambda_{ac}\epsilon^{cb}\right) q_b \qquad \frac{\delta E}{N_\Phi} = \tfrac{1}{2} G^{abcd} \lambda_{ab} \lambda_{cd}$$

symmetric

guiding-center shear modulus

- Motivated by Feynman’s theory of the roton in ^{4}He, GMP used the single-mode approximation as a variational ansatz for the collective excitation:

$$|\Psi^{\mathrm{smp}}(\boldsymbol{q})\rangle = \rho(\boldsymbol{q})|\Psi_0^{\mathrm{exact}}\rangle \qquad \varepsilon(\boldsymbol{q}) = E(q) - E_0^{\mathrm{exact}} = \frac{A(\boldsymbol{q})}{S(\boldsymbol{q})}$$

$$A(\boldsymbol{q}) = \frac{1}{2}\int \frac{d^2\boldsymbol{q}'\ell_B^2}{2\pi} v(\boldsymbol{q}')\left(S(\boldsymbol{q}'+\boldsymbol{q}) + S(\boldsymbol{q}'-\boldsymbol{q}) - 2S(\boldsymbol{q}')\right)\left(2\sin\tfrac{1}{2}\boldsymbol{q}\times\boldsymbol{q}'\ell_B^2\right)^2$$

- GMP did not interpret A(q), but evaluated it numerically, using the Laughlin state S(q) obtained by Monte Carlo methods.

in fact

$$\lim_{\lambda\to 0} A(\lambda\boldsymbol{q}) \to \lambda^4 G^{abcd} q_a q_b q_c q_d \ell_B^4$$

- In the long wavelength limit, the GMP result can be written as
  $$\varepsilon^{\text{exact}}(\boldsymbol{q})S(\boldsymbol{q}) \leq G^{abcd}q_aq_bq_cq_d\ell_B^4$$
- This turns out to be an equality for systems with a single collective mode (single-component FQHE states, as well as Wigner-lattice states with one electron per unit cell)
- As GMP recognized, if the collective mode is gapped (i.e., the state is incompressible), S(q) must be quartic at long wavelengths. This was their fundamental insight into FQHE incompressibility.

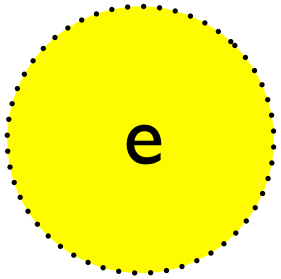
- From this we learn that the fundamental stiffness of the incompressible FQHE states is the their resistance to area-preserving distortions that changes the shape of the correlation hole around a guiding center from the shape that minimizes the energy.

- The collective degrees of freedom can be described as one (or more) UNIMODULAR positive definite real-symmetric spatial metric tensor fields

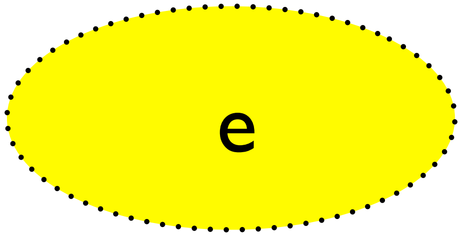
$$g_{ab}^{(\alpha)}(\boldsymbol{r},t) = \begin{pmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{pmatrix} \qquad \det g = 1$$

As a quadratic form, this describes a local "shape of a circle"

- Physically, it is the <u>shape</u> of the “attached flux” of the “composite bosons” that condense if the Chern-Simons “flux attachment picture:



region with 3 flux quanta surrounding the electron. Other electrons are excluded from this region (analogy is a Hubbard model lattice site)



area-preserving shape deformation of the exclusion region costs correlation energy

at 1/3 filling, an electron with 3 “attached” flux quanta behaves like a neutral boson

- The metric has non-commuting elements that fluctuate around a value

$$\langle 0|g_{ab}(\boldsymbol{r},t)|0\rangle = g^0_{ab}$$

- Charge fluctuations associated with this metric are given by

a "guiding-center spin" (2s = integer, topologically-quantized by Gauss-Bonnet in incompressible states)

elementary fractional charge e/q

$$\delta\rho(\boldsymbol{r},t) = \frac{e^*}{2\pi} s^\alpha K^\alpha(\boldsymbol{r},t)$$

Gaussian curvature

$$K = -\tfrac{1}{2}\partial_a\partial_b g^{ab} + \tfrac{1}{8}(\partial_a g^{ce})(\partial_b g^{df})\epsilon^{ab}\epsilon_{cd}g_{ef}$$

(Brioschi formula)

This is why S(q) ~ q^4

- A 2+1-d space-time metric is given by

$$g_{\mu\nu}(\boldsymbol{r},t)=\left(\begin{array}{ccc} g_{11}(\boldsymbol{r},t) & g_{12}(\boldsymbol{r},t) & 0 \\ g_{21}(\boldsymbol{r},t) & g_{22}(\boldsymbol{r},t) & 0 \\ 0 & 0 & g_{00} \end{array}\right)$$

- There is nothing that propagates on the Hall surface with speed c, and there is no Lorentz symmetry. Absolute simultaneity (unretarded Coulomb interaction) is allowed in the non-relativisic model.

- Wen and Zee (1992) provided the infrastructure necessary to construct the extension to the topological Lagrangian when they described coupling to the extrinsically-derived curvature of an embedded surface on which the electrons move
- In their case, the embedded surface in 3D Euclidean space and its normal are $\vec{\boldsymbol{r}}(x), \vec{\boldsymbol{n}}(x)$

$$g_{ab} = \partial_a \vec{\boldsymbol{r}} \cdot \partial_a \vec{\boldsymbol{r}}$$

$$\vec{n} \cdot \partial_a \vec{\boldsymbol{n}} \times \partial_b \vec{\boldsymbol{n}} = \epsilon_{ab} K \sqrt{(\det g)}$$

- Wen and Zee extended (Abelian) CS theory:

$$S = \int d^3x \frac{\hbar}{4\pi} K^{IJ} \epsilon^{\mu\nu\lambda} a_{I\mu} \partial_\nu a_{J\lambda} + \frac{e}{2\pi} \epsilon^{\mu\nu\lambda} t^I A_\mu \partial_\nu a_{i\lambda}$$

$$+ \frac{\hbar}{2\pi} s^I \epsilon^{\mu\nu\lambda} \Omega_\mu(g) \partial_\nu a_{I\lambda}$$

Coupling to curvature is a quantized spin

“spin connection”

$$\Omega^i_{\mu j} = \Omega_\mu \epsilon^{ik} \eta_{jk}$$

curvature gauge field

$$\epsilon^{\mu\nu\lambda} \partial_\nu \Omega_\lambda(g)$$

Gaussian curvature current (conserved)

- Wen and Zee treated the metric induced on the surface by the flat 3D Euclidean metric of the space in which the surface was embedded, with a local SO(2) isotropy of rotations around the normal.
- They though the (2D orbital "spin" was only meaningful if this rotational invariance was broken and the new features were only of formal interest for compactification of FQHE on a sphere ("shifts", etc.), and would not survive disorder. This turns out to be false: "Spin" is quantuzed here by Gauss-Bonnet (topological), not dependent on rotational invariance

- In the new interpretation, the metric(s) are tensor fields describing t**he shape fluctuations of the Composite bosons**.

- The physical (embedded) surface is flat. (atomically clean surfaces like graphene or eptaxially grown surfaces strongly resist Gaussian curvature of their physical shape).

- Happily, Gaussian curvature formulas are intrinsic, and do not depend on the physical origin of the metric.

$$S = \int d^3x \frac{\hbar}{4\pi} K^{IJ} \epsilon^{\mu\nu\lambda} a_{I\mu} \partial_\nu a_{J\lambda} + \frac{e}{2\pi} \epsilon^{\mu\nu\lambda} t^I A_\mu \partial_\nu a_{i\lambda}$$

$$+ \frac{\hbar}{2\pi} \sum_\alpha s^I_\alpha \epsilon^{\mu\nu\lambda} \Omega_\mu(g_\alpha) \partial_\nu a_{I\lambda} - \int dt\, H(\{g_\alpha\})$$

One metric and its spin vector for each independent "composite boson" in the multicomponent case.

At last!
a Hamiltonian!

- The Hamiltonian

$$H = \sum_{\alpha} \int \frac{d^2\boldsymbol{r}}{2\pi\ell_B^2} u_\alpha(g_\alpha)$$

correlation energy

$$+\tfrac{1}{2} \int d^2\boldsymbol{r} \sum_{\alpha\alpha'} \gamma_{\alpha\alpha'} K_\alpha K_{\alpha'}$$

quadratic in local curvatures

$$+\tfrac{1}{2} \int d^2\boldsymbol{r} \int d^2\boldsymbol{r}' V(\boldsymbol{r},\boldsymbol{r}')\delta\rho(\boldsymbol{r})\delta\rho(\boldsymbol{r}')$$

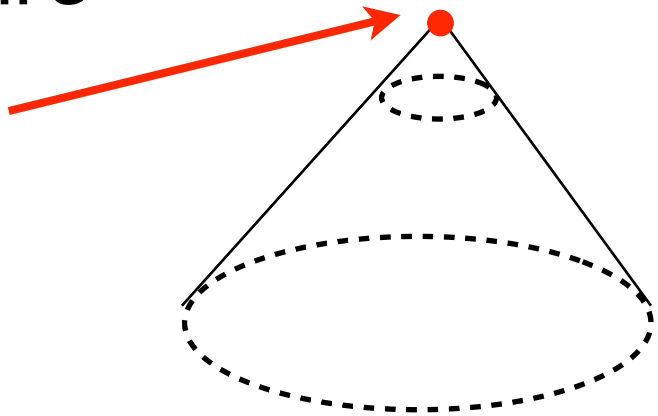
long-range unretarded Coulomb

charge

$$e^* = \pm\frac{e}{q}$$

Gaussian curvature

$$K = \pm\frac{4\pi}{2\bar{s}}$$



- charge e/q quasiparticles are rational cone singularities of guiding-center metric field

- competition between $u(g)$ and $V(r,r')$ will smooth out point singularity at "tip" of "cone"

  - $u(g)$ geometry-dependent correlation-energy
  - $V(r,r')$ Coulomb interaction potential

- This seems to reproduce all the basic phenomenology of incompressibility:
- For single CS field model (Laughlin-type), the Girvin-MacDonald-Platzman collective mode spectrum is reproduced at long wavelengths
- The Hall viscosity and Q^4 long wavelength behavior of the guiding-center structure factor S(Q) are reproduced
- quasiparticles/holes are rational cone singularities of the metric field
- Effective theory will allow calculation of the energies of different types of topological excitations in terms of its parameters

Back to "square one": rexamine the description of the FQHE

- electrons moving on a 2D surface with a magnetic flux density passing through it

$$\pi_{ia} = p_{ia} - eA_a(\boldsymbol{r}_i) \qquad \epsilon^{ab}\partial_a A_b(\boldsymbol{r}) = B$$

$$\boxed{H = \sum_i \frac{1}{2m} g^{ab} \pi_{ia}\pi_{ib} + \frac{1}{A}\sum_{\boldsymbol{q}} V(\boldsymbol{q}) \sum_{i<j} e^{i\boldsymbol{q}\cdot(\boldsymbol{r}_i - \boldsymbol{r}_j)}}$$

cyclotron
effective mass m

inverse of "unimodular"
Galileian metric
determines shape of
Landau orbit

$$\det g = 1$$

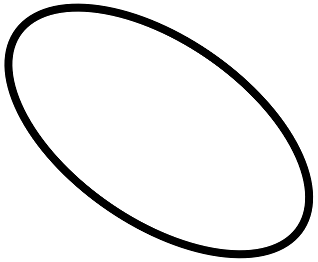
Fourier transform of two-body Coulomb interaction $V(\boldsymbol{q})$

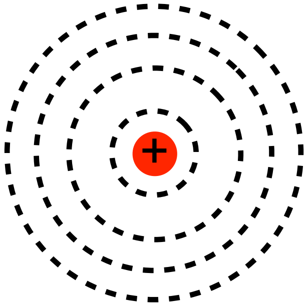
periodic boundary conditions on a region with area A

- what physical properties define a 2D spatial metric g_{ab} in this problem?
- a “UNIMODULAR metric” means $\det g$ = 1.
- a unimodular metric defines the “shape of a circle” $g_{ab}r^ar^b$ = const..

- **Two** distinct physical definitions of “the circle”:
  - The shape of the **Landau orbit** is defined by the unimodular “**Galileian metric”** (the effective mass tensor is proportional to it)
  - The shape of **Coulomb equipotentials around a point charge** is defined by the unimodular “**Coulomb metric**”.

Landau orbit





Coulomb equipotentials

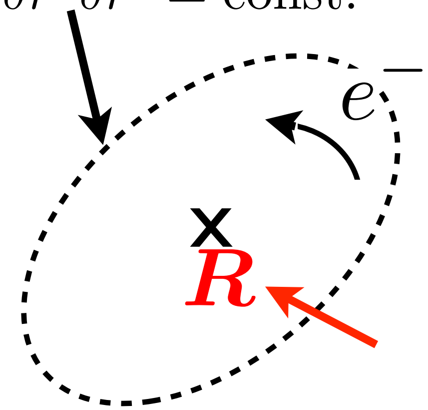
- Most work on the fractional Hall effect implicitly assumes that the "circles" defined by the Coulomb point charge equipotentials and the Landau orbits are congruent
- There is NO general reason for this to be true, one derives from the 3D dielectric tensor, the other from 2D bandstructure.

My claim: the "simplification" of treating the Coulomb and Galileian metrics as identical (which gives the system rotational invariance) has hidden a fundamental geometric property of the FQHE

decomposition into "guiding centers" and dynamical momenta

$m_{ab} = m g_{ab}$

$m_{ab}\delta r^a \delta r^b = \text{const.}$



$\pi_a = p_a - eA_a(\boldsymbol{r}) = m_{ab}v^b$

dynamical momentum in a magnetic field

"guiding center"

2D Landau orbits

$$\boxed{[\pi_a, R^b] = 0}$$

$r^a = R^a + \delta r^a \quad eB\delta r^a = \epsilon^{ab}\pi_b$

- Non-commutative geometry of guiding centers!

$$[R^a, R^b] = -i\ell_B^2\epsilon^{ab}$$

High-field limit

$$\hbar\omega_B \equiv \frac{\hbar^2}{m\ell_B^2} \gg \frac{1}{A}\sum_{\boldsymbol{q}} V(\boldsymbol{q}) f(\boldsymbol{q})^2$$

Landau energy

form-factor of lowest
Landau level
(depends on Galileian metric)

$$f(\boldsymbol{q}) = \exp -\tfrac{1}{4} g^{ab} q_a q_b \ell_B^2$$

$$[\pi_a, R^b] = 0$$

"left-handed" $\boldsymbol{\pi}$	"right-handed" $\boldsymbol{R}$
act in Hilbert space $\mathcal{H}_L$	act in Hilbert space $\mathcal{H}_R$
$[\pi_a, \pi_b] = i\epsilon_{ab}\frac{\hbar^2}{\ell_B^2}$	$[R^a, R^b] = -i\epsilon^{ab}\ell_B^2$

$$|\Psi\rangle \rightarrow |\Psi_0^L(g)\rangle \otimes |\Psi_\alpha^R\rangle$$

- in the high-field limit, the low-energy eigenstates of H become **unentangled products** of states in "left" and "right" Hilbert spaces

$$|\Psi_0^L(g)\rangle \otimes |\Psi_\alpha^R\rangle$$

- "Left variable" state is a trivial fully-symmetric coherent harmonic oscillator state that depends only on the Galilean metric

$$a_i(g)|\Psi_0^L(g)\rangle = 0, i = 1, \ldots, N$$

- "Right variable" state is a non-trivial eigenstate of

$$H_R = \frac{1}{2A}\sum_{\boldsymbol{q}} V(\boldsymbol{q}) f(\boldsymbol{q})^2 \rho_{\boldsymbol{q}} \rho_{-\boldsymbol{q}} \qquad \rho_{\boldsymbol{q}} = \sum_{i=1}^{N} e^{i\boldsymbol{q}\cdot\boldsymbol{R}_i}$$

$$[\rho_{\boldsymbol{q}}, \rho_{\boldsymbol{q}'}] = 2i\sin(\tfrac{1}{2}\boldsymbol{q} \times \boldsymbol{q}'\ell_B^2)\rho_{\boldsymbol{q}+\boldsymbol{q}'}$$

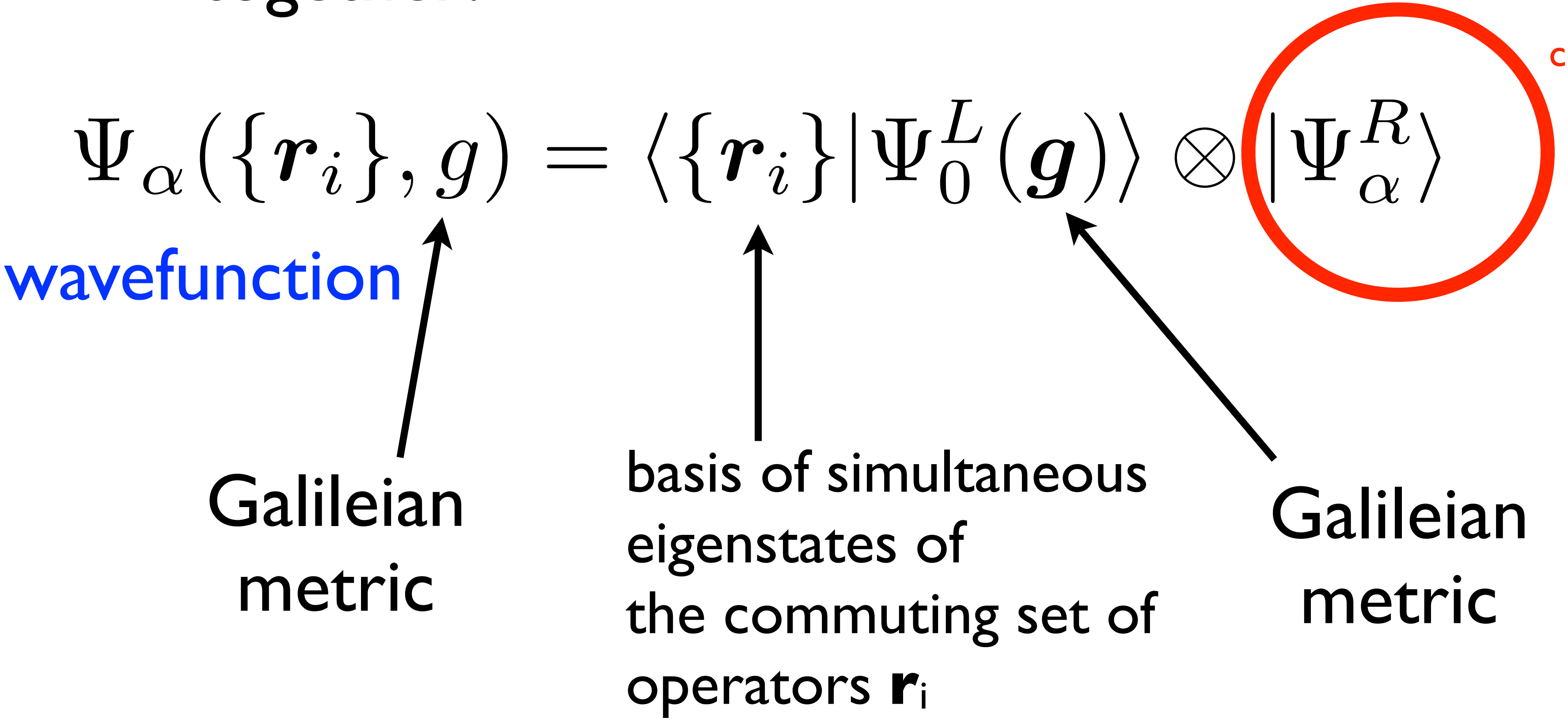
Quantum geometry

$$(\text{ Algebra } \mathcal{A}, \text{ Hilbert space } \mathcal{H}, \text{Hamiltonian } H \in \mathcal{A})$$

(the "triple" in Alain Connes definition of quantum geometry)

- Discard the trivial "left variables", and work only in the Hilbert space of the "right variables" (guiding centers). This makes numerical diagonalization tractable for finite N, N_Φ .
- **Without the "left variables", the notion of locality needed by classical geometry, and Schrödinger's formulation of quantum mechanics is lost!**

- <u>**Wavefunctions**</u> are only possible after "left" and "right" states are "glued" back together!



$$\Psi_\alpha(\{\boldsymbol{r}_i\}, g) = \langle\{\boldsymbol{r}_i\}|\Psi_0^L(\boldsymbol{g})\rangle \otimes |\Psi_\alpha^R\rangle$$

<u>This</u> is where the guiding center physics is!

wavefunction

Galilean metric

basis of simultaneous eigenstates of the commuting set of operators $\boldsymbol{r}_i$

Galilean metric

- The wavefunction depends on the Galilean metric g_{ab} in addition to $|\Psi^{\mathrm{R}}\alpha\rangle$
- **This means it is not a "pure" representation of $|\Psi^R\alpha\rangle$ because it involves extraneous elements**!
- The "pure" FQHE physics should only derive from guiding center physics

$$H_R = \frac{1}{2A}\sum_{\boldsymbol{q}} V(\boldsymbol{q}) f(\boldsymbol{q})^2 \rho_{\boldsymbol{q}}\rho_{-\boldsymbol{q}} \qquad \rho_{\boldsymbol{q}} = \sum_{i=1}^{N} e^{i\boldsymbol{q}\cdot\boldsymbol{R}_i}$$

$$[\rho_{\boldsymbol{q}}, \rho_{\boldsymbol{q}'}] = 2i\sin(\tfrac{1}{2}\boldsymbol{q}\times\boldsymbol{q}'\ell_B^2)\rho_{\boldsymbol{q}+\boldsymbol{q}'}$$

(+ Hilbert space)

- But.... a great part of successful FQHE theory is based on wavefunctions! (e.g. Laughlin)

$$\Psi(\{\boldsymbol{r}_i\}) = F(\{z_i\}) \prod_i e^{-\frac{1}{2} z_i^* z_i}$$

holomorphic

$$z_i = \frac{\omega_a(g) r_i^a}{\sqrt{2}\ell_B}$$

$$a_i = \frac{1}{2} z_i + \frac{\partial}{\partial z_i^*}$$

The definition of z_i depends on the Galileian metric

$$a_j \Psi(\{\boldsymbol{r}_i\}) = 0, \quad j = 1, \ldots, N$$

(lowest Landau level condition)

Laughlin wavefunction

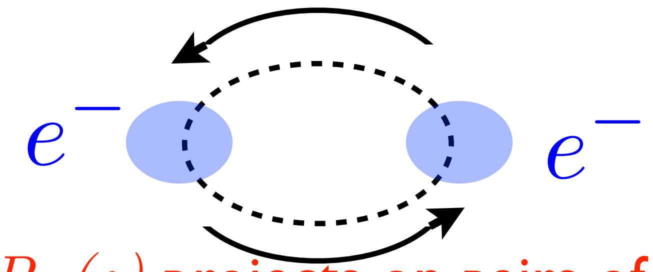
$$F_L^q(\{z_i\}) = \prod_{i<j}(z_i - z_j)^q$$

- It was initially proposed as a "trial wavefunction" with no continuously-variable variational parameters that achieved a much lower energy than Hartree-Fock approximations ("keeps particles apart better")
- The Laughlin wavefunction has q zeros as a function of z_i at each other coordinate z_j. Is this its defining property? (This was used when generalizing Laughlin to the torus (pbc)).

- I just argued that holomorphic wavefunctions were **not** “faithful” representations of quantum-geometric guiding-center states because they involve the Galileian metric
- Another definition of Laughlin state follows from “Haldane pseudopotentials”. At filling 1/q, it is the only zero-energy eigenstate of:

$$H_R(g) = \sum_{m=0}^{q-1} V_m P_m(g), \quad V_m > 0$$

$$P_m(g) = \frac{1}{2N_\Phi} \sum_{\boldsymbol{q}} L_m(q_g^2 \ell_B^2) e^{-\frac{1}{2} q_g^2 \ell_B^2} \rho_{\boldsymbol{q}} \rho_{-\boldsymbol{q}}$$



$P_m(g)$ projects on pairs of guiding centers with relative angular momentum m

Laguerre polynomial

argument depends on a metric!

$$q_g^2 = g^{ab} q_a q_b$$

(inverse) metric

- This defines a **family** of Laughlin states parametrized by a unimodular "guiding center metric" $\bar{g}_{ab}$ note the bar above g

$$P_m(\bar{g})|\Psi^q_{L\alpha}(\bar{g})\rangle = 0, \quad m = 0, 1, \ldots, q-1.$$

- The guiding center metric parametrizes an elliptic deformation of the shape of the "correlation hole" surrounding the electrons relative to the "circular" shape of the Landau orbits.
- The guiding center metric is NOT fixed by any one-particle physics, but should be viewed as a true variational parameter that is chosen to minimize the correlation energy!

$$H_R = \frac{1}{2A}\sum_{\boldsymbol{q}} V(\boldsymbol{q}) f(\boldsymbol{q})^2 \rho_{\boldsymbol{q}} \rho_{-\boldsymbol{q}}$$

depends on
Coulomb metric

depends on
Galileian metric

- If the Coulomb and Galileian metrics coincide, the correlation energy is lowest when the guiding-center metric (as a variational parameter) is equal to them both
- If they are not equal, the energy is minimized by choosing the guiding center metric intermediate between them.

- While TQFT can classify different "vortices" by electric charge and braiding statistics, it cannot say what their relative energies are, or how the process of fusion of two vortices proceeds.
- Till now, there has been no viable analytically tractable effective theory that can "explain" the origin of incompressibility, and its properties.
- Heuristic ideas include:
  - Analogy with superfluids (Ginzburg-Landau +Chern Simons)
  - Composite fermions fill "Landau levels" in analogy with the integer effect, where the Pauli principle explains incompressibility
- Unfortunately, these are "narratives" rather than tractable effective theories.

The Laughlin state: two different interpretations

- The wavefunction:

$$\Psi_L = \prod_{i<j} (z_i - z_j)^q \prod_i e^{-\frac{1}{2} z_i z_i^*}$$

$$a_i^\dagger = \tfrac{1}{2} z_i^* - \frac{\partial}{\partial z_i}$$ Landau-orbit raising operator

$$a_i = \tfrac{1}{2} z_i + \frac{\partial}{\partial z_i^*}$$ Landau-orbit lowering operator

$$a_i \Psi_L = 0$$ lowest-Landau-level condition

$$\Psi_L = \prod_{i<j} (z_i - z_j)^q \prod_i e^{-\frac{1}{2} z_i z_i^*}$$

- The narrative explanation of its success

“The Laughlin state places q zeroes of the wavefunction as a function of any z_i at the locations of every other particle. This keeps the particles apart, so lowers the correlation energy.”

The quantitative explanation

- introduce guiding center operators

$$\bar{a}_i^\dagger = \tfrac{1}{2}\bar{z}_i^* - \frac{\partial}{\partial \bar{z}_i}$$

$$\bar{a}_i = \tfrac{1}{2}\bar{z}_i + \frac{\partial}{\partial \bar{z}_i^*}$$

Guiding centers

$$a_i^\dagger = \tfrac{1}{2}z_i^* - \frac{\partial}{\partial z_i}$$

$$a_i = \tfrac{1}{2}z_i + \frac{\partial}{\partial z_i^*}$$

Landau orbits

Conventional choice: $\bar{z}_i = z_i^*$

- The Laughlin state can be rewritten as

$$\Psi_L = \prod_{i<j} \left(\bar{a}_i^\dagger - \bar{a}_j^\dagger \right)^q \Psi_0 \quad \Psi_0 = \prod_i e^{-\frac{1}{2} z_i \bar{z}_i}$$

$$a_i \Psi_0 = 0 \quad \bar{a}_i \Psi_0 = 0$$

- If , for any i,j this state is expanded in eigenstates of relative guiding-center angular momentum

$$L_{ij} = \tfrac{1}{2}(\bar{a}_i^\dagger - \bar{a}_j^\dagger)(\bar{a}_i - \bar{a}_j) = 0, 1, 2, \ldots$$

No pair of particles has $L_{ij} < q$

This is a fundamentally "Heisenberg" description of the Laughlin state formulated completely in terms of guiding-center physics, unlike the previous "Schrodinger" one

$\bar{z}$ does **not** have to be z^* !

- The guiding center geometry is fixed by the geometry of Coulomb equipotentials of a point charge, which do not have to be congruent to the shape of the Landau orbits
- The second (but not the original) definition of the Laughlin state remains valid if

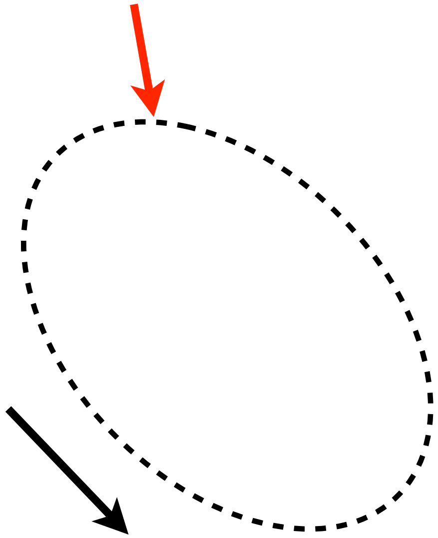
$$\begin{pmatrix} \bar{z} \\ \bar{z}^* \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \beta^* & \alpha^* \end{pmatrix} \begin{pmatrix} z^* \\ z \end{pmatrix}$$

$$\alpha^*\alpha - \beta^*\beta = 1$$

(Bogoliubov SU(1,1) transformation)

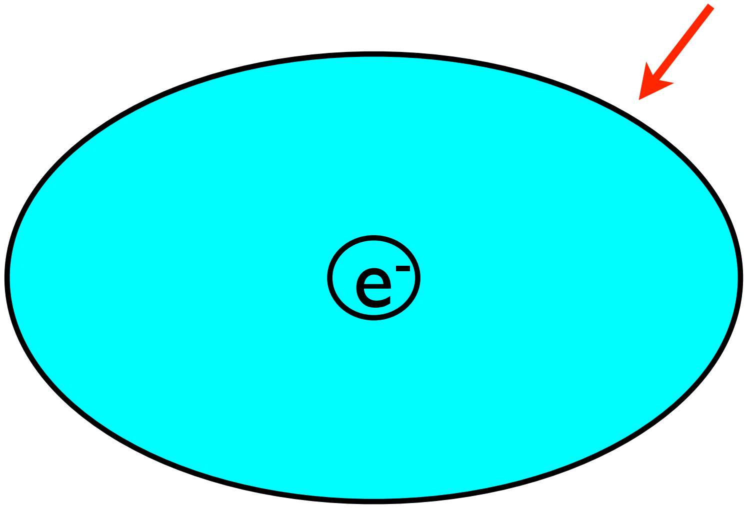
Physically

$|z| = \text{constant}$



shape of the
Landau orbit

$|\bar{z}| = \text{constant}$



Shape of the "excluded region" of q flux quanta surrounding each electron

The shape of the excluded region will self-select to minimize the correlation energy, but can fluctuate about that shape

The quantum geometry

- The shape $|\bar{z}|$ = constant is defined by a UNIMODULAR 2D spatial metric $\bar{g}_{ab}(\boldsymbol{r}, t), \quad \det \bar{g} = 1$

- Its Gaussian curvature is the curl of a DYNAMICAL spin-connection gauge field analogous to the STATIC spin-connection described by Wen and Zee (1992) in their treatment of FQHE on a static extrinsically-curved surface (like the sphere used in numerical diagonalization)

- Here the surface is flat, the curved metric is NOT the induced metric from 3D Euclidean space, as in Wen and Zee

Some key results

- The filling factor p/q and the elementary fractional charge e* = e/q are joined by a third topological parameter, the GUIDING CENTER SPIN, s, which is quantized to integer or half-integer in an incompressible state by Gauss-Bonnet.
- Charge density

$$\rho_e(\boldsymbol{r}) = \sigma_H B(\boldsymbol{r}) + \frac{e^* s}{2\pi} K^g(\boldsymbol{r})$$

$$\sigma_H = \frac{pe^2}{qh}$$

Gaussian curvature of $g_{ab}(\boldsymbol{r})$

- leading term in Gaussian curvature is second derivative of metric. The zero-point fluctuations of the metric naturally reproduces the k^4 behavior of the structure function at long-wavelength found by Girvin MacDonald and Platzman.
- The long-wavelength energy gap agrees quantitatively with GMP, in terms of the deformation stiffness of the state.
- quasiparticles are "cone singularities" of the metric field with rational quantized curvature in units $2\pi/s$

- Core structures of quasi particles are calculable in terms of the shape-dependent correlation energy and the effective interaction between charge fluctuations (Gaussian curvature fluctuations).
- The multicomponent case seems to have an independent metric field for each component

Conclusion: the effective theory of FQHE is finally revealed as a marriage of Chern-Simons topology with (2D spatial) quantum geometry.