Rigorous construction of Luttinger liquids through Ward Identities

G. Benfatto

Università di Roma “Tor Vergata”

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Outline

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- The flow of the effective coupling (the beta function) is the same, up to exponentially small terms, as the analogous flow for the spinless Tomonaga model (that is the Luttinger model with ultraviolet cutoff and local interaction, which is equivalent to the Thirring model with fixed ultraviolet cutoff).
- The beta function for this special model (which is not solvable) is asymptotically vanishing, so that the effective coupling on large scales is essentially constant and of the same order of the coupling on small scales.

The role of the Ward Identities

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This is an old approach in the physical literature, but its rigorous implementation in an RG scheme is not trivial at all, because the ultraviolet and infrared cutoffs destroy local Gauge invariance and produce not negligible correction terms with respect to the formal Ward identities.

The strategy

The solution of the problem is in the use of a new set of identities, that we call Correction Identities, relating the corrections to the Schwinger functions.

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By combining Ward and Correction identities with a Dyson equation, the vanishing of the Beta function follows, so that the infrared cutoff can be removed.

As a byproduct, even the ultraviolet cutoff can be removed, after a suitable ultraviolet renormalization, so that a Euclidean Quantum Field Theory corresponding to the Thirring model at imaginary time is constructed, for any value of the mass.

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References

- G. Benfatto, G. Gallavotti: Perturbation Theory of the Fermi Surface in a Quantum Liquid. A General Quasiparticle Formalism and One-Dimensional Systems, J. Stat. Phys. **59**, 541–664 (1990).

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- G. Benfatto, P. Falco, V. Mastropietro: Functional Integral Construction of the Thirring model: axioms verification and massless limit, Comm. Math. Phys. **273**, 67–118, 2007.

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Let $\mathcal{D}$ be the set of *space-time momenta*

$$\mathbf{k} = (k, k_0), \quad k = \frac{2\pi}{L}\left(n + \frac{1}{2}\right), \quad k_0 = \frac{2\pi}{\beta}\left(n_0 + \frac{1}{2}\right)$$

With each $\mathbf{k} \in \mathcal{D}$ we associate four Grassmannian variables

$$\hat{\psi}^{\sigma}_{\mathbf{k},\omega}, \quad \sigma, \omega \in \{+, -\}$$

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In space coordinates:

$$\psi^{\sigma}_{\mathbf{x},\omega} = (L\beta)^{-1} \sum_{\mathbf{k}} e^{i\sigma\mathbf{k}\mathbf{x}} \hat{\psi}^{\sigma}_{\mathbf{k},\omega}, \quad \mathbf{x} = (x, x_0)$$

The *free model* is described by the free *Gaussian* measure

$$P(d\psi) = \mathcal{D}\psi \, \frac{1}{\mathcal{N}} \exp\left\{ -\frac{Z_0}{L\beta} \sum_{\omega=\pm 1} \sum_{\mathbf{k}\in\mathcal{D}} C^{\varepsilon}_{h,0}(\mathbf{k})(-ik_0 + \omega k)\hat{\psi}^{+}_{\mathbf{k},\omega}\hat{\psi}^{-}_{\mathbf{k},\omega} \right\}$$

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$$g_\omega(\mathbf{x}-\mathbf{y}) = < \psi^{-}_{\mathbf{x},\omega} \psi^{+}_{\mathbf{y},\omega'} > = \frac{\delta_{\omega,\omega'}}{L\beta} \sum_{\mathbf{k}} \frac{\chi^{\varepsilon}_{h,0}(\mathbf{k})}{-ik_0 + \omega k} e^{-i\mathbf{k}(\mathbf{x}-\mathbf{y})}$$

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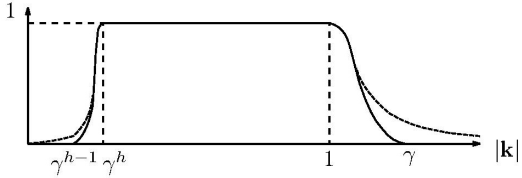
$$\chi^{\varepsilon}_{h,0} = [C^{\varepsilon}_{h,0}]^{-1}$$



$\chi^{\varepsilon}_{h,0}(\mathbf{k})$ is a smooth function, which, for $\varepsilon = 0$, has support in the interval $\{\gamma^{h-1} \le |\mathbf{k}| \le \gamma\}$, $\gamma > 1$, and is equal to 1 in the interval $\{\gamma^h \le |\mathbf{k}| \le 1\}$.

The interacting model

The correlation functions of density and field operators can be obtained by the generating functional

$$\mathcal{W}(\phi, J) = \log \int P(d\psi) \exp \Bigg\{ -V(\psi) + \sum_{\omega} \int d\mathbf{x} \left[J_{\mathbf{x},\omega} Z_0^{(2)} \rho_{\mathbf{x},\omega} + \phi^+_{\mathbf{x},\omega} \psi^-_{\mathbf{x},\omega} + \psi^+_{\mathbf{x},\omega} \phi^-_{\mathbf{x},\omega} \right] \Bigg\}$$

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$$V(\psi) = \lambda (Z_0)^2 \int d\mathbf{x}\, \psi^+_{\mathbf{x},+} \psi^-_{\mathbf{x},+} \psi^+_{\mathbf{x},-} \psi^-_{\mathbf{x},-}, \quad \rho_{\mathbf{x},\omega} = \psi^+_{\mathbf{x},\omega} \psi^-_{\mathbf{x},\omega}$$

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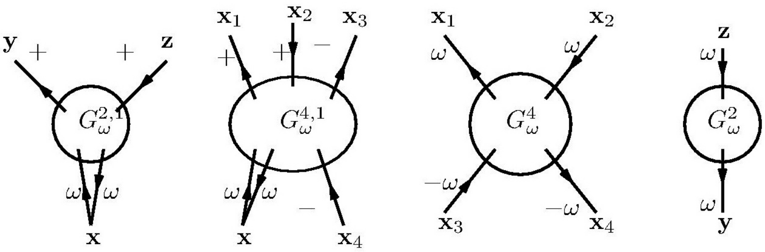
For example: $Z_0^{(2)} = Z_0 = 1$

Examples od correlation functions



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The scale decomposition

$$\chi_{h,0}(\mathbf{k}) = \sum_{j=h}^{0} f_j(\mathbf{k}) \Rightarrow \hat{\psi}^{\pm}_{\mathbf{k},\omega} = \sum_{j=h}^{0} \hat{\psi}^{\pm(j)}_{\mathbf{k},\omega}$$

$$\text{supp } f_j(\mathbf{k}) = \{\gamma^{j-1} \leq |\mathbf{k}| \leq \gamma^{j+1}\}, \quad h \leq j \leq 0$$

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$$|g^{(j)}_{\omega}(\mathbf{x}-\mathbf{y})| \leq C_M \frac{\gamma^j}{1 + [\gamma^j |\mathbf{x}-\mathbf{y}|]^M}, \quad \forall M \geq 0$$

The *effective potential on scale j* is defined iteratively so that

$$e^{\mathcal{W}(\phi,J)} = e^{-L\beta E_j} \int P_{\tilde{Z}_j, C_{h,j}}(d\psi) e^{-V^{(j)}(\sqrt{Z_j}\psi) + \mathcal{B}^{(j)}(\sqrt{Z_j}\psi,\phi,J)}$$

where $\tilde{Z}_j = \tilde{Z}_j(\mathbf{k})$, $j = h, \ldots, 0$ are suitable functions of $\mathbf{k}$, independent of the IR cutoff h for $j > h$,

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and $P_{\tilde{Z}_j,C_{h,j}}(d\psi)$ is defined as $P(d\psi)$, with $\tilde{Z}_j$ in place of Z_0 and $C_{h,j}$ in place of $C_{h,0}$.

If $j = 0$, we have

$$\mathcal{B}^{(0)}(\psi, \phi, J) = \sum_{\omega} \int d\mathbf{x} \left[J_{\mathbf{x},\omega} Z_0^{(2)} \psi^{+}_{\mathbf{x},\omega} \psi^{-}_{\mathbf{x},\omega} + \phi^{+}_{\mathbf{x},\omega} \psi^{-}_{\mathbf{x},\omega} + \psi^{+}_{\mathbf{x},\omega} \phi^{-}_{\mathbf{x},\omega} \right]$$

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First integration step

$$e^{\mathcal{W}(\phi,J)} = \int P_{Z_0, C_{h,-1}}(d\psi) \int P_{Z_0, f_0^{-1}}(d\psi^{(0)}) \cdot$$
$$\cdot\, e^{-V^{(0)}\left(\sqrt{Z_0}[\psi+\psi^{(0)}]\right) + \mathcal{B}^{(0)}\left(\sqrt{Z_0}[\psi+\psi^{(0)}], \phi, J\right)}$$
$$= e^{-L\beta E_{-1}} \int P_{Z_0, C_{h,-1}}(d\psi) e^{-V^{(-1)}\left(\sqrt{Z_0}\psi\right) + \mathcal{B}^{(-1)}\left(\sqrt{Z_0}\psi, \phi, J\right)}$$

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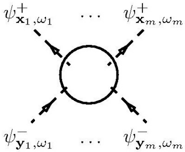
This defines E_{-1}, $V^{(-1)}$, $\mathcal{B}^{(-1)}$ and $\tilde{Z}_{-1}(\mathbf{k}) = Z_0$.

$$V^{(-1)}(\psi) = \sum_{\substack{m \geq 1 \\ \omega_1, \ldots, \omega_m}} \int d\mathbf{x}_1 \ldots d\mathbf{x}_{2m} W^{(-1)}_{2m,\underline{\omega}}(\underline{\mathbf{x}}) \prod_{i=1}^{2m} \psi^{\sigma_i}_{\mathbf{x}_i,\omega_i}$$

$$W^{(-1)}_{2m,\underline{\omega}}(\underline{\mathbf{x}}) =$$



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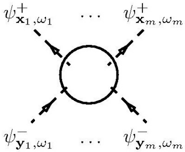

$$y_i = x_{m+i}$$

A similar representation is valid for $\mathcal{B}^{(-1)}(\psi, \phi, J)$, with at least one external line of type J or φ, but possibly no external line of type ψ.

The localization operation

$$(L\beta)^{-1}\int d\underline{\mathbf{x}}|W^{(-1)}_{2m,\underline{\omega}}(\underline{\mathbf{x}})| \leq A_m \sum_{n>m/2} (C|\lambda|)^n \gamma^{(-1)\frac{4n-2m}{2}-2(-1)(n-1)}$$

$$= A_m \gamma^{-D_{2m}} \sum_{n>m/2} (C|\lambda|)^n, \quad D_{2m} = 2-m$$

Note: no $n!$ in the bound

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The terms with *dimension* $D_{2m} \geq 0$ need to be localized.

$$\mathcal{L}\int d\underline{\mathbf{x}} W^{(-1)}_{4,\underline{\omega}}(\underline{\mathbf{x}}) \prod_{i=1}^{4} \psi^{\sigma_i}_{\mathbf{x}_i,\omega_i} = \begin{cases} \int d\underline{\mathbf{x}} W^{(-1)}_{4,\underline{\omega}}(\underline{\mathbf{x}}) \prod_{i=1}^{4} \psi^{\sigma_i}_{\mathbf{x}_1,\omega_i} & \text{if } \sum_{\omega_i} = 0 \\ 0 & \text{otherwise} \end{cases}$$

$$\mathcal{L}\int d\mathbf{x}d\mathbf{y}\, W^{(-1)}_{2,\omega}(\mathbf{x},\mathbf{y})\psi^+_{\mathbf{x},\omega}\psi^-_{\mathbf{y},\omega} =$$

$$\int d\mathbf{x}d\mathbf{y}\, W^{(-1)}_{2,\omega}(\mathbf{x},\mathbf{y})\psi^+_{\mathbf{x},\omega}[\psi^-_{\mathbf{x},\omega} + (\mathbf{y}-\mathbf{x})\nabla\psi^-_{\mathbf{x},\omega}] =$$

$$\int d\mathbf{x}\psi^+_{\mathbf{x},\omega}\nabla\psi^-_{\mathbf{x},\omega}\int d\mathbf{y}\,(\mathbf{y}-\mathbf{x})\, W^{(-1)}_{2,\omega}(\mathbf{x},\mathbf{y})$$

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$$\int d\mathbf{x}\psi^+_{\mathbf{x},\omega}\nabla\psi^-_{\mathbf{x},\omega}\int d\mathbf{y}\,(\mathbf{y}-\mathbf{x})W_{2,\omega}^{(-1)}(\mathbf{x},\mathbf{y})$$

$$V^{(-1)}(\psi) = \mathcal{L}V^{(-1)}(\psi) + \mathcal{R}V^{(-1)}(\psi), \quad \mathcal{R} \equiv 1 - \mathcal{L}$$

$$\mathcal{L}V^{(-1)}(\psi) = \zeta_{-1}F_\zeta(\psi) + l_{-1}F_\lambda(\psi)$$

$$F_\zeta(\psi) = \sum_\omega \int d\mathbf{x}\,\psi^+_{\mathbf{x},\omega}[-\partial_{x_0} + i\omega\partial_x]\psi^-_{\mathbf{x},\omega}$$

$$F_\lambda(\psi) = \int d\mathbf{x}\,\psi^+_{\mathbf{x},+}\psi^-_{\mathbf{x},+}\psi^+_{\mathbf{x},-}\psi^-_{\mathbf{x},-}$$

A similar procedure is applied to $\mathcal{B}^{(-1)}(\sqrt{Z_{-1}}\psi, \phi, J)$. If $\phi = 0$, we find another marginal term:

$$\mathcal{B}_J^{(-1,2)}(\sqrt{Z_{-1}}\psi) = Z_{-1} \sum_{\omega} \int d\mathbf{x} d\mathbf{y} d\mathbf{z} B_{\omega}(\mathbf{x}, \mathbf{y}, \mathbf{z}) J_{\mathbf{x},\omega} \psi^{+}_{\mathbf{y},\tilde{\omega}} \psi^{-}_{\mathbf{z},\tilde{\omega}}$$

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$$\mathcal{L}\mathcal{B}_J^{(-1,2)}(\sqrt{Z_{-1}}\psi) = \sum_{\omega} \frac{Z_{-1}^{(2)}}{Z_{-1}} \int d\mathbf{x} J_{\mathbf{x},\omega} (\sqrt{Z_{-1}}\psi^{+}_{\mathbf{x},\omega})(\sqrt{Z_{-1}}\psi^{-}_{\mathbf{x},\omega})$$

We now *renormalize* $P_{Z_{-1},C_{h,-1}}(d\psi)$, by adding to it part of the quadratic part of $\mathcal{L}V^{(-1)}$:

$$\int P_{Z_{-1},C_{h,-1}}(d\psi)\, e^{-\mathcal{V}^{(-1)}(\sqrt{Z_{-1}}\psi)+\mathcal{B}^{(-1)}\left(\sqrt{Z_{-1}}\psi,\phi,J\right)} =$$
$$e^{-L\beta t_{-1}} \int P_{\tilde{Z}_{-2},C_{h,-1}}(d\psi)\, e^{-\tilde{\mathcal{V}}^{(-1)}(\sqrt{Z_{-1}}\psi)+\mathcal{B}^{(-1)}\left(\sqrt{Z_{-1}}\psi,\phi,J\right)}$$

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$$\tilde{Z}_{-2}(\mathbf{k}) = Z_{-1}[1+\chi_{h,-1}(\mathbf{k})\zeta_{-1}], \quad Z_{-2} = Z_{-1}[1+\zeta_{-1}]$$

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The factor $\exp(-L\beta t_j)$ in takes into account the different normalization of the two functional integrals.

We now *rescale* the field so that

$$\tilde{\mathcal{V}}^{(-1)}(\sqrt{Z_{-1}}\psi) = \hat{\mathcal{V}}^{(-1)}(\sqrt{Z_{-2}}\psi)$$

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$P_{Z_{-2}, \tilde{f}_{-1}^{-1}}(d\psi_0)$ is the integration withpropagator

$$\hat{g}_\omega^{(-1)}(\mathbf{k}) = \frac{1}{Z_{-2}} \frac{\tilde{f}_{-1}(\mathbf{k})}{D_\omega(\mathbf{k})}, \quad \tilde{f}_j(\mathbf{k}) = f_j(\mathbf{k}) \frac{Z_{j-1}}{\tilde{Z}_{j-1}(\mathbf{k})}$$

$$\int P_{Z_{-2},\tilde{f}_{-1}^{-1}}(d\psi_0)\, e^{-\hat{\mathcal{V}}^{(-1)}(\sqrt{Z_{-2}}[\psi+\psi_0])+\hat{\mathcal{B}}^{(-1)}\left(\sqrt{Z_{-2}}[\psi+\psi_0],\phi,J\right)}$$

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Note that the propagator is independent of the infrared cutoff for $j > h$ and $\tilde{f}_j(\mathbf{k}) \leq f_j(\mathbf{k})(1+\zeta_j)$, so that $\hat{g}^{(i)}$ satisfies the same bound as $g^{(i)}$.

Outline

The flow of the running coupling constants and the renormalization constants

$$\text{Let} \quad \varepsilon_j = \max_{i \geq j} |\lambda_i|, \quad \lambda_0 = \lambda; \quad \text{then}$$

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The main difficulty is to prove that ε_j stays bounded and of order λ, uniformly in $j \geq h$ and in the infrared cutoff h.

Outline

The tree expansion

At the end of the integration procedure we get

$$\mathcal{W}(\varphi, J) = -L\beta E_{L,\beta} + \sum_{m^\phi + n^J \geq 1} S^{(h)}_{2m^\phi, n^J}(\phi, J)$$

We can expand the functional $S^{(h)}_{2m^\phi, n^J}(\phi, J)$, the effective potential and the beta function as a sum of terms. Each term is associated with a *tree*, which describes how this term is produced along the iterative integration procedure.

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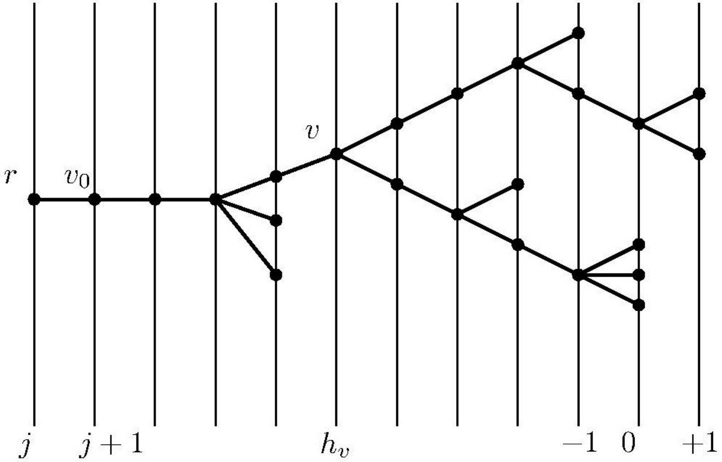
Let us consider, in particular, the effective potential on scale j.
We get:

$$\mathcal{V}^{(j)}(\sqrt{Z_j}\psi) + L\beta\tilde{E}_{j+1} =$$

$$\sum_{n=1}^{\infty} \sum_{\tau \in \mathcal{T}_{j,n}} \sum_{\mathbf{P} \in \mathcal{P}_\tau} \sqrt{Z_j}^{|P_{v_0}|} \int d\mathbf{x}_{v_0} \tilde{\psi}(P_{v_0}) K^{(j+1)}_{\tau,\mathbf{P}}(\mathbf{x}_{v_0})$$

$$\tilde{\psi}(P_v) = \prod_{f \in P_v} \psi^{\sigma(f)}_{\mathbf{x}(f), \omega(f)}$$

$K^{(j+1)}_{\tau,\mathbf{P}}(\mathbf{x}_{v_0})$ is a suitable function, which is obtained by summing the values of all the Feynmann graphs compatible with **P**, and applying iteratively in the vertices of the tree, different from the endpoints and v_0, the $\mathcal{R}$-operation, starting from the vertices with higher scale.



The main bound

In order to control, uniformly in L and β, the various sums in the tree expansion, one has to exploit in a careful way the $\mathcal{R}$ operation acting on the vertices of the tree.

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In absence of the $\mathcal{R}$ operation, one gets the dimensional bound (here it is essential that the particles are fermions):

$$\int d\mathbf{x}|K^{(j+1)}_{\tau,\mathbf{P}}(\mathbf{x})| \le L\beta\,(C\varepsilon_{j+1})^n\gamma^{-j(-2+|P_{v_0}|/2)}\ .$$

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This bound allows us to associate a factor $\gamma^{2-|P_v|/2}$ with any trivial or non trivial vertex of the tree. This would allow us to control the sums over the scale labels and $\mathcal{P}_\tau$, provided $|P_v|$ were larger than 4 in all vertices, which is however not true.

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Roughly, this follows from the fact that, in the Taylor expansion of the external ψ field in a vertex of scale h_v, each derivative includes in the bound the difference between two points of the corresponding cluster, hence a *bad* factor γ^{-h_v}, while the field derivative produces a *good* factor $\gamma^{h_{v'}}$ in the vertex v' where it is contracted.

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It is easy to see that, in the new bound, we can associate to any vertex a factor γ^{d_v}, with $d_v = 2 - |P_v|/2 - r_v < 0$.

Outline

Existence of the infrared limit

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$$\lambda_j \xrightarrow[j\to-\infty]{} \lambda_{-\infty}(\lambda), \quad \frac{Z_{j-1}^{(i)}}{Z_j^{(i)}} \xrightarrow[j\to-\infty]{} \eta_i(\lambda_{-\infty})$$

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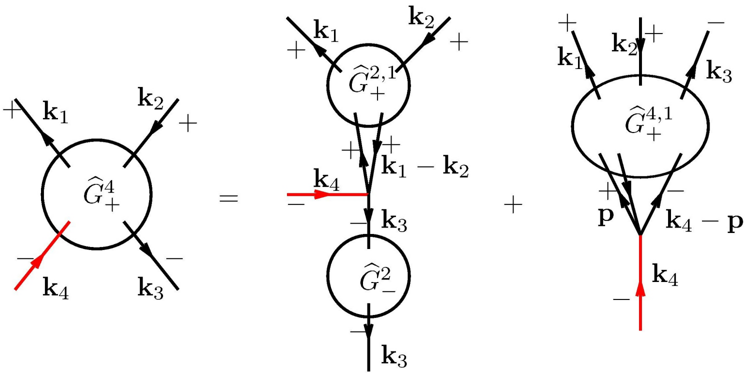
$\lambda_{-\infty}(\lambda)$ and $\eta_i(\lambda_{-\infty})$ are analytic functions.

Outline

The Green function with 4 external legs

$$-\hat{G}^4_+(\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4) = \lambda\hat{g}_-(\mathbf{k}_4)\Big[\hat{G}^2_-(\mathbf{k}_3)\hat{G}^{2,1}_+(\mathbf{k}_1-\mathbf{k}_2,\mathbf{k}_1,\mathbf{k}_2)+$$
$$+\frac{1}{L\beta}\sum_{\mathbf{p}} G^{4,1}_+(\mathbf{p};\mathbf{k}_1,\mathbf{k}_2,\mathbf{k}_3,\mathbf{k}_4-\mathbf{p})\Big]$$



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In particular, if we fix the external momenta in the Dyson equation so that

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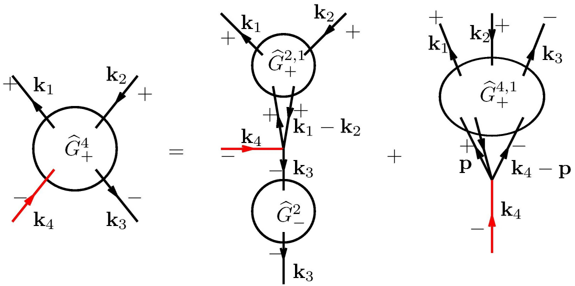
and ε_h is small enough, we get

$$\hat{G}^2_\omega(\bar{\mathbf{k}}) = \frac{1}{Z_h^{(1)} D_\omega(\bar{\mathbf{k}})}[1 + O(\varepsilon_h^2)], \quad D_\omega(\mathbf{k}) = -ik_0 + \omega k$$

$$\hat{G}^{2,1}_\omega(2\bar{\mathbf{k}}, \bar{\mathbf{k}}, -\bar{\mathbf{k}}) = -\frac{Z_h^{(2)}}{\left(Z_h^{(1)}\right)^2 D_\omega(\bar{\mathbf{k}})^2}[1 + O(\varepsilon_h^2)]$$

$$\hat{G}^4_+(\bar{\mathbf{k}}, -\bar{\mathbf{k}}, -\bar{\mathbf{k}}, \bar{\mathbf{k}}) = \frac{1}{\left(Z_h^{(1)}\right)^2 |\bar{\mathbf{k}}|^4}[-\lambda_h + O(\varepsilon_h^2)]$$



$+$ $\mathbf{k}_1$ $\mathbf{k}_2$ $+$
$\widehat{G}^4_+$
$-$ $\mathbf{k}_4$ $\mathbf{k}_3$ $-$
$=$
$+$ $\mathbf{k}_1$ $\mathbf{k}_2$ $+$
$\widehat{G}^{2,1}_+$
$+$ $+$ $\mathbf{k}_1 - \mathbf{k}_2$
$\mathbf{k}_4$ $-$
$-$ $\mathbf{k}_3$
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$+$ $\mathbf{p}$ $-$ $\mathbf{k}_4 - \mathbf{p}$
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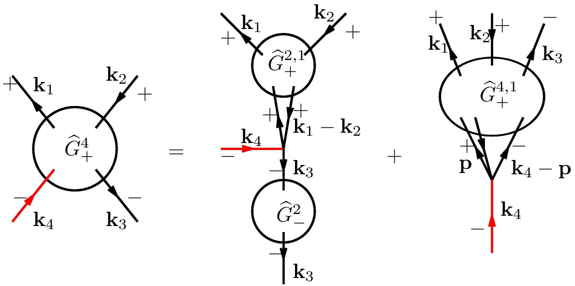

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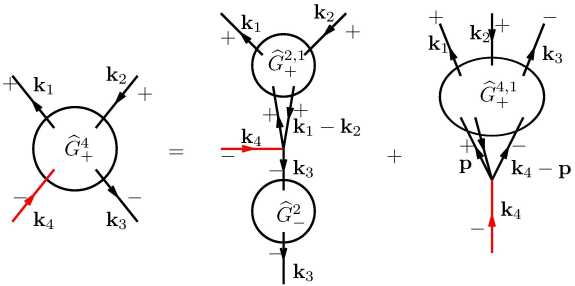

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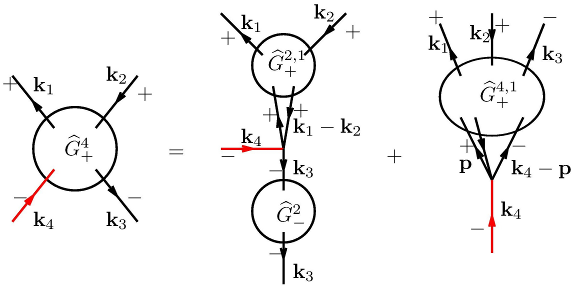

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By local gauge invariance $\quad \dfrac{Z_h^{(2)}}{Z_h^{(1)}} = 1 + O(\varepsilon_h)$

Were we able to bound the second term in the r.h.s. as

$$C\frac{\varepsilon_h^2}{\left(Z_h^{(1)}\right)^2 |\bar{\mathbf{k}}|^4}$$

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However, the RG analysis only allows us to bound such term as

$$C \frac{\varepsilon_h^2}{\left(Z_h^{(1)}\right)^2 |\bar{\mathbf{k}}|^4} \left[\frac{\gamma^{C\varepsilon_h |h|} - 1}{\varepsilon_h} \right]$$

which is of course not sufficient.

The natural guess is that the origin of the problem is in the fact that one is not taking into account some crucial cancellations related with the gauge invariance.

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Hence, inspired by the analysis in the physical literature (W. Metzner and C. Di Castro, PRB 47, 1993), we rewrite $\hat{G}^{4,1}_{\omega}$ in terms of $\hat{G}^{4}_{+}$ by suitable Ward identities, that is the identities obtained by applying the chiral Gauge transformation

$$\psi^{\pm}_{\mathbf{x},+} \to e^{\pm i\alpha_{\mathbf{x}}}\psi^{\pm}_{\mathbf{x},+}, \quad \psi^{\pm}_{\mathbf{x},-} \to \psi^{\pm}_{\mathbf{x},-}$$

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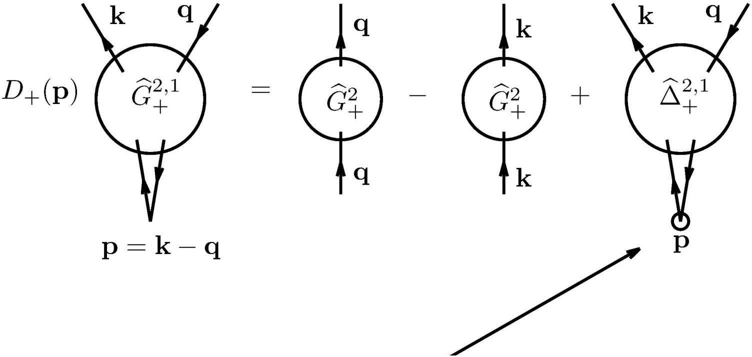
The problem is finally solved by using other identities, which we call correction identities.

Outline

The first Ward identity



$$D_\omega(\mathbf{p}) = -ip_0 + \omega p, \quad \int d\mathbf{k} C_+(\mathbf{k}, \mathbf{k} - \mathbf{p}) \psi^+_{\mathbf{k},+} \psi^-_{\mathbf{k}-\mathbf{p},+}$$

$$C_\omega(\mathbf{k}^+, \mathbf{k}^-) = [C_{h,0}(\mathbf{k}^-) - 1] D_\omega(\mathbf{k}^-) - [C_{h,0}(\mathbf{k}^+) - 1] D_\omega(\mathbf{k}^+)$$

At graph level, the Ward identities follow from the trivial identity

$$\frac{1}{D_\omega(\mathbf{k})} - \frac{1}{D_\omega(\mathbf{k}+\mathbf{p})} = \frac{D_\omega(\mathbf{p})}{D_\omega(\mathbf{k})D_\omega(\mathbf{k}+\mathbf{p})}$$

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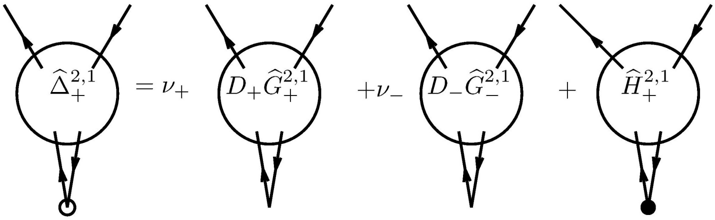
However, this is not true, but we have the *first correction identity*



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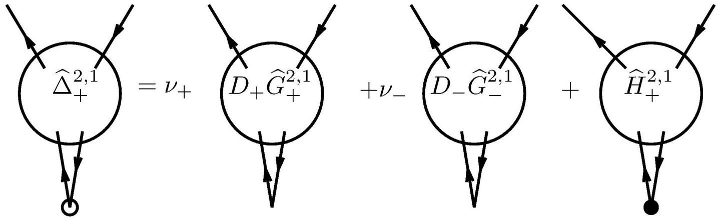

The filled point represents

$$\int d\mathbf{k} C_+(\mathbf{k},\mathbf{k}-\mathbf{p})\psi^+_{\mathbf{k},+}\psi^-_{\mathbf{k}-\mathbf{p},+} - \sum_\omega \nu_\omega D_\omega(\mathbf{p}) \int d\mathbf{k}\psi^+_{\mathbf{k},\omega}\psi^-_{\mathbf{k}-\mathbf{p},\omega}$$

- ν_+, ν_- are $O(\lambda)$ and weakly dependent on h.
- the term $H_+^{2,1}$ is indeed negligible, in the sense that, if we can make the limit $h \to -\infty$, its contribution goes to 0 as the external momenta go to 0 (of course staying much larger than γ^h).

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If we insert the correction identity in the WI, we get

$$(1-\nu_+)D_+(\mathbf{p})\hat{G}_+^{2,1}(\mathbf{p},\mathbf{k},\mathbf{q}) - \nu_- D_-(\mathbf{p})\hat{G}_-^{2,1}(\mathbf{p},\mathbf{k},\mathbf{q}) =$$

$$= \hat{G}_+^2(\mathbf{q}) - \hat{G}_+^2(\mathbf{k}) + \hat{H}_+^{2,1}(\mathbf{p},\mathbf{k},\mathbf{q})$$

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The presence of $G_+^{2,1}$ in the correction identity is not a problem. In fact, this function satisfies another Ward identity and a corresponding correction identity, involving the same constants ν_+, ν_-, and we get

$$(1-\nu_+)D_-(\mathbf{p})\hat{G}_-^{2,1}(\mathbf{p},\mathbf{k},\mathbf{q}) - \nu_- D_+(\mathbf{p})\hat{G}_+^{2,1}(\mathbf{p},\mathbf{k},\mathbf{q}) = \hat{H}_-^{2,1}(\mathbf{p},\mathbf{k},\mathbf{q})$$

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$$(1-\nu_+)D_+(\mathbf{p})\hat{G}^{2,1}_+(\mathbf{p},\mathbf{k},\mathbf{q}) - \nu_- D_-(\mathbf{p})\hat{G}^{2,1}_-(\mathbf{p},\mathbf{k},\mathbf{q}) =$$

$$= \hat{G}^2_+(\mathbf{q}) - \hat{G}^2_+(\mathbf{k}) + \hat{H}^{2,1}_+(\mathbf{p},\mathbf{k},\mathbf{q})$$

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$$(1-\nu_+)D_-(\mathbf{p})\hat{G}^{2,1}_-(\mathbf{p},\mathbf{k},\mathbf{q}) - \nu_- D_+(\mathbf{p})\hat{G}^{2,1}_+(\mathbf{p},\mathbf{k},\mathbf{q}) = \hat{H}^{2,1}_-(\mathbf{p},\mathbf{k},\mathbf{q})$$

Hence, we can represent $\hat{G}^{2,1}_+$ as a linear combination of propagators, as in the case of the formal WI, up to negligible terms.

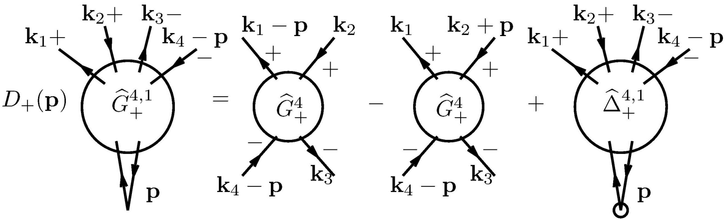
The first WI can be used to prove that

$$\frac{Z_h^{(2)}}{Z_h^{(1)}} = 1 + O(\varepsilon_h)$$

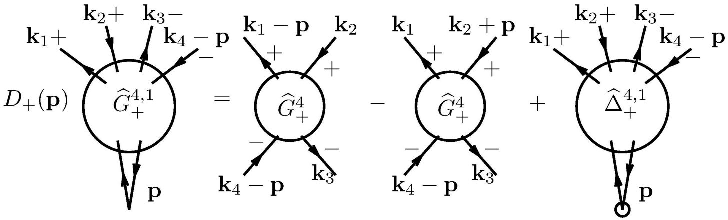
In order to get this result, we put $\mathbf{k} = -\mathbf{q} = \bar{\mathbf{k}}$, with $|\bar{\mathbf{k}}| = \gamma^h$. For these values of the external momenta, $\hat{H}_-^{2,1}(\mathbf{p}, \mathbf{k}, \mathbf{q})$ is not negligible, but one can show that

$$\left| \frac{\hat{\Delta}_+^{2,1}(2\bar{\mathbf{k}}, \bar{\mathbf{k}}, -\bar{\mathbf{k}})}{D_+(2\bar{\mathbf{k}})} \right| \le C\gamma^{-2h}\varepsilon_h \frac{Z_h^{(2)}}{\left(Z_h^{(1)}\right)^2}$$

Outline

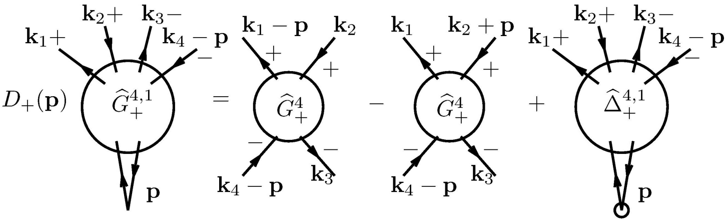


$$0=\mathbf{k}_1+\mathbf{k}_3-\mathbf{k}_2-\mathbf{k}_4$$



$$0 = \mathbf{k}_1 + \mathbf{k}_3 - \mathbf{k}_2 - \mathbf{k}_4$$

If one inserts this identity in the Dyson equation, the two terms containing the four point function give the right bound, but the correction term $\Delta_+^{4,1}/D_+(\mathbf{p})$ has the same bad bound as the original one, so making apparently useless the WI.



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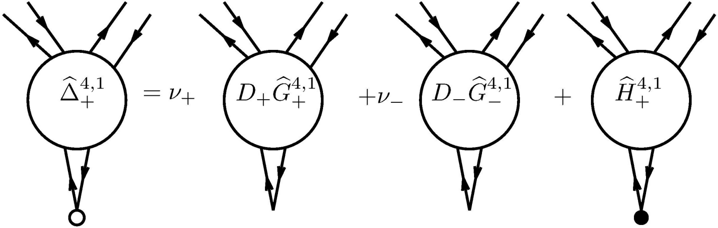
However, there is again a correction identity, that allows us to solve this problem.

Outline

The second correction identity



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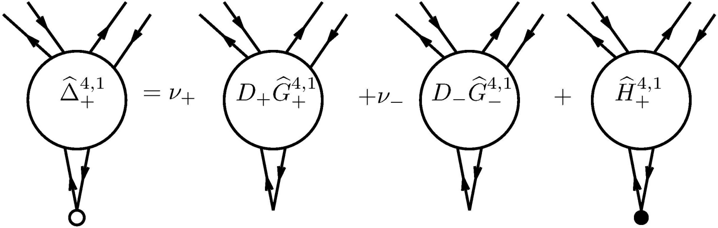
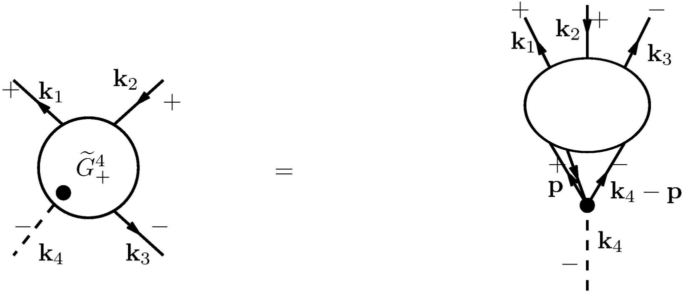


In this case, to show that the contribution of $\hat{H}^{4,1}_{+}$ has the right bound is not so simple, since we need to evaluate it for external momenta of order γ^h.

It turns out that we have to evaluate a correlation very similar to the four point function with one of the external vertices substituted with the correction vertex and the free propagator entering this special vertex.



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The presence of a ”special” vertex has however the effect that a new running coupling appears, associated with the local part of the terms with four external lines among which one is the dotted line, to which the bare propagator $\hat{g}(\mathbf{k}_4)$ is associated; we will call this new running coupling constant $\tilde{\lambda}_j$.

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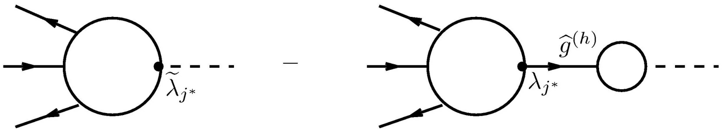
One then gets for $\tilde{G}_4$ a bound very similar to the one for $\hat{G}_4$, except that λ_h is replaced by $\tilde{\lambda}_h$ and there is no wave function renormalization associated to the external line with momentum $\mathbf{k}_4$.

We can however identify two class of terms in the expansion of $\tilde{G}_4$ and summing them has the effect that also the external line with momentum $\mathbf{k}_4$ is dressed by the interaction.



$$\tilde{\lambda}_{j^*}\frac{1}{D_-(\mathbf{k}_4)} - \lambda_{j^*}\frac{1}{Z_h D_-(\mathbf{k}_4)}\left[\sum_{j=h+1}^{0}\tilde{z}_j Z_j D_-(\mathbf{k}_4)\right]\frac{1}{D_-(\mathbf{k}_4)}$$

$$|\tilde{\lambda}_j - \alpha\lambda_j| \leq c\varepsilon_h\gamma^{j/2}, \quad |\tilde{z}_j Z_j - \alpha z_j Z_j| \leq c\varepsilon_h\gamma^{j/2}$$

Hence, we get:

$$\alpha\lambda_{j^*}\frac{1}{D_-(\mathbf{k}_4)}\left[1-\frac{\sum_{j=h+1}^{0} z_j Z_j}{Z_h}\right]+\frac{O(\varepsilon_h)}{Z_h D_-(\mathbf{k}_4)}$$

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$$|\lambda\tilde{G}_4| \le C\frac{\varepsilon_h^2}{\left(Z_h^{(1)}\right)^2 |\bar{\mathbf{k}}|^4}$$

Remark - The improvement of the bound is due to the cancellations implied by the inductive hypothesis that λ_j does not grow as $j \to -\infty$. In order to use this property we had to choose $\nu_\pm$ so that $\tilde{\lambda}_j \simeq \alpha\lambda_j$.