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Magnetic pairing

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Abstract

The Dirac field in 2+1 dimension in presence of a constant magnetic field is an exactly solvable problem. If analysed in terms of the degrees of freedom of the free field a pairing structure emerges for any value of the mass, as shown long ago in a paper with Francesca Marchetti. This provides an explanation for chiral symmetry breaking in NJL models in presence of a magnetic field and very weak nonlinearity. Such a phenomenon, studied by Gusynin, Miransky and Shovkovy and called magnetic catalysis, has acquired a novel interest in connection with the physics of graphene.

NJL model

The Lagrangian of the NJL model is

$$\mathcal{L} = -\bar{\psi}\gamma_\mu\partial_\mu\psi + g\left[(\bar{\psi}\psi)^2 - (\bar{\psi}\gamma_5\psi)^2\right]. \tag{1}$$

It is invariant under ordinary and chiral gauge transformations

$$\psi \to e^{i\alpha}\psi, \qquad \bar{\psi} \to \bar{\psi}e^{-i\alpha} \tag{2}$$

$$\psi \to e^{i\alpha\gamma_5}\psi, \qquad \bar{\psi} \to \bar{\psi}e^{i\alpha\gamma_5}. \tag{3}$$

A simple mean field approximation gives the equation for the mass

$$m = 2g[< \bar{\psi}\psi > -\gamma_5 < \bar{\psi}\gamma_5\psi >] \tag{4}$$
$$= -2g[trS^{(m)}(0) - tr\gamma_5 S^{(m)}(0)], \tag{5}$$

where $S^{(m)}$ is the propagator of the Dirac field of mass m, or more explicitely,

$$\frac{2\pi^2}{g\Lambda^2} = 1 - \frac{m^2}{\Lambda^2} \ln \left(1 + \frac{\Lambda^2}{m^2}\right), \tag{6}$$

where Λ is the invariant cut-off. This equation is very similar to the gap equation in BCS theory. If $\frac{2\pi^2}{g\Lambda^2} < 1$ there exists a solution $m > 0$.

The vacuum

$$|m>=\prod_{p,s}\{[\frac{1}{2}(1+\beta_p)]^{\frac{1}{2}}-[\frac{1}{2}(1-\beta_p)]^{\frac{1}{2}}a^{(0)\dagger}_{p,s}b^{(0)\dagger}_{-p,s}\}|0> \qquad (7)$$

where $\beta_p=\frac{|p|}{E_p}$, and $E_p=(p^2+m^2)^{\frac{1}{2}}$. The operators acting on this vacuum for a particle or an antiparticle of mass m are related to the zero mass operators by a Bogolyubov transformation

$$a^{(m)}_{p,s}=[\frac{1}{2}(1+\beta_p)]^{\frac{1}{2}}a^{(0)}_{p,s}+[\frac{1}{2}(1-\beta_p)]^{\frac{1}{2}}b^{(0)\dagger}_{-p,s} \qquad (8)$$

$$b^{(m)}_{p,s}=[\frac{1}{2}(1+\beta_p)]^{\frac{1}{2}}b^{(0)}_{p,s}-[\frac{1}{2}(1-\beta_p)]^{\frac{1}{2}}a^{(0)\dagger}_{-p,s} \qquad (9)$$

Magnetic catalysis

V. P. Gusynin, V. A. Miransky and I. A. Shovkovy, *Phys. Rev. D.* **52**, 4718 (1995); *Phys. Rev. D.* **52**, 4747 (1995); *Nucl. Phys.* **B462**, 249 (1996).

NJL models, where SSB of chiral symmetry takes place for the nonlinear coupling over a certain threshold, in presence of a magnetic field exhibit SSB for any value of the coupling both in $2+1$ and $3+1$ dimensions. This phenomenon is called *magnetic catalysis*. In $2+1$ they found

$$m_{\text{dyn}}^2 = |eB| \frac{N_c^2 g^2 |eB|}{4\pi^2} \ , \tag{10}$$

where N_c is the number of fermion colors, and

$$m_{\text{dyn}}^2 = \frac{|eB|}{\pi} \exp\left(- \frac{4\pi^2(1-G)}{|eB| N_c g} \right) \ , \tag{11}$$

where $G \equiv N_c g \Lambda^2/(4\pi^2)$, in (3+1) dimensions.

The explanation provided by these authors emphasizes a dimensional reduction taking place in this phenomenon

"The essence of the present effect is that in a constant magnetic field, the dynamics of fermion pairing is one-dimensional: the pairing takes place essentially for fermions in the (degenerate) lowest Landau level.".

We thought that relating the operators in presence of a magnetic field to those in absence and calculating the new vacuum could clarify further the phenomenon. It turned out that this problem can be solved exactly by Bogolyubov transformations.

The Dirac field in $2+1$ dimensions

In the following we use units with $\hbar = c = 1$ where c is the velocity of light, as their values do not play a role in our discussion. Consider the Lagrangian $\mathcal{L}$:

$$\begin{aligned} \mathcal{L} &= \bar{\psi}^B(x)\left[\gamma^\mu \mathcal{D}_\mu - m\right]\psi^B(x) \\ \mathcal{D}_\mu &= \partial_\mu - eA_\mu \ , \end{aligned} \tag{12}$$

where $\bar{\psi}^B(x)$ is the Dirac quantized field in presence of the vector potential A_μ and e is the modulus of the electron charge. Since the magnetic field is constant and homogeneous we can choose the Landau gauge:

$$A_\mu = -\delta_{\mu 1} B x_2 \ . \tag{13}$$

Notice that introducing the magnetic length $l = 1/(eB)^{1/2}$ and using it as a unit for the space time variables, the magnetic field can be rescaled to the value 1.

In $2+1$ dimensions there are two inequivalent minimal versions of the Dirac algebra

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} \qquad \mu = 0,1,2$$
$$\gamma^{0\dagger} = \gamma^0 \qquad \gamma^{i\dagger} = -\gamma^i \quad i = 1,2\ .$$

We shall use the representations

$$\widetilde{\gamma}^0 = \sigma_3 \qquad \widetilde{\gamma}^1 = i\sigma_1 \qquad \widetilde{\gamma}^2 = i\sigma_2$$

$$\widehat{\gamma}^0 = -\sigma_3 \qquad \widehat{\gamma}^1 = -i\sigma_1 \qquad \widehat{\gamma}^2 = -i\sigma_2$$

where σ_i are the Pauli matrices

A *chiral* version is obtained as the direct sum of the two inequivalent representations

$$\gamma^\mu = \begin{pmatrix} \widetilde{\gamma}^\mu & 0 \\ 0 & \widehat{\gamma}^\mu \end{pmatrix} = \begin{pmatrix} \widetilde{\gamma}^\mu & 0 \\ 0 & -\widetilde{\gamma}^\mu \end{pmatrix} \qquad \psi(x) = \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \end{pmatrix} .$$

The lagrangian in the chiral version is connected to the lagrangians in the minimal versions by

$$\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2 \qquad \mathcal{L}_i = \bar{\psi}_i^B(x) \left[i\widetilde{\gamma}^\mu \mathcal{D}_\mu - \alpha_i m \right] \psi_i^B(x)$$

$$\alpha_i = \begin{cases} 1 & i = 1 \\ -1 & i = 2 \end{cases} \qquad \bar{\psi}_i^B(x) = \psi_i^{B\dagger}(x)\sigma_3 \qquad \forall i = 1, 2 .$$

The mass term has opposite sign in the two versions

Introducing the following anticommuting matrices

$$\gamma^3 = i \begin{pmatrix} 0 & \mathbb{I} \\ \mathbb{I} & 0 \end{pmatrix} \qquad \gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = i \begin{pmatrix} 0 & \mathbb{I} \\ -\mathbb{I} & 0 \end{pmatrix} . \tag{14}$$

in the chiral version we can introduce a $U(2)$ symmetry group

$$U(\omega) \in U(2): \; U(\omega) = e^{i\omega^\alpha T_\alpha} \qquad \alpha = 0, ..., 3 \tag{15}$$

$$T_0 = \mathbb{I} \qquad T_1 = \gamma^5 \qquad T_2 = -i\gamma^3 \qquad T_3 = \gamma^3\gamma^5 . \tag{16}$$

which is broken by the mass term. The symmetry is not recovered in the limit $m \to 0$. This is revealed by

$$\lim_{m\to 0} \langle B|\bar{\psi}^B(x)\psi^B(x)|B\rangle = -\frac{|eB|}{2\pi} \tag{17}$$

The pairing structure of the vacuum induced by a magnetic field in $2+1$-dimensional Dirac field theory

G. Jona-Lasinio, F. M. Marchetti, Phys. Lett. B **459**, 208 (1999).

The free Dirac field minimally coupled to a homogeneous magnetic field can be written

$$\psi^B(x) = \begin{pmatrix} \psi_1^B(x) \\ \psi_2^B(x) \end{pmatrix} \tag{18}$$

$$\psi_1^B(x) = \sum_{n=0}^{\infty} \sum_{p_1} \{u_{np_1}(x) a_{np_1} + v_{n-p_1}(x) b^\dagger_{np_1}\} \tag{19}$$

$$\psi_2^B(x) = \sum_{n=0}^{\infty} \sum_{p_1} \{u^{(2)}_{np_1}(x) c_{np_1} + v^{(2)}_{n-p_1}(x) d^\dagger_{np_1}\} \tag{20}$$

$$
\begin{aligned}
u_{np_1}(x) &= \frac{1}{\sqrt{lL_1}} e^{-E_n t} e^{p_1 x_1} \begin{pmatrix} A_n w_n(\xi_{x_2}^{p_1}) \\ -B_n w_{n-1}(\xi_{x_2}^{p_1}) \end{pmatrix} \\
v_{np_1}(x) &= \frac{1}{\sqrt{lL_1}} e^{+E_n t} e^{p_1 x_1} \begin{pmatrix} B_n w_n(\xi_{x_2}^{p_1}) \\ +A_n w_{n-1}(\xi_{x_2}^{p_1}) \end{pmatrix}
\end{aligned} \tag{21}
$$

$$
u_{np_1}^{(2)}(x) = (-1)^n v_{n-p_1}(-x) \quad v_{np_1}^{(2)}(x) = (-1)^n u_{n-p_1}(-x) \tag{22}
$$

$$
A_n = \sqrt{\frac{E_n + m}{2E_n}} \qquad B_n = \sqrt{\frac{E_n - m}{2E_n}} \tag{23}
$$

$$
E_n = \sqrt{m^2 + 2neB} \qquad \xi_{x_2}^{p_1} = \frac{x_2}{l} + lp_1 = \sqrt{eB}x_2 + \frac{p_1}{\sqrt{eB}} \tag{24}
$$

$$
w_n(\xi) = c_n e^{-\xi_2/2} H_n(\xi) = \left(2^n n! \sqrt{\pi}\right)^{-1/2} e^{-\xi^2/2} H_n(\xi) \ . \tag{25}
$$

The operators $a_{np^1}, b_{np^1}, \ldots$ satisfy the usual canonical anticommutation relations and $H_n(\xi)$ are the Hermite polynomials.

One of the main properties of the theory in the minimal versions is the spectrum asymmetry of the lowest level (called lowest Landau level LLL). For example if $m > 0$ and $eB > 0$ the energy spectrum is

$$E = \pm\sqrt{m^2 + (2n + 1 - \sigma)eB} \qquad \begin{cases} \sigma = +1 & \text{if } E > 0 \\ \sigma = -1 & \text{if } E < 0 \end{cases}$$

The situation is inverted for the second minimal version

The problem solved thirteen years ago was the calculation of the formal relationship between the Dirac field in presence of magnetic field and the free Dirac field,

$$\psi_1(x) = \sum_{\boldsymbol{p}} \sqrt{\frac{m}{L_1 L_2 E_{\boldsymbol{p}}}} \left\{ u(\boldsymbol{p}) e^{-p\cdot x} a_{\boldsymbol{p}} + v(\boldsymbol{p}) e^{p\cdot x} b^\dagger_{\boldsymbol{p}} \right\} \tag{26}$$

$$u(\boldsymbol{p}) = \sqrt{\frac{E_{\boldsymbol{p}} + m}{2m}} \begin{pmatrix} 1 \\ \frac{p_2 - p_1}{E_{\boldsymbol{p}} + m} \end{pmatrix} \quad v(\boldsymbol{p}) = \sqrt{\frac{E_{\boldsymbol{p}} + m}{2m}} \begin{pmatrix} \frac{p_2 + p_1}{E_{\boldsymbol{p}} + m} \\ 1 \end{pmatrix} , \tag{27}$$

where $E_{\boldsymbol{p}} = (m^2 + |\boldsymbol{p}|^2)^{1/2}$. Similarly for ψ_2.
Following NJL we obtained the desired relationship by imposing the same initial condition on the Dirac equations describing the free field and the one in presence of the external magnetic field,

$$\psi_i(0, x) = \psi_i^B(0, x) \quad i = 1, 2 \; . \tag{28}$$

Relationship between plane waves and Hermite functions

To connect the $a_{np^1}, b_{np^1}, \ldots$ operators in presence of the magnetic field with the $a_{\mathbf{p}}, b_{\mathbf{p}}$ of the free field, we need the relationship between plane waves and Hermite functions

$$e^{ip_2x_2} = e^{-il^2p_1p_2}\sqrt{2\pi}\sum_{n=0}^{\infty}(i)^n w_n(\xi_{x_2}^{p_1})w_n(lp_2) \tag{29}$$

$$w_n(\xi_{x_2}^{p_1}) = \frac{\sqrt{2\pi}(-i)^n l}{L_2}\sum_{p_2} w_n(lp_2)e^{ip_2x_2}e^{il^2p_1p_2}\ . \tag{30}$$

which follow by formal analytic continuation to $t = i$ from

$$\sqrt{\pi}\sum_{n=0}^{\infty} t^n w_n(x)w_n(y) = \frac{1}{\sqrt{1-t^2}}e^{\frac{x^2-y^2}{2}-\frac{(x-yt)^2}{1-t^2}} \quad |t| < 1 \tag{31}$$

Recall that $\xi_{x_2}^{p_1} = \frac{x_2}{l} + lp_1$.

Constructing the Hilbert space

There is a natural ambient space for the construction of the vacuum of the Dirac field in $2+1$ dimensions with constant magnetic field. Let us define

$$A_p = \sqrt{\frac{E_p + m}{2Ep}}(a_p - C_p b^\dagger_{-p}) \tag{32}$$

$$B_p = \sqrt{\frac{E_p + m}{2Ep}}(C_p a^\dagger_p + b_{-p}) \ , \tag{33}$$

with

$$C_{\boldsymbol{p}} = \frac{p_2 + ip_1}{E_{\boldsymbol{p}} + m} \qquad . \tag{34}$$

The operators $A_p, A^\dagger_p$ and $B_p, B^\dagger_p$ satisfy the usual canonical anticommutation relations (CAR).

The corresponding vacuum is

$$|\hat{0}\rangle = \prod_{\boldsymbol{h}} \frac{E_{\boldsymbol{h}} + m}{2E_{\boldsymbol{h}}} \left(1 + C_{\boldsymbol{h}} a^{\dagger}_{\boldsymbol{h}} b^{\dagger}_{-\boldsymbol{h}}\right) |0\rangle \ . \qquad (35)$$

By applying the operators $A^{\dagger}_{p}, B^{\dagger}_{p}$ to $|\hat{0}\rangle$ we generate a new Fock space where a first pairing appears. Notice that pairs carry a phase which depends on the momentum.

We next introduce the creation and destruction operators

$$\begin{aligned} \widehat{a}_{np_1} &= A(f_{np_1}) = \sum_{p_2} f_{np_1}(p_2) A_p \\ \widehat{b}_{n-p_1} &= B(-if_{n-1\;-p_1}) = \sum_{p_2} -if_{n-1\;-p_1}(p_2) B_p \;, \end{aligned} \tag{36}$$

where

$$f_{np_1}(p_2) = i^n \sqrt{2\pi l} e^{-l^2 p_1 p_2} w_n(lp_2) \;, \tag{37}$$

The operators a_{np_1}, b_{np_1} describing Dirac particles in a magnetic field are finally given by the Bogolyubov transformation

$$\begin{cases} a_{np_1} = A_n \widehat{a}_{np_1} - B_n \widehat{b}^\dagger_{n-p_1} \\ b_{n-p_1} = A_n \widehat{b}_{n-p_1} + B_n \widehat{a}^\dagger_{np_1} \ . \end{cases} \tag{38}$$

A similar analysis can be done for ψ_2^B introducing operators $\widehat{c}_{np_1}$ and $\widehat{d}_{np_1}$

The complete expression of the vacuum

$$|B\rangle = \prod_{n\geq 1}\prod_{p_1}\left(A_n + B_n \widehat{a}^{\dagger}_{np_1}\widehat{b}^{\dagger}_{n-p_1}\right)\left(A_n - B_n \widehat{c}^{\dagger}_{np_1}\widehat{d}^{\dagger}_{n-p_1}\right)|\hat{0}\rangle \ , \tag{39}$$

where,

$$|\hat{0}\rangle = \prod_{\boldsymbol{h}} \frac{E_{\boldsymbol{h}} + m}{2E_{\boldsymbol{h}}}\left(1 + C_{\boldsymbol{h}} a^{\dagger}_{\boldsymbol{h}} b^{\dagger}_{-\boldsymbol{h}}\right)\left(1 + C^{*}_{\boldsymbol{h}} c^{\dagger}_{\boldsymbol{h}} d^{\dagger}_{-\boldsymbol{h}}\right)|0\rangle \ . \tag{40}$$

We emphasize that this expression holds for any value of m. By applying the conjugate of the operators defined in (38), and the corresponding ones associated to ψ_2, to the vacuum $|B\rangle$ we generate the full Hilbert space of the Dirac field in a magnetic field. A simple calculation shows that $\langle 0|\hat{0}\rangle = 0$ and $\langle\hat{0}|B\rangle = 0$. Therefore three Fock spaces orthogonal to each other are involved in the diagonalization of the Hamiltonian of a Dirac field in presence of a constant magnetic field in $2+1$ dimensions.

In order to clarify the meaning of the auxiliary operators let us write the hamiltonian in presence of magnetic field in terms of these operators

$$\mathsf{H}^B = \widehat{\mathsf{H}} - \sum_{n=1}^{\infty} \sum_{p_1} \sqrt{2neB} \left(\widehat{a}^{\dagger}_{np_1} \widehat{b}^{\dagger}_{n-p_1} + \widehat{b}_{n-p_1} \widehat{a}_{np_1} \right)$$
$$\widehat{\mathsf{H}} \equiv m \sum_{n=0}^{\infty} \sum_{p_1} \left(\widehat{a}^{\dagger}_{np_1} \widehat{a}_{np_1} + \widehat{b}^{\dagger}_{np_1} \widehat{b}_{np_1} \right) \quad . \tag{41}$$

The hamiltonian $\hat{H}$ describes auxiliary particles with degenerate energy m, that is the energy of the lowest Landau level. The magnetic field induces creation and destruction of auxiliary particle-antiparticle pairs and this removes the degeneracy giving the usual Landau levels.

It is easy to see that $\widehat{H}$ does not depend on the magnetic field B. Using the completeness of Hermite functions it can be rewritten as $m \sum_p (A_p^\dagger A_p + B_p^\dagger B_p)$. A decomposition similar to (41) holds for the free Hamiltonian of the massive field in NJL models without magnetic field: the creation and destruction operators of the massless field play the role of the hatted operators and the mass replaces the magnetic field in the quadratic interaction.

Conclusion

The magnetic catalysis appears as a special consequence of the more general phenomenon represented by magnetic pairing. A characteristic feature of SSB with a mechanism akin to supeconductivity is a pairing of particles and antiparticles. This is what happens in the NJL model where the pairing is due to the attractive nonlinear interaction provided it is sufficiently strong.

We have shown that pairing is induced in a free Dirac field in $2+1$ dimensions by a constant magnetic field for any value of the mass. There is no surprise therefore that when we switch on the nonlinear interaction in the limit of zero mass chiral SSB takes place even for very small values of the coupling. In $2+1$ dimensions the dependence of the dynamical mass on the nonlinear coupling is analytic.

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E. V. Gorbar, V. P. Gusynin, V. A. Miransky, Phys. Rev. B66, 045108, Phys. Rev.B74, 195429, and a recent one, Phys. Rev. B81, 155451, on bilayer graphene.

Experiments in Princeton (Phys.Rev. B79, 115434) support magnetic catalysis. Similar experiments of the Harvard group in bilayer graphene (Nature Phys. 5, 889) also discuss magnetic catalysis scenario for a quasiparticle gap.