RG flows and phase diagrams from lattice simulations

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RG flows on the lattice

- Studies of the RG flow in lattice gauge theories are ubiquitous
- Different techniques have been developed (EoS, SF, MCRG)
- Existence of nontrivial fixed points, determination of the critical exponents
- Flow of the couplings in different schemes
- Applications to 3D theories of fermions (GN, Thirring, *graphene*)
- Applications to 4D gauge theories coupled to matter (QCD, BSM models)

Fixed points

In a neighbourhood of a fixed point: $\beta(g^*) = 0$

$$\delta g_i = g_i - g_i^* \,, \qquad \mu \frac{d}{d\mu}\left(\delta g_i\right) = M_{ij}\delta g_j + O\left(\delta g^2\right)$$

$$M_{ij} = \left.\frac{\partial \beta_i}{\partial g_j}\right|_{g^*}$$

Choosing a basis of eigenvalues of M we find:

$$\frac{d}{d\mu} u_i = -y_i u_i + O(u^2)\,, \qquad u_i(\mu) = \left(\frac{\mu}{\mu'}\right)^{-y_i} u_i(\mu')$$

Define the scaling dimension of the associated operator:

$$\Delta_i \equiv d_i + \gamma_i = D - y_i$$

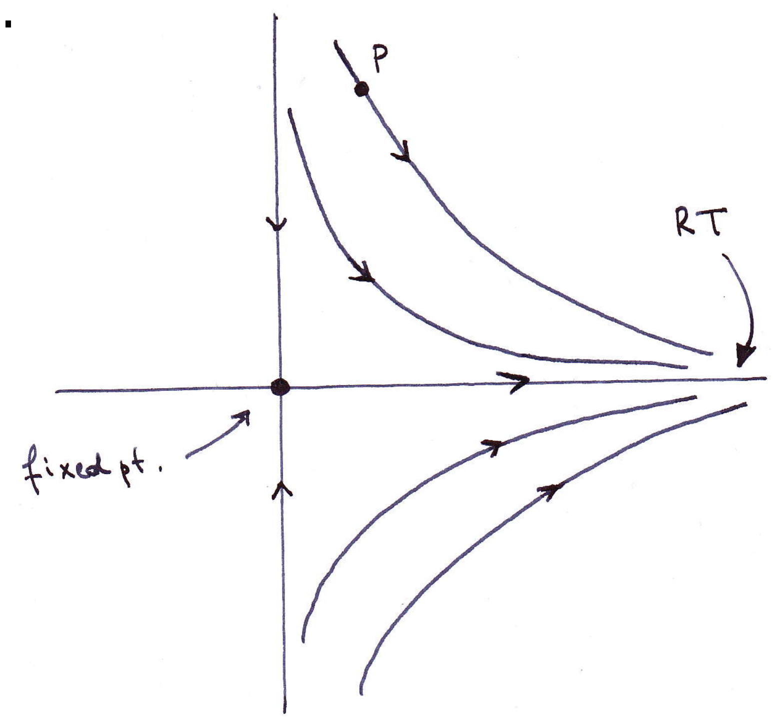
Relevant couplings

Power-law scaling in the neighbourhood of a fixed point suggests a classification of the couplings:

1. $y_i > 0$: **relevant** couplings, diverge in the IR, need to be fine tuned at the cut-off scale.

2. $y_i < 0$: **irrelevant** couplings, their value at the cut-off scale has no influence on the low-energy physics.

3. $y_i = 0$: **marginal** couplings.



Scheme-dependence

Consider the couplings in two different schemes:

$$g_i(\mu) = g_i\left(\tilde{g}(\tilde{\mu}), \tilde{\mu}/\mu\right)$$

$$0 = \tilde{\mu}\frac{d}{d\tilde{\mu}} g_i(\mu) = \sum_a \frac{\partial g_i}{\partial \tilde{g}_a}\tilde{\beta}_a + \frac{\tilde{\mu}}{\mu}\frac{\partial g_i}{\partial(\tilde{\mu}/\mu)}$$

Rewrite the second term in the sum using:

$$\beta_i = \frac{d}{d\mu} g_i = -\frac{\tilde{\mu}}{\mu}\frac{\partial g_i}{\partial(\tilde{\mu}/\mu)}$$

and substitute in the eq. above:

$$\beta_i = \sum_a \frac{\partial g_i}{\partial \tilde{g}_a}\tilde{\beta}_a$$

beta functions transform **covariantly** under a change of scheme

Invariance of the critical exponents

Taking one more derivative wrt $\tilde{g}_b$

$$\sum_j \frac{\partial \beta_i}{\partial g_j} \frac{\partial g_j}{\partial \tilde{g}_b} = \sum_a \left[\frac{\partial^2 g_i}{\partial \tilde{g}_a \partial \tilde{g}_b} \tilde{\beta}_a + \frac{\partial g_i}{\partial \tilde{g}_a} \frac{\partial \tilde{\beta}_a}{\partial \tilde{g}_b} \right]$$

Introducing the matrix S:

$$S_{ia} = \left. \frac{\partial g_i}{\partial \tilde{g}_a} \right|_{\tilde{g}^*}$$

and evaluating everything at the critical point, yields:

$$\sum_j M_{ij} S_{jb} = \sum_a S_{ia} \tilde{M}_{ab}$$

M and $\tilde{M}$ have the **same** eigenvalues

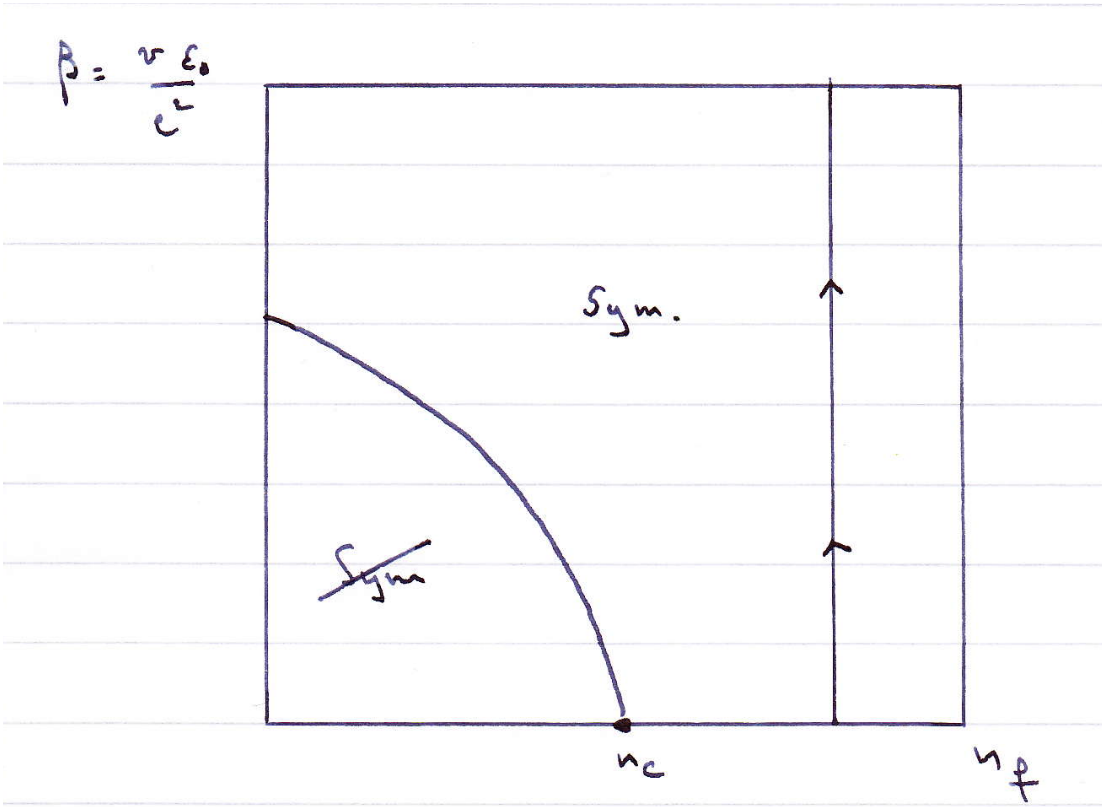
Recent numerical work on graphene

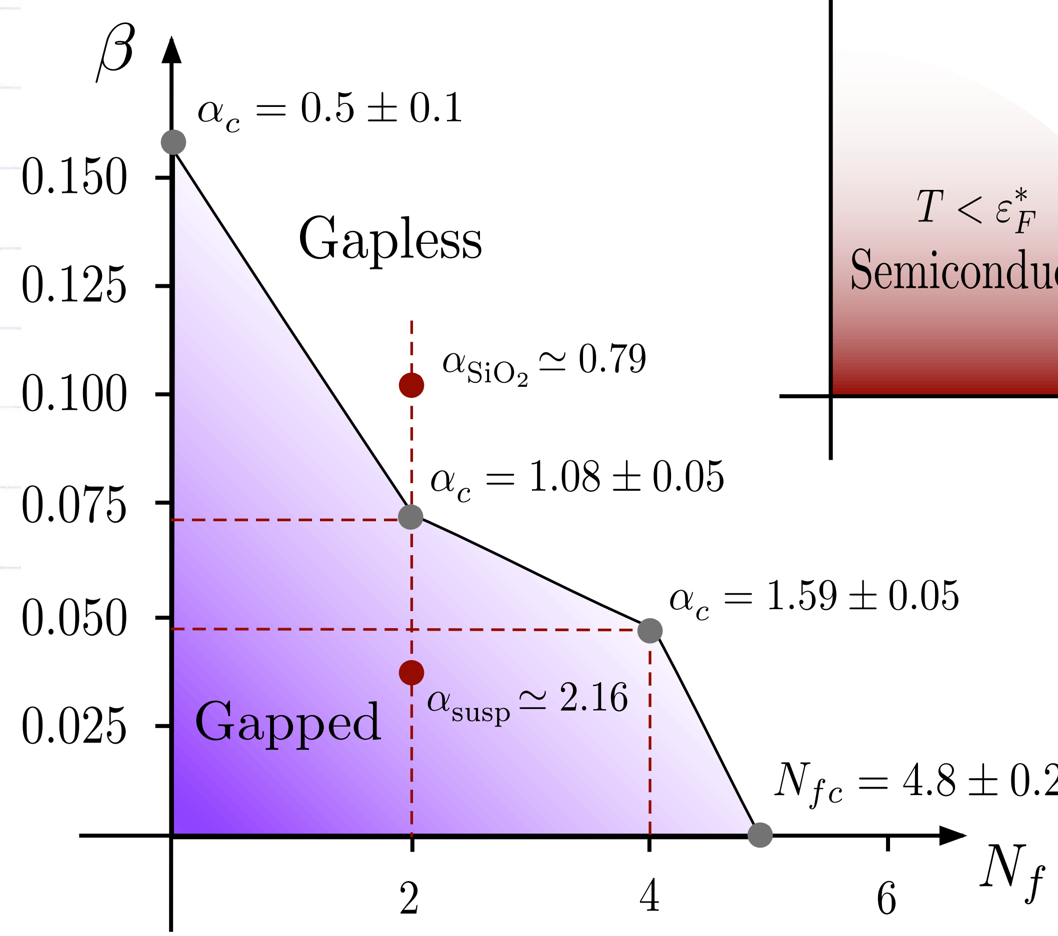
- Low-energy excitations: linear dispersion relation at the corners of the BZ
- Dirac Hamiltonian for 2 flavors of 4 components massless spinors [G Semenoff, 84]
- Electron speed $v \simeq c/300$; breaking of Lorentz symmetry
- 4-fermi interactions + Coulomb [IF Herbut, 06]
- 3-dim Coulomb interaction [DT Son, 07; J Drut & DT Son, 08]
- Alternative formulation as a deformed Thirring model [SJ Hands, 08]
- Strongly coupled theory, analytical results in the large n_f limit
- Identification of a RG fixed point, separating gapless/gapped phase
- Nonperturbative studies: lattice/Schwinger-Dyson eqs

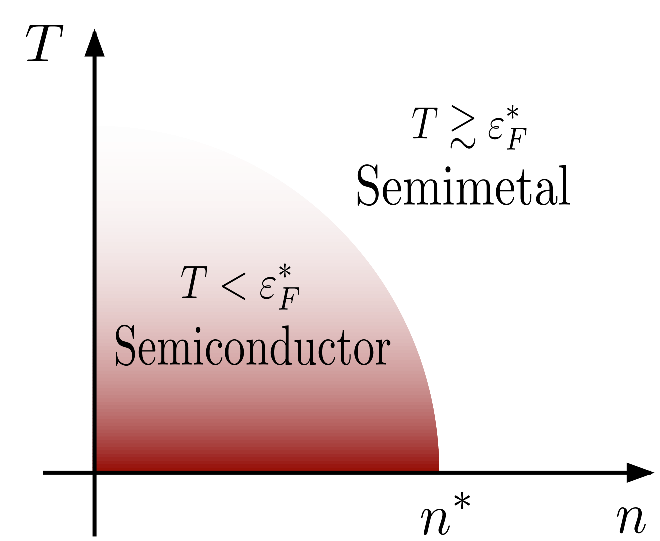
Phase diagrams

$$S_E = -\int d^2x dt\, \bar{\psi} D\psi + \frac{\varepsilon_0}{2e^2}\int d^3x dt\, (\partial A)^2$$

[DT Son 07, J Drut 09]

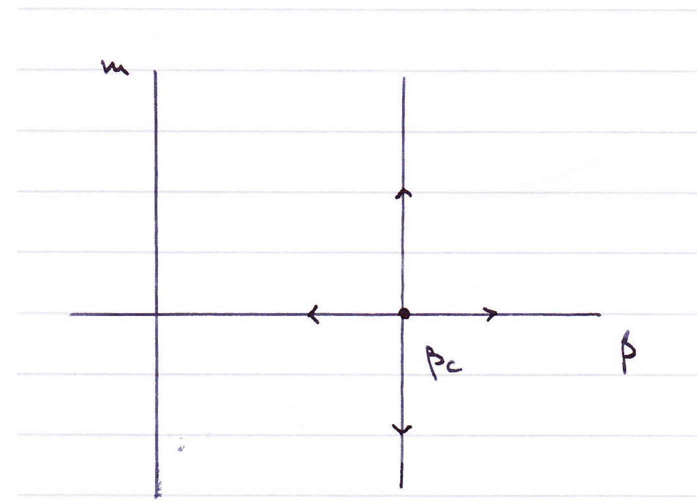






RG flows & scaling laws

Order parameter: $\langle \bar{\psi}\psi \rangle$

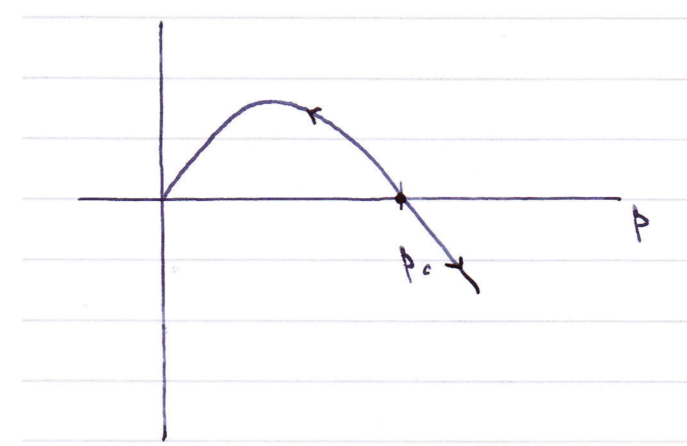


Relevant parameters: $m,\ t = 1/g^2 - 1/g_c^2$

Scaling of the order parameter is dictated by the linearized RGE
(in the neighbourhood of the fixed point)

$$m = \langle \bar{\psi}\psi \rangle^{\delta} \mathcal{F}\left(t \langle \bar{\psi}\psi \rangle^{-1/\beta} \right)$$

$$\langle \bar{\psi}\psi \rangle \sim t^{\beta} \qquad \langle \bar{\psi}\psi \rangle \sim m^{1/\delta}$$



Critical exponents are determined by the e.v. of the linearized RG flow:

$$\beta = \frac{D - y_m}{y_t}, \qquad \delta = \frac{y_m}{D - y_m}$$

Lattice simulations

Lattice simulations of fermionic systems are computationally challenging:

$$Z = \int \mathcal{D}\phi \, e^{-S[\phi]} \, (\det D)$$
$$= \int \mathcal{D}\phi \mathcal{D}\chi \mathcal{D}\chi^* \, e^{-S[\phi] - \chi^* (D^\dagger D)^{-1} \chi}$$

Simulations are performed at **finite** m and **finite** volume
System is always driven away from the fixed point

Fixed points can be identified from the scaling laws

Input parameters: g, m

g_c needs to be determined together with the critical exponents

Observables:

$$\langle \bar{\psi}\psi \rangle, M_H, \frac{\partial}{\partial m} \langle \bar{\psi}\psi \rangle, \frac{\partial \log \langle \bar{\psi}\psi \rangle}{\partial \log m}$$

Scaling laws

RG equations imply simple scaling rules [JB Kogut, MP Lombardo,G Schierholz, LDD 90s]

$$m = B\langle\bar{\psi}\psi\rangle^{\delta} + At\langle\bar{\psi}\psi\rangle^{(\delta-1/\beta)}$$

$$R = \frac{m}{\langle\bar{\psi}\psi\rangle}\frac{\langle\bar{\psi}\psi\rangle}{m} = \frac{1}{\left(\delta - \frac{1}{\beta}\right) + \frac{B\langle\bar{\psi}\psi\rangle^{\delta}}{\beta m}}$$

$$\lim_{m\to 0} R = \begin{cases} 0, & g^2 < g_c^2 \\ 1/\left(\delta - \frac{1}{\beta}\right), & g^2 > g_c^2 \end{cases}$$

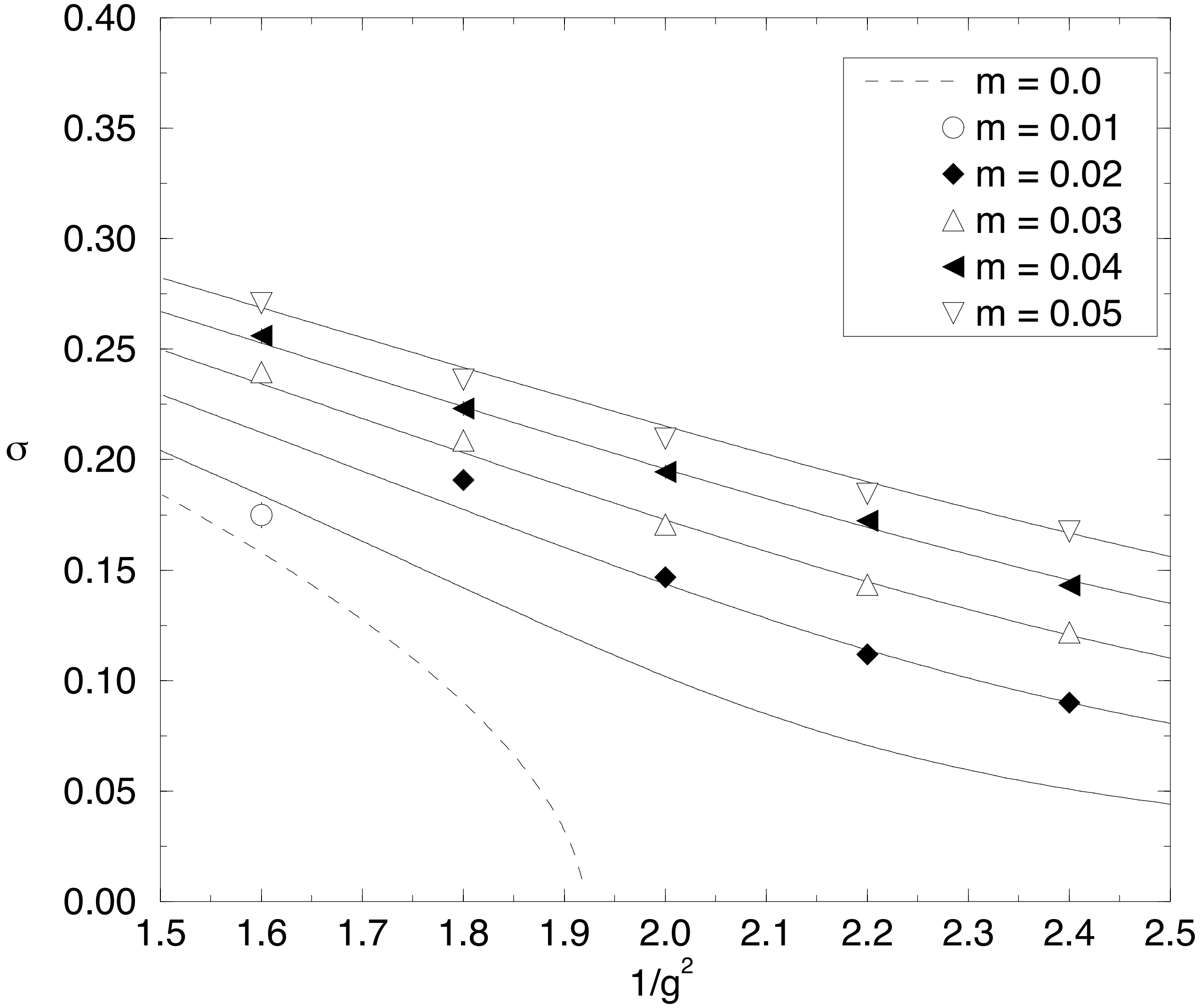
$$R\big|_{g=g_c} = 1/\delta$$

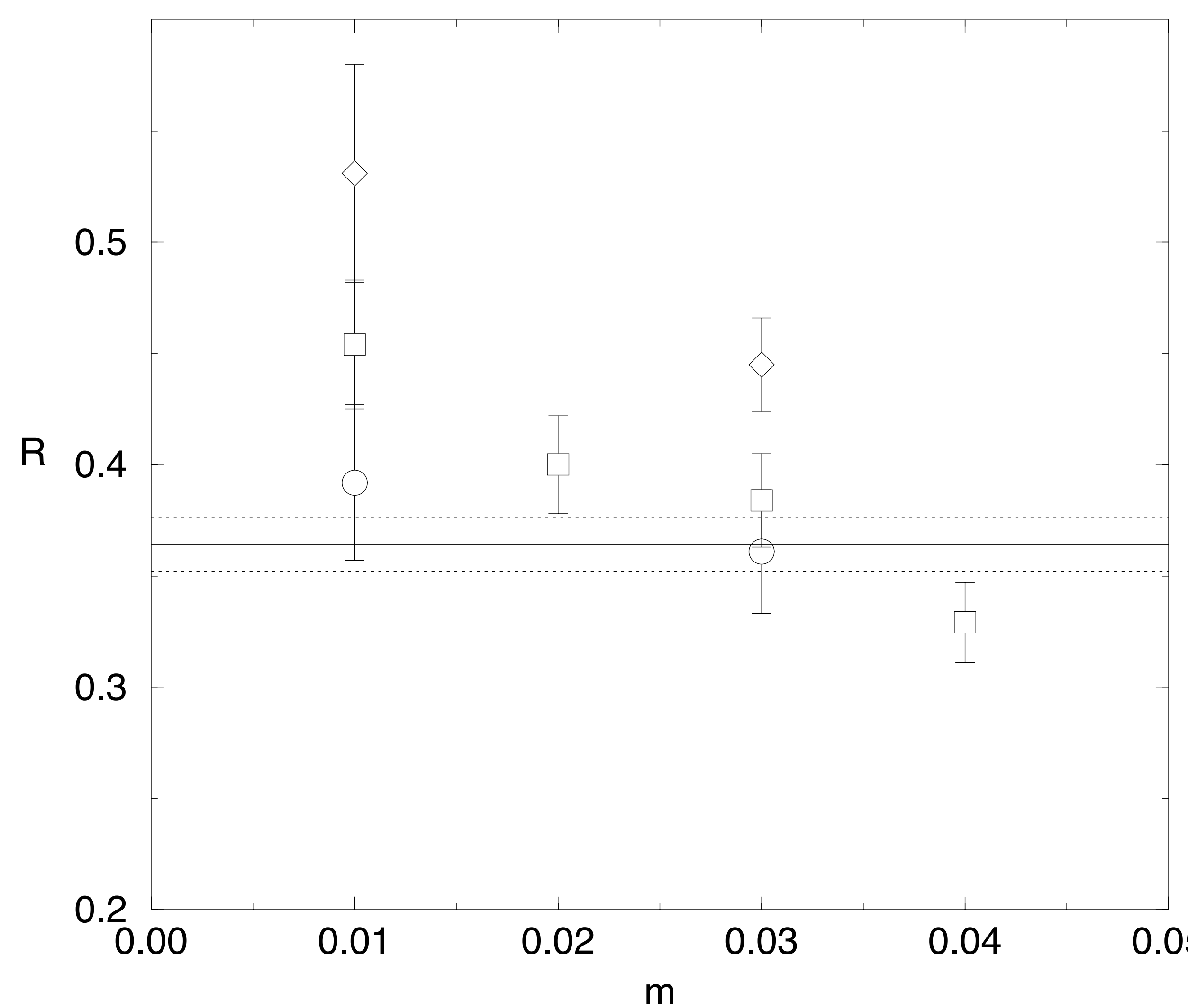
Check the **consistency** of lattice data with the existence of a fixed point

(old) Results for the Thirring model

$$S = \int d^3x \left[\bar{\psi}(x) D \psi(x) + \frac{g^2}{2n_f} \left(\bar{\psi}(x) \gamma_i \psi(x) \right)^2 \right]$$

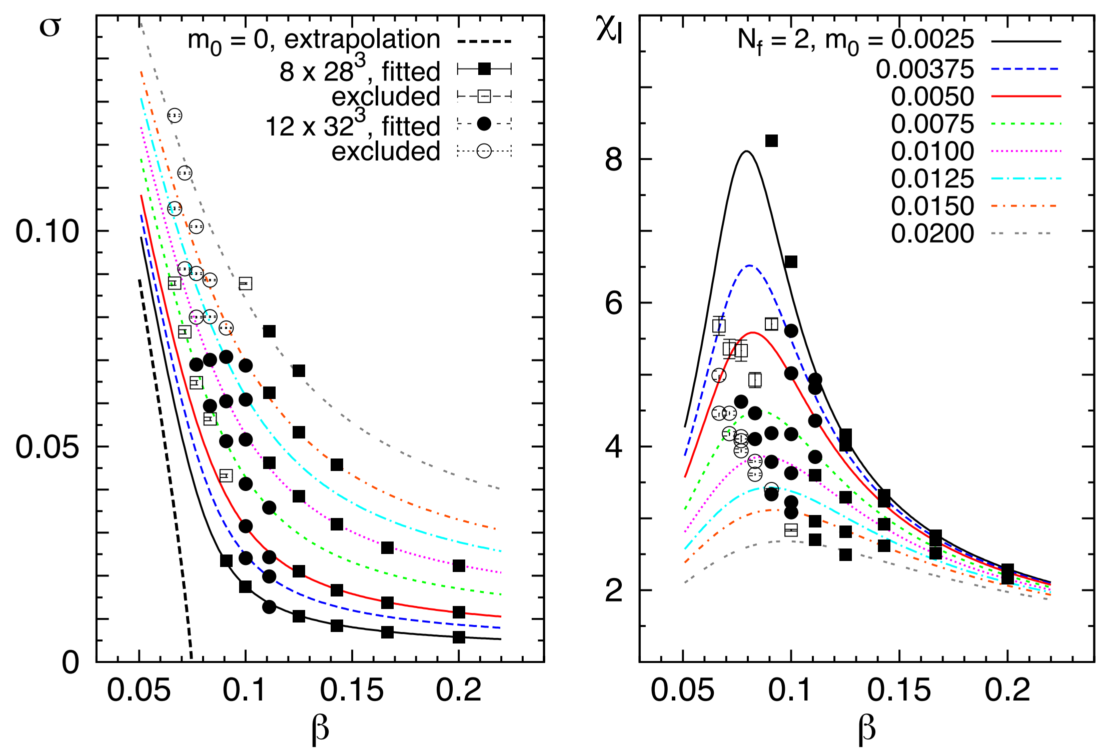
[LDD & SJ Hands 97]

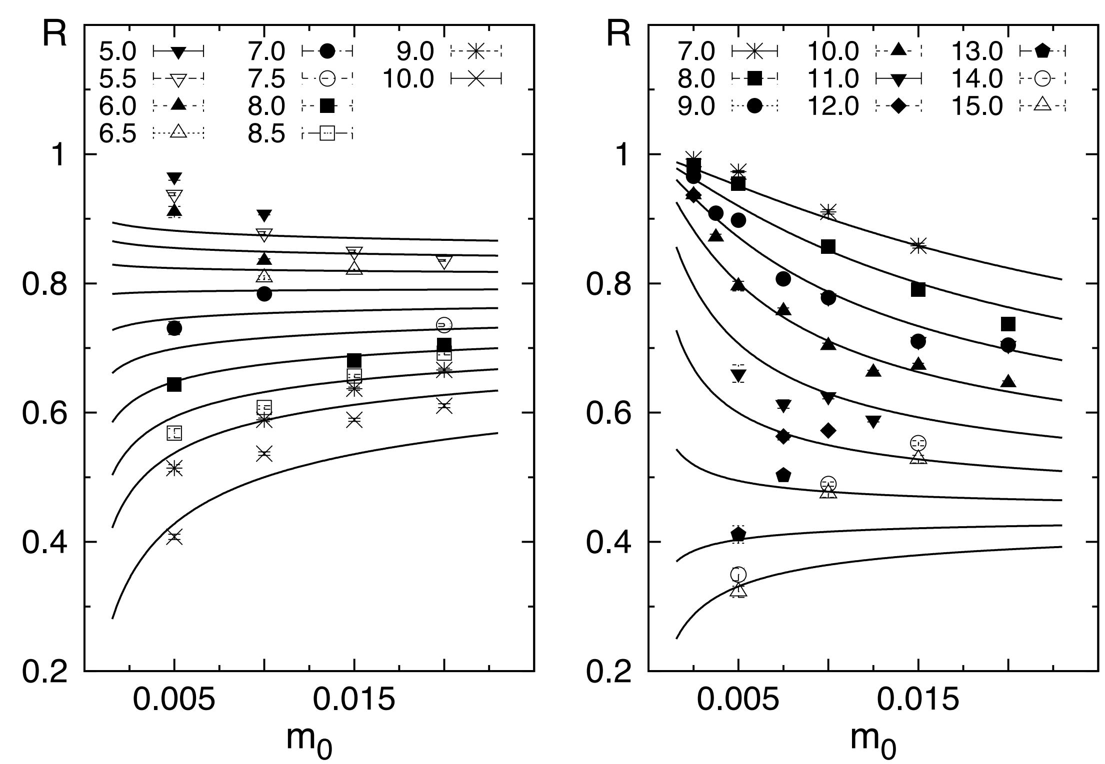




(new) Results for graphene

[J Drut 09]





Tuesday, 6 September 2011

Finite-size scaling

Finite-size effects can be incorporated in the RG analysis [LDD & SJ Hands 97]

$$m = \langle\bar{\psi}\psi\rangle^{\delta}\mathcal{F}\left(t\langle\bar{\psi}\psi\rangle^{-1/\beta}, L^{-1/\nu}\langle\bar{\psi}\psi\rangle^{-1/\beta}\right)$$

Taylor expansion of the scaling function yields for the EoS

$$m = B\langle\bar{\psi}\psi\rangle^{\delta} + A(t + CL^{-1/\nu})\langle\bar{\psi}\psi\rangle^{(\delta-1/\beta)}$$

The exponents determined from this fit are the ones in the thermodynamical limit

Recently used to analyse the n_f dependence of the order parameter at the critical coupling:

$$m = B\langle\bar{\psi}\psi\rangle^{\delta} + A[(n_f - n_c) + CL_t^{-1/\nu_t}]\langle\bar{\psi}\psi\rangle^{(\delta-1/\beta)}$$

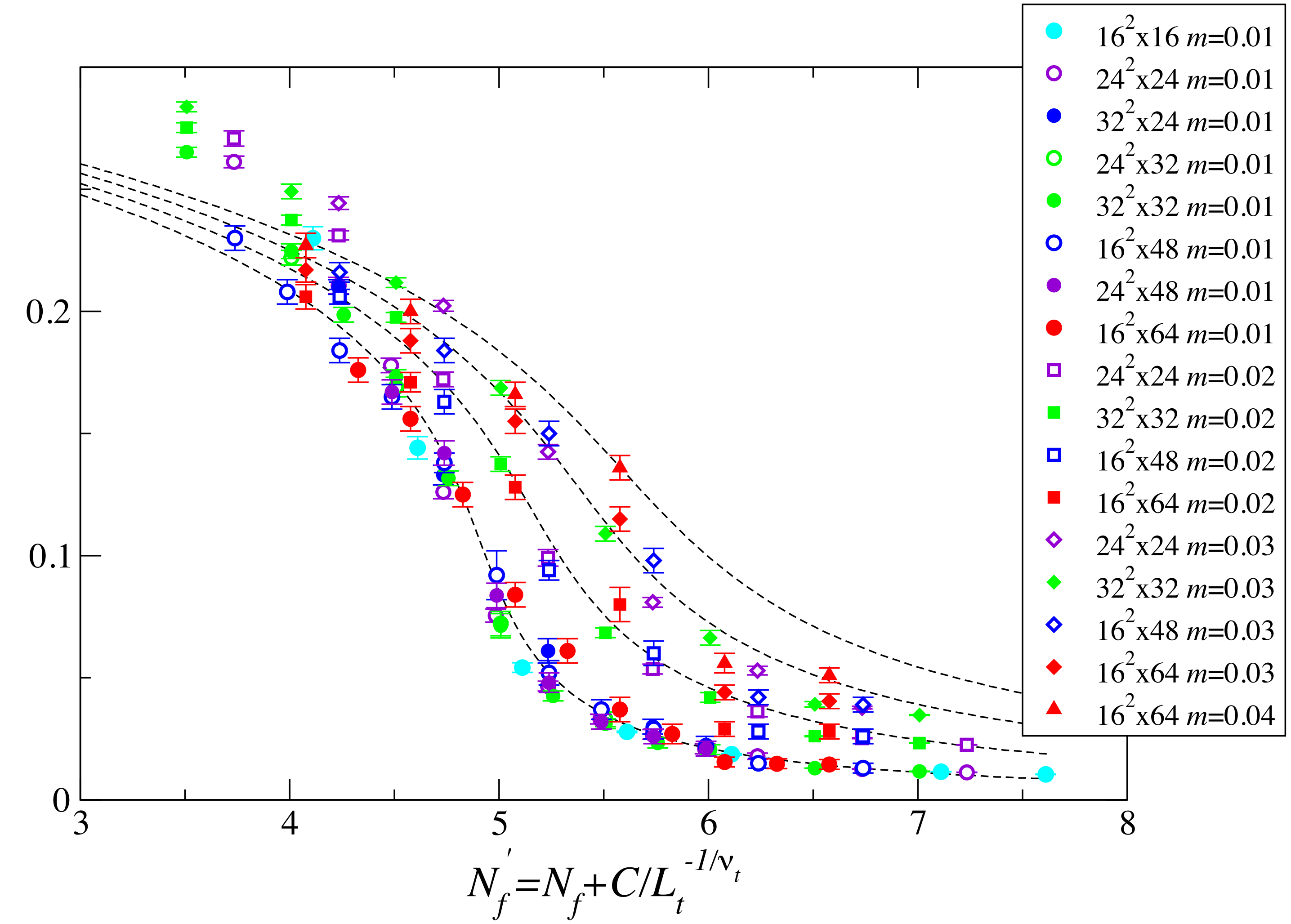
[SJ Hands et al 08]

Finite-size scaling for graphene

$$S = \int d^3x \left[\bar{\psi}(x) D\psi(x) + \frac{g^2}{2n_f} \left(\bar{\psi}(x)\gamma_0\psi(x)\right)^2 \right]$$

[SJ Hands 08]

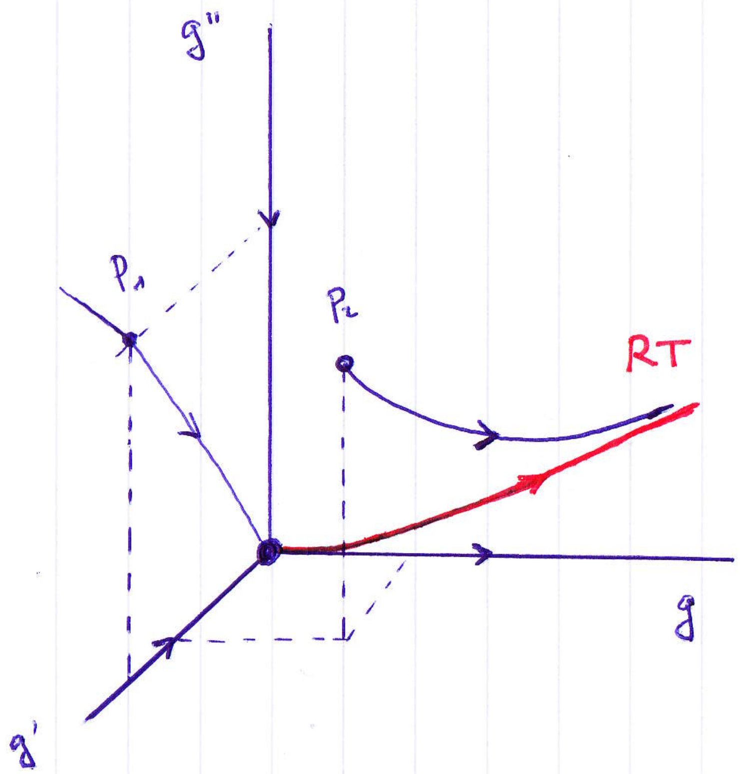
n_c=4.8(2)



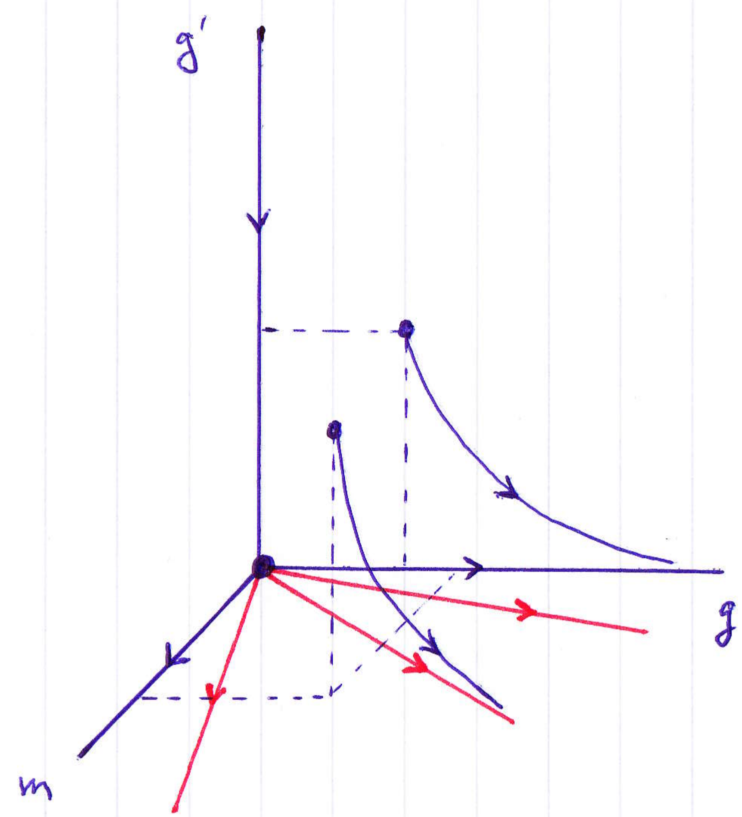
Comments

- Assess the systematic errors involved in the numerical simulations
- Fermion discretization, staggered, chiral fermions
- Square lattice vs hexagonal lattice?
- First results on hex lattices have been presented at the Lattice 2011 conference
- Finite-size scaling
- Importance of the fixed point
- Other interactions?

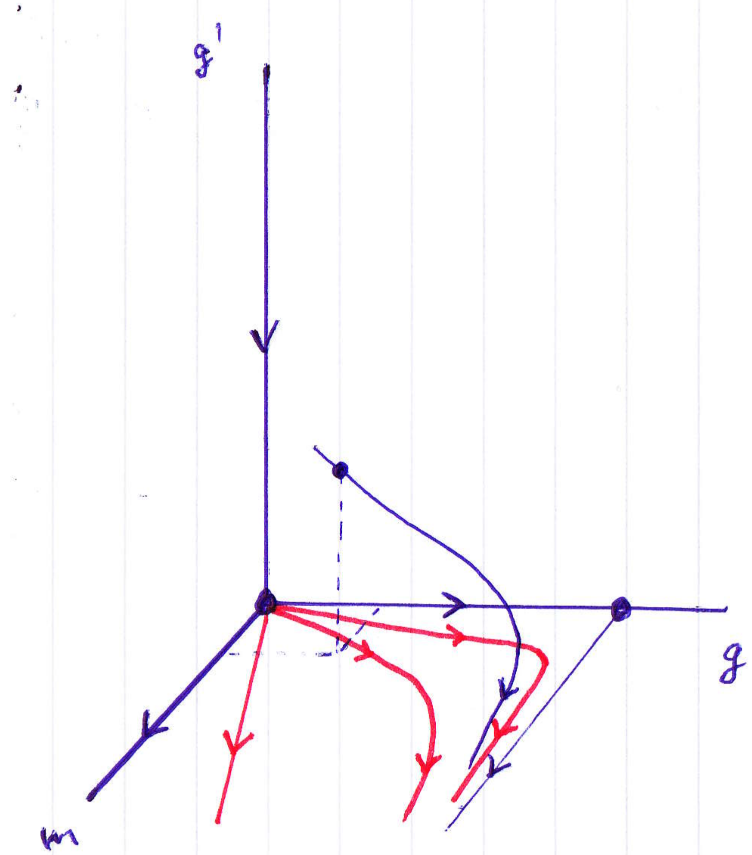
RG flows for 4D gauge theories



Pure YM

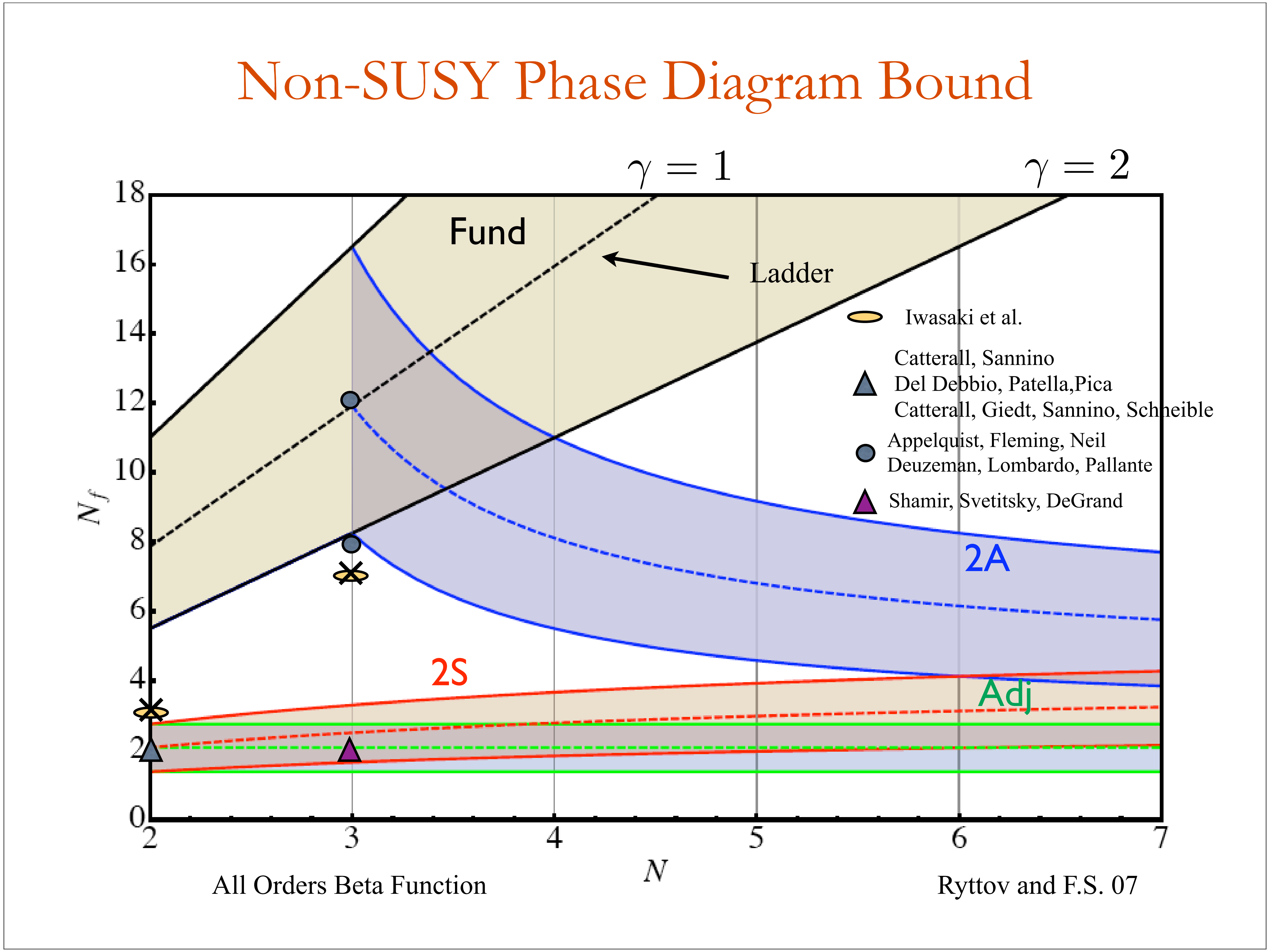


QCD-like



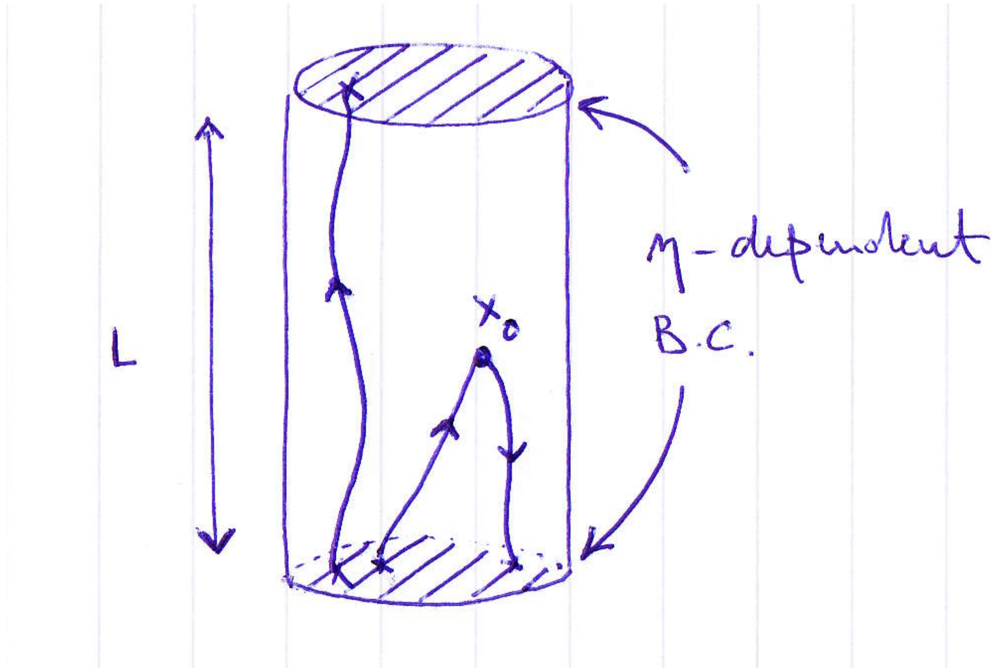
IRFP

Conformal window in SU(N) gauge theories



Schrödinger functional

Finite-volume renormalization scheme:
size of the system defines the renormalization scale,
impose Dirichlet boundary conditions



$$\bar{g}^2(L) = k \left\langle \frac{\partial S}{\partial \eta} \right\rangle^{-1}$$

Normalization to match perturbation theory.

The renormalized charge is an ***observable***, i.e. it can be measured by numerical simulations.

The definition above extends outside the perturbative regime and yields a **nonperturbative** coupling.

[M Luscher et al 90s]

The coupling depends on one scale only, the finite size of the system.

SF - running of the coupling

The running of the coupling as the scale is varied by a factor s is encoded in the step scaling function:

$$\Sigma(u, s, a/L) = \overline{g}^2(g_0, sL/a)\Big|_{\overline{g}^2(g_0, L/a)=u}$$

Lattice step scaling is affected by lattice artefacts, i.e. depends on the details of the UV regulator. We can define a continuous step scaling:

$$\sigma(u, s) = \lim_{a/L \to 0} \Sigma(u, s, a/L)$$

L/a: *resolution* of the theory

$$-2 \log s = \int_u^{\sigma(u,s)} \frac{dx}{\sqrt{x}\beta(\sqrt{x})}$$

SF - running of the coupling

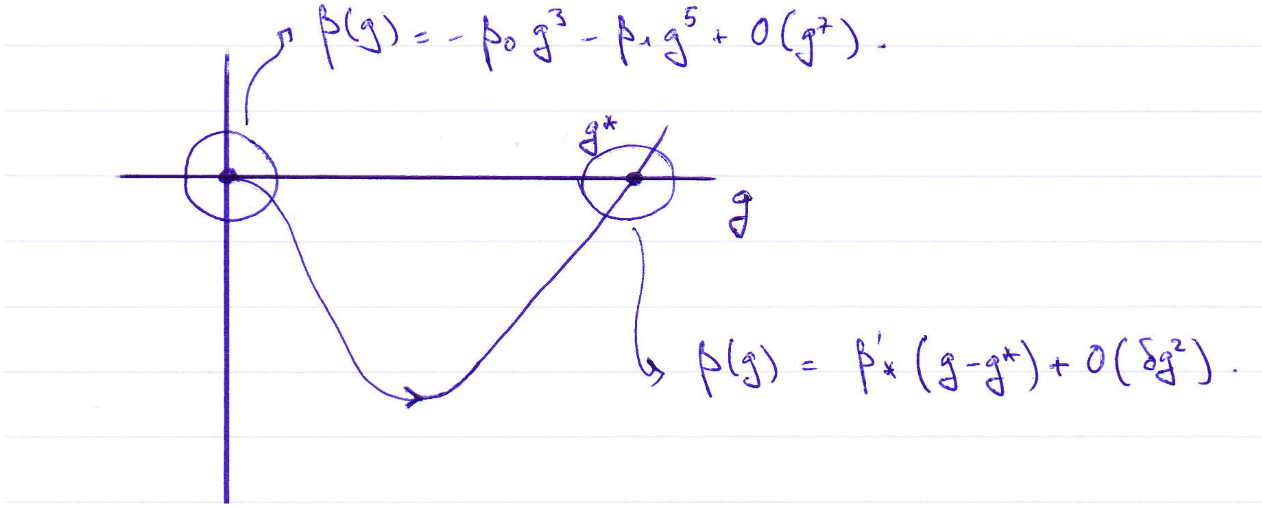
At the fixed point:

$$\sigma(u,s) = u \quad \Longleftrightarrow \quad \sigma(u,s)/u = 1$$

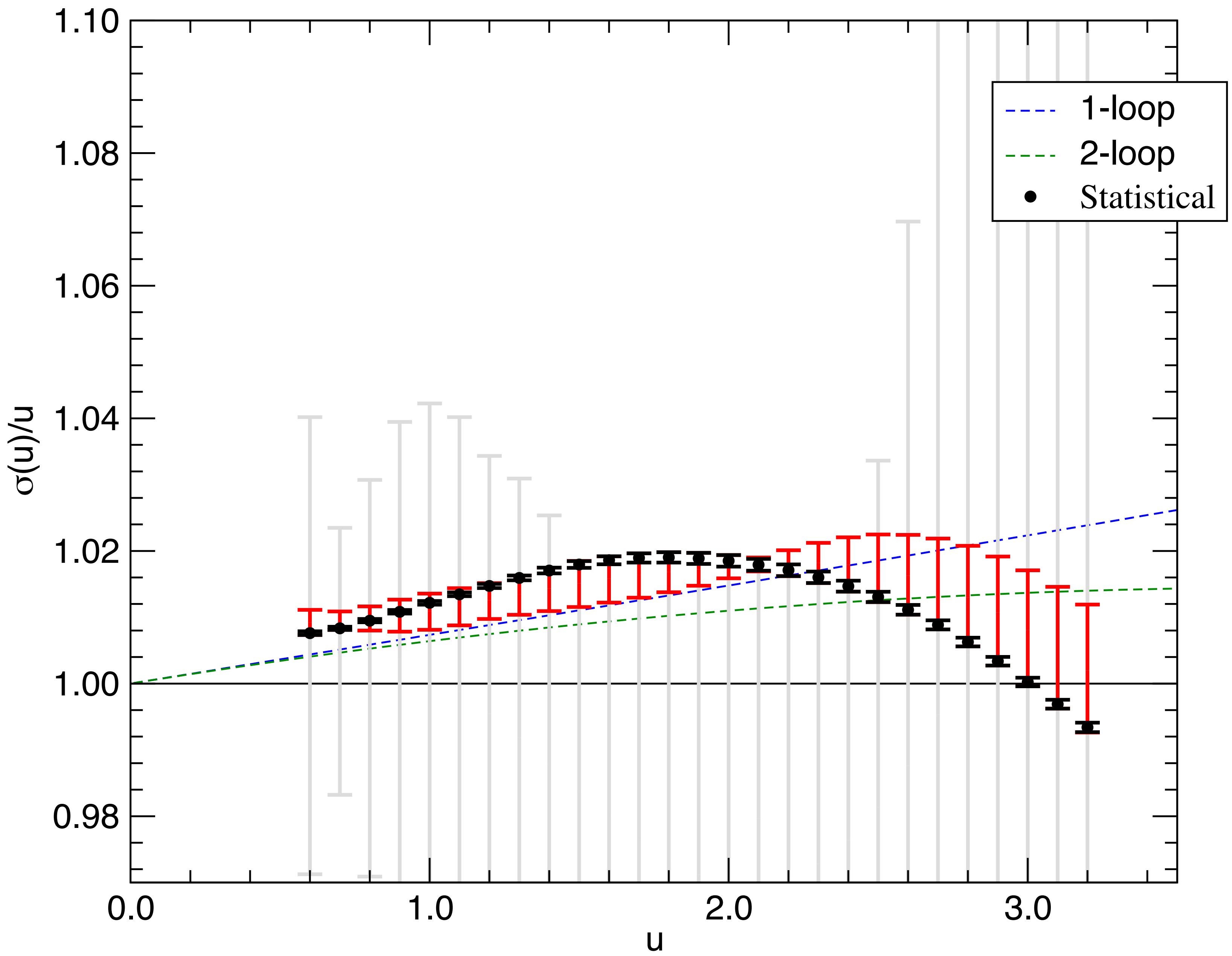
In the neighbourhood of the fixed point, the running is very slow.

Need a high statistical accuracy in order to resolve the physically interesting behaviour.

$$\frac{\sqrt{\sigma(u,s)} - g^*}{\sqrt{u} - g^*} = s^{-\beta'_*}$$



Lattice data



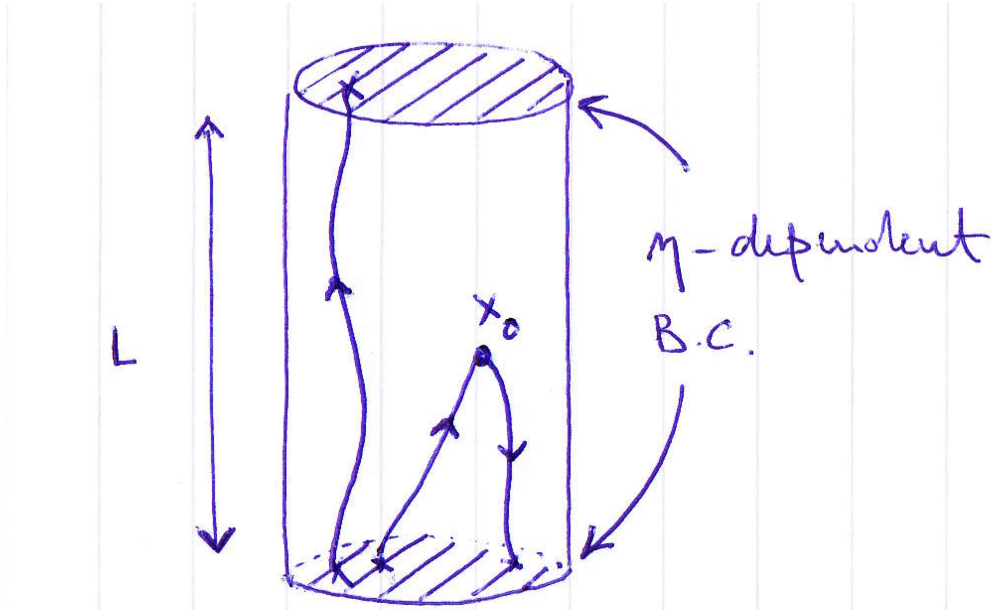
SF - running of the mass

The renormalized mass is defined as:

$$\bar{m}(\mu) = \frac{Z_A}{Z_P(\mu)} m$$

In order to study its running we need to compute nonperturbatively:

$$Z_P(L) = \sqrt{3f_1}/f_P(L/2)$$



$$f_1 = -1/12L^6 \int d^3u\, d^3v\, d^3y\, d^3z\, \langle \overline{\zeta}'(u)\gamma_5\tau^a\zeta'(v)\overline{\zeta}(y)\gamma_5\tau^a\zeta(z)\rangle\,,$$

$$f_P(x_0) = -1/12 \int d^3y\, d^3z\, \langle \overline{\psi}(x_0)\gamma_5\tau^a\psi(x_0)\overline{\zeta}(y)\gamma_5\tau^a\zeta(z)\rangle\,.$$

SF - running of the mass

Step scaling functions for the mass:

$$\Sigma_P(u,s,a/L) = \left.\frac{Z_P(g_0,sL/a)}{Z_P(g_0,L/a)}\right|_{\overline{g}^2(L)=u}$$

$$\sigma_P(u,s) = \lim_{a\to 0}\Sigma_P(u,s,a/L)$$

Relation to the anomalous dimension:

$$\sigma_P(u) = \left(\frac{u}{\sigma(u)}\right)^{(d_0/(2\beta_0))} \exp\left[\int_{\sqrt{u}}^{\sqrt{\sigma(u)}} dx \left(\frac{\gamma(x)}{\beta(x)} - \frac{d_0}{\beta_0 x}\right)\right]$$

SF - running of the mass

In the neighbourhood of the fixed point:

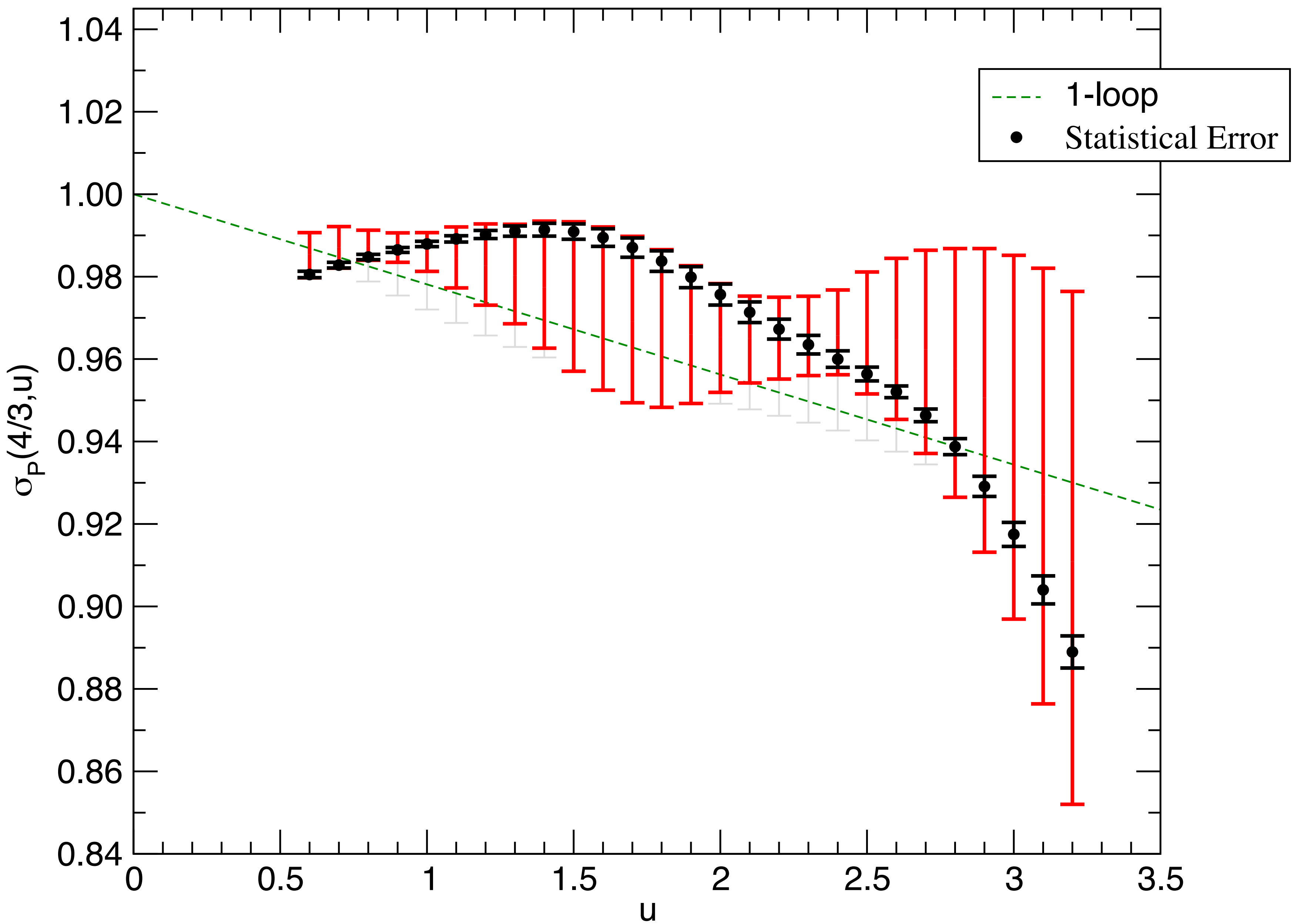
$$\int_{\overline{m}(\mu)}^{\overline{m}(\mu/s)} \frac{dm}{m} = -\gamma_* \int_{\mu}^{\mu/s} \frac{dq}{q}$$

$$\log|\sigma_P(s,u)| = -\gamma_* \log s$$

Hence we can define an estimator for the anomalous dimension:

$$\boxed{\hat{\gamma}(u) = -\frac{\log|\sigma_P(u,s)|}{\log|s|}}$$

Lattice data



MCRG

Follow the flow of bare couplings under RG transformations (blocking)
Each blocking step changes the scale by a factor s.

Two methods:

1. Compute directly the matrix M that defines the linearized RG flow

2. Find the RG-transformed couplings by matching observables

Computation of the linearized RG transformation

Theory is defined by the action:

$$S = \sum_i g_i S_i$$

$$S_i = \int d^D x \, \mu^{D-d_i} \partial^{p_i} \phi^{n_i}$$

RG transformation:

$$S^{(n+1)} = R_s S^{(n)} = \sum_i g_i^{(n+1)} S_i^{(n+1)}$$

Fixed point:

$$S^* = R_s S^* = \sum_i g_i^* S_i^*$$

Computation of the linearized RG transformation

Close to the fixed point:

$$g_i^{(n+1)} - g_i^* = \sum_j T_{ij}^* \, (g_j^{(n)} - g_j^*)$$

and therefore:

$$\frac{\partial \langle S_i^{(n)} \rangle}{\partial g_j^{(n-1)}} = \sum_k \frac{\partial g_k^{(n)}}{\partial g_j^{(n-1)}} \frac{\partial \langle S_i^{(n)} \rangle}{\partial g_k^{(n)}} = \sum_k T_{kj} \, \frac{\partial \langle S_i^{(n)} \rangle}{\partial g_k^{(n)}}$$

We can evaluate the vev in the LHS and RHS:

$$\frac{\partial \langle S_i^{(n)} \rangle}{\partial g_j^{(n-1)}} = \langle S_i^{(n)} S_j^{(n-1)} \rangle - \langle S_i^{(n)} \rangle \langle S_j^{(n-1)} \rangle \equiv A_{ij}^{(n)}$$

$$\frac{\partial \langle S_i^{(n)} \rangle}{\partial g_j^{(n)}} = \langle S_i^{(n)} S_j^{(n)} \rangle - \langle S_i^{(n)} \rangle \langle S_j^{(n)} \rangle \equiv B_{ij}^{(n)}$$

Computation of the linearized RG transformation

and finally:

$$\Rightarrow \boxed{T = B^{-1} A}$$

Blocking transformation:

$$\vartheta(n_B) = b \sum_{n \in n_B} \phi(n) + \frac{\zeta}{\sqrt{N a_W}}$$

satisfies

$$e^{-S[\vartheta]} = \int d[\phi] e^{-S[\phi]} e^{-\frac{a_W}{2} \sum_{n_B} \left[\vartheta(n_B) - b \sum_{n \in n_B} \phi(n)\right]^2}$$

Conclusions

- MC simulations have already been used for the study of graphene
- RG fixed points can be identified from the scaling of the order parameter (and susceptibilities) in the neighbourhood of the fixed point
- study of systematic errors - hexagonal lattice, different types of fermions
- other methods have been used for the study of 4D field theory
- these new methods could be used for condensed matter systems
- graphene as a testing ground?

Gauge theories with fermions

$$\beta(g) = \mu \frac{d}{d\mu} g(\mu) = -\beta_0 g^3 - \beta_1 g^5 + O(g^7)$$

$$\beta_0 = \frac{1}{(4\pi)^2} \left[\frac{11}{3} C_s(A) - \frac{4}{3} T_R n_f \right]$$

$$\beta_1 = \frac{1}{(4\pi)^4} \left[\frac{34}{3} C_2(A)^2 - \left(\frac{20}{3} C_2(A) - 4 C_2(R) \right) T_R n_f \right]$$

Asymptotic freedom:

$$n_f < n_f^+ = \frac{11N}{4T_R} \qquad\qquad n_f \to n_f^+ \Longrightarrow \beta_0 \to 0$$

Gauge theories with fermions

$$\beta_1 = \beta_{1g} - \beta_{1f} n_f$$

$$\beta_1 < 0 \implies \frac{\beta_{1g}}{\beta_{1f}} = n_f^- < n_f$$

$n_f^- < n_f < n_f^+$ non-trivial zero of the beta function $g_*^2 = -\frac{\beta_0}{\beta_1} > 0$