How to calculate GPDs on the lattice

Paul Rakow

THE UNIVERSITY *of* LIVERPOOL

Acknowledgements

Reporting on work by QCDSF LHPC SESAM
Special thanks
Ph Hägler, D Pleiter, W Schroers and J Zanotti

This talk: Discusses methods

Next talk: Phillip Hägler talks about results.

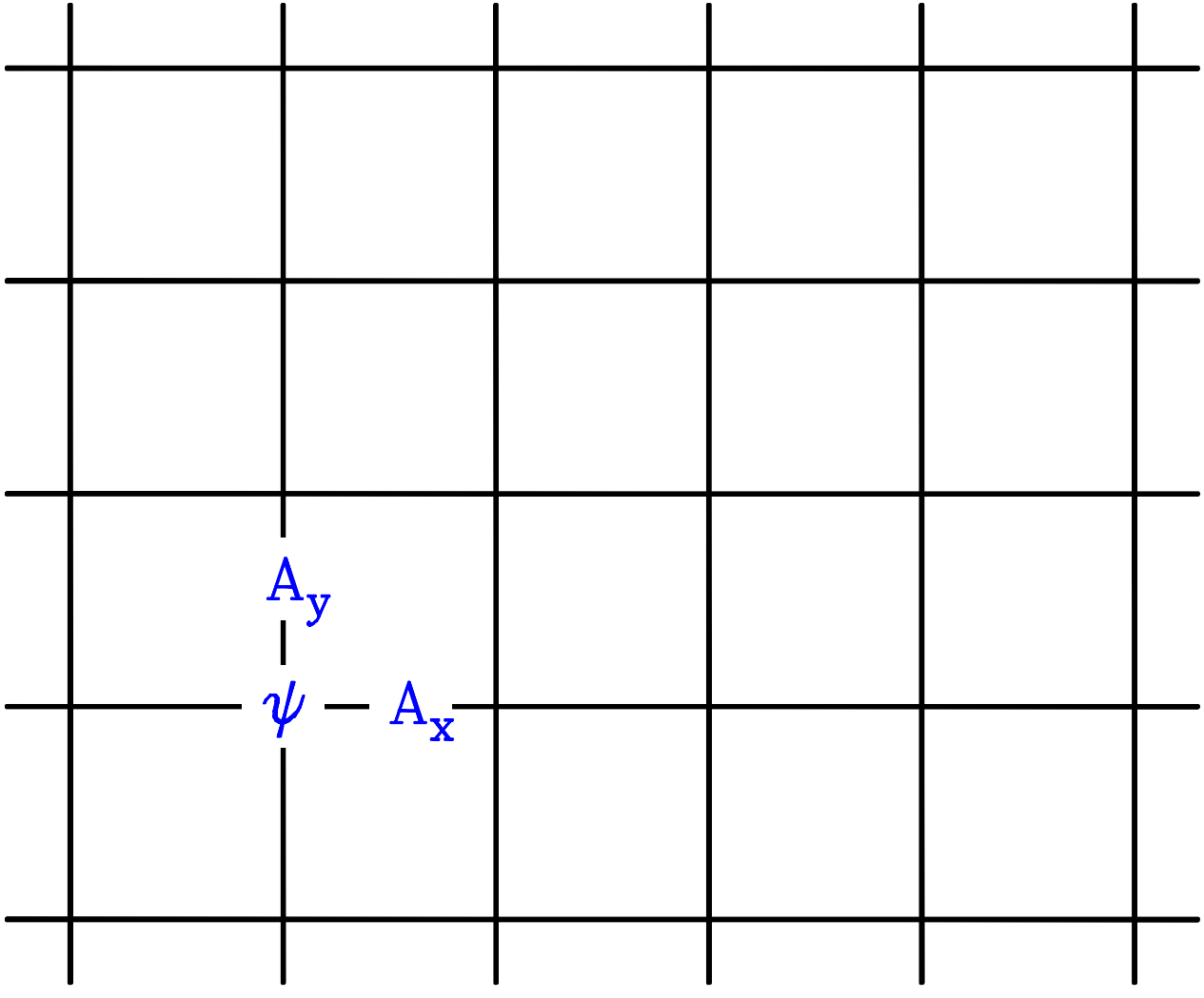
Introduction

- Lattice Theory
- Finding hadron propagators and masses
- Finding forward matrix elements (eg structure functions)
- Non-forward matrix elements (GPD)
- Extrapolations
- Conclusions

What is lattice gauge theory?

Computational field theory.
We can't put a continuum in a computer — divide up space and time into discrete steps (like solving a differential equation, or doing an integral).
Discrete 4-d Euclidean lattice, lattice spacing a.
Quark fields ψ on the sites.
Gauge fields A_μ on the links.
A regularisation of field theory, preserving exact gauge invariance.

What is lattice gauge theory?



What is lattice gauge theory?

Lattice gauge theory really is a first-principles calculation, the only input is the action of QCD.
Output: masses and matrix elements.
Complications: Lack of computer power means we have to do various extrapolations. Realistically light quarks are still too expensive, so we have to extrapolate towards the physical u and d masses.

Path Integral

We want to “measure”

$$\langle F \rangle = \frac{1}{Z} \int dA \; d\psi \; d\bar{\psi} \; F(A, \psi, \bar{\psi}) e^{-S_g(A) - \bar{\psi} M_f(A) \psi}$$

The action is bilinear in the fermion field – we can do the fermionic integral analytically. (Fortunate, because it avoids the issue of dealing with Grassmann quantities on the computer).

$$Z = \int dA \; d\psi \; d\bar{\psi} \; e^{-S_g(A) - \bar{\psi} M_f(A) \psi}$$

$$= \int dA \; \det(M_f(A))^{N_f} e^{-S_g(A)}$$

We have algorithms that generate A_μ configurations with the probability

$$\det(M_f(A))^{N_f} e^{-S_g(A)} \quad \text{Expensive} \quad \text{“Dynamical”}$$

$$e^{-S_g(A)} \quad \text{Cheap} \quad \text{“Quenched”}$$

Quenched is equivalent to dropping Feynman diagrams with sea-quark loops.
The integral becomes a statistical expectation value.

Transfer matrix decomposition

$$\langle Q \rangle = \frac{\sum_i \langle i|Q|i\rangle e^{-E_i T}}{\sum_i \langle i|i\rangle e^{-E_i T}} \to \langle 0|Q|0\rangle$$

Like decomposing into Hamiltonian eigenstates in QM, but because the lattice is Euclidean we have exponential decays, not phases.
These sums look like the state sums in thermodynamics — this is more than just an analogy, simulating on lattices with small T allows us to find out what QCD does at finite temperature. (Deconfinement phase transition, quark-gluon plasma, etc.)

Transfer matrix decomposition

Same decomposition for a two-point function.

$$\langle A(0)B(t)\rangle = \frac{\sum_{i,j}\langle i|A|j\rangle\langle j|B|i\rangle e^{-E_j t}e^{-E_i(T-t)}}{\sum_i \langle i|i\rangle e^{-E_i T}}$$

$$\rightarrow \langle 0|A|j_0\rangle\langle j_0|B|0\rangle e^{-E_{j_0}t} + \langle i_0|A|0\rangle\langle 0|B|i_0\rangle e^{-E_{i_0}(T-t)}$$

Measure energy relative to vacuum, so $\langle 0|0\rangle e^{-E_0 T} \rightarrow 1$.

Two point functions

Baryon propagator in terms of the transfer matrix:

$$\langle B(t)\ \bar{B}(0)\rangle = Tr\left[e^{-H(T-t)}Be^{-Ht}\bar{B}\right]$$

T is the lattice length, t is the separation of the operators. Introduce a complete set of states

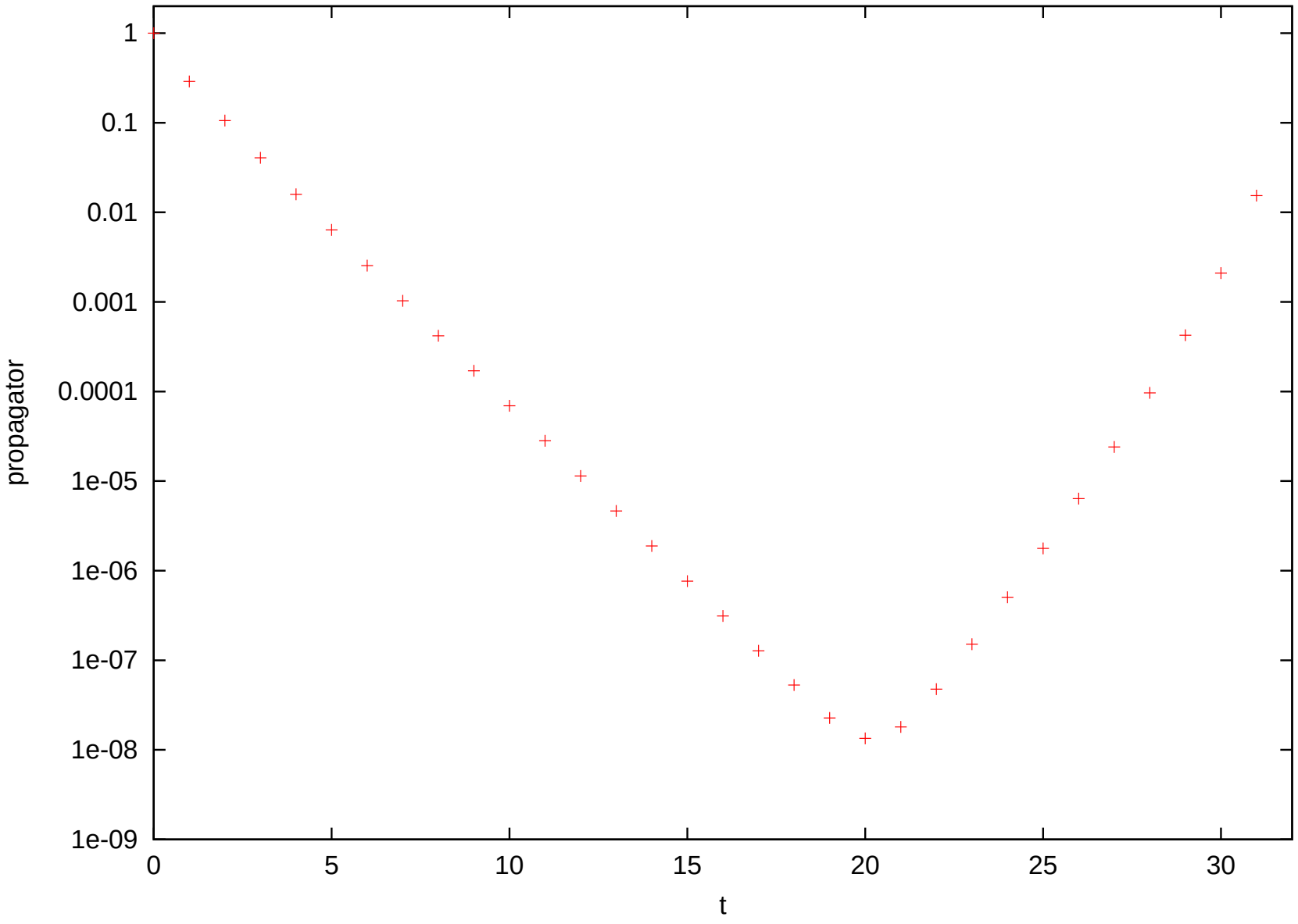
$$\langle B(t)\ \bar{B}(0)\rangle = \sum_{i,j} e^{-E_i(T-t)}\langle i|B|j\rangle e^{-E_j t}\langle j|\bar{B}|i\rangle$$

Two point functions

When T, t are both large, the lowest energy states dominate the sum

$$e^{-E_N t}\langle 0|B|N\rangle\langle N|\bar{B}|0\rangle + e^{-E_{\overline{N^\star}}(T-t)}\langle \overline{N^\star}|\bar{B}|0\rangle\langle 0|B|\overline{N^\star}\rangle$$

The propagator decay rate gives the baryon energy (mass, dispersion relation, check of restoration of Lorentz invariance).



Nucleon $\overline{\mathrm{N}^{\star}}$

Three-point functions

Follow the same argument for the forward expectation value

$$\langle B(t)\, O(\tau)\, \bar{B}(0)\rangle$$

$$= Tr\left[e^{-H(T-t)} B e^{-H(t-\tau)} O e^{-H\tau} \bar{B}\right]$$

The dominant contribution when $0 \ll \tau \ll t \ll T$ is again given by the lowest energy state

$$e^{-E_N t}\langle 0|B|N\rangle\langle N|O|N\rangle\langle N|\bar{B}|0\rangle$$

The three point function is proportional to the interesting quantity $\langle N|O|N\rangle$, but we need to remove the other factors which multiply it.

Three-point functions

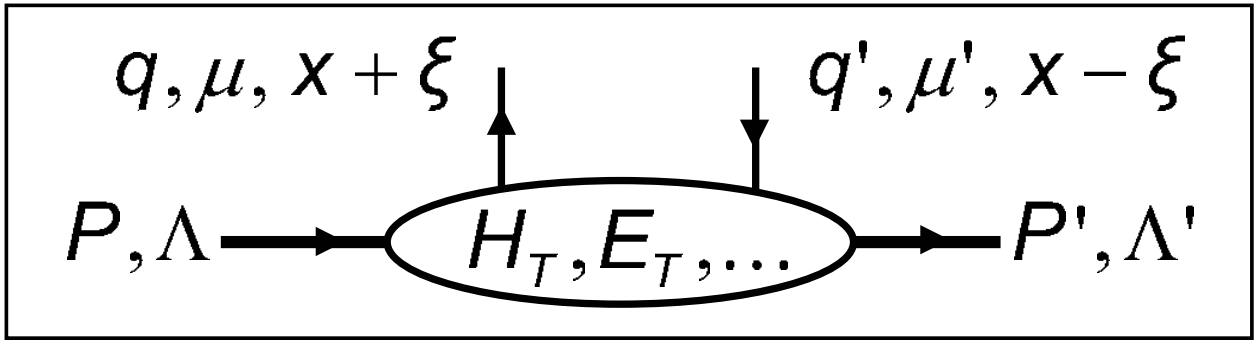
$$e^{-E_N t}\langle 0|B|N\rangle\langle N|O|N\rangle\langle N|\bar{B}|0\rangle$$

Comparing this with the expression for the two-point function.

$$e^{-E_N t}\langle 0|B|N\rangle\langle N|\bar{B}|0\rangle$$

We see that the ratio three-point/two-point gives $\langle N|O|N\rangle$, independent of τ.
If we don't have $0 \ll \tau \ll t \ll T$ (all operators separated by large times) there will still be contrbutions from excited states, plateau won't be perfect.

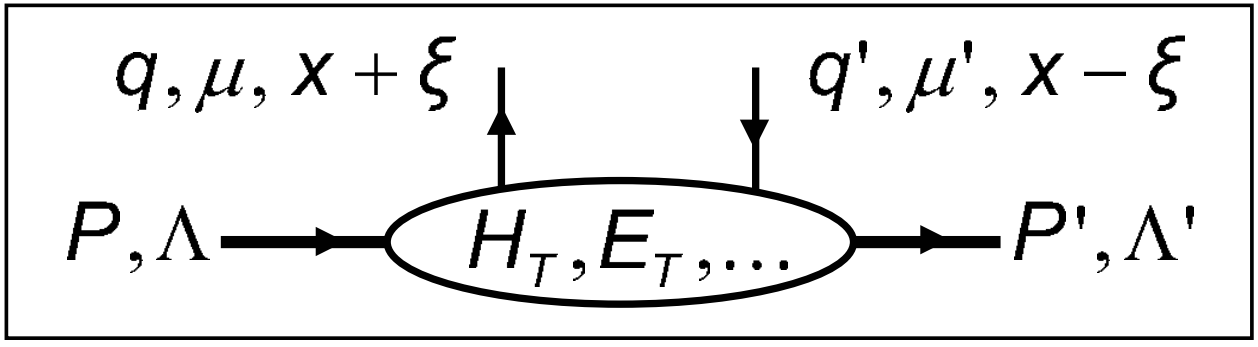
Three-point functions



Net momentum transfer $\Delta = q' - q = p' - p$.

On the lattice we can get some information on the GPDs by measuring Generalised Form Factors (GFFs).

Three-point functions



Form factor using (eg)

$$\left(\frac{i}{2}\right)^n \bar{\psi}\gamma_{\mu_1}\overleftrightarrow{D}_{\mu_2}\cdots\overleftrightarrow{D}_{\mu_n}\psi$$

symmetrised and made traceless.
Moments of GPDs from operator insertions with momentum transfer.

Three-point functions

$$\langle B(t,\vec{p'})\, O(\tau,\vec{\Delta})\, \bar{B}(0,\vec{p})\rangle$$

$$= Tr\left[e^{-H(T-t)} B e^{-H(t-\tau)} O e^{-H\tau} \bar{B}\right]$$

We can calculate three-point functions with momentum transfer $\vec{\Delta}$ at the operator. The operator changes a proton with momentum $\vec{p}$ into one with 3-momentum $\vec{p'} \equiv \vec{p} + \vec{\Delta}$.

Three-point functions

$$\langle B(t,\vec{p'})\, O(\tau,\vec{\Delta})\, \bar{B}(0,\vec{p})\rangle$$

$$= Tr\left[e^{-H(T-t)} B e^{-H(t-\tau)} O e^{-H\tau} \bar{B}\right]$$

We choose 3-momenta, 4th component is fixed by kinematics (energy of a proton of given momentum). Since this usually means that the scattered proton has a different energy from the incoming proton, the non-forward three-point function has a more complicated dependence on the times t and τ, and it isn't quite as easy to remove all the external factors to leave the matrix element alone.

Three-point functions

$$C^{2pt}(t,\vec{p}) = \langle B(t,\vec{p})\bar{B}(0,\vec{p})\rangle \approx A^2(\vec{p})e^{-E_p t}$$

$$C^{3pt}(t,\vec{p'},\vec{p}) = \langle B(t,\vec{p'})\; O(\tau,\vec{\Delta})\; \bar{B}(0,\vec{p})\rangle$$

$$\approx A(\vec{p'})e^{-E_{p'}(t-\tau)}\langle N(\vec{p'})|O|N(\vec{p})\rangle e^{-E_p\tau} A(\vec{p})$$

Can we combine 2pt functions to cancel the amplitudes and exponentials, and just get the matrix element?

Three-point functions

$$C^{3pt}(t,\vec{p'},\vec{p}) \approx A(\vec{p'})e^{-E_{p'}(t-\tau)}\langle N(\vec{p'})|O|N(\vec{p})\rangle e^{-E_p\tau}A(\vec{p})$$

$$C^{2pt}(t,\vec{p}) \approx A^2(\vec{p})e^{-E_p t}$$

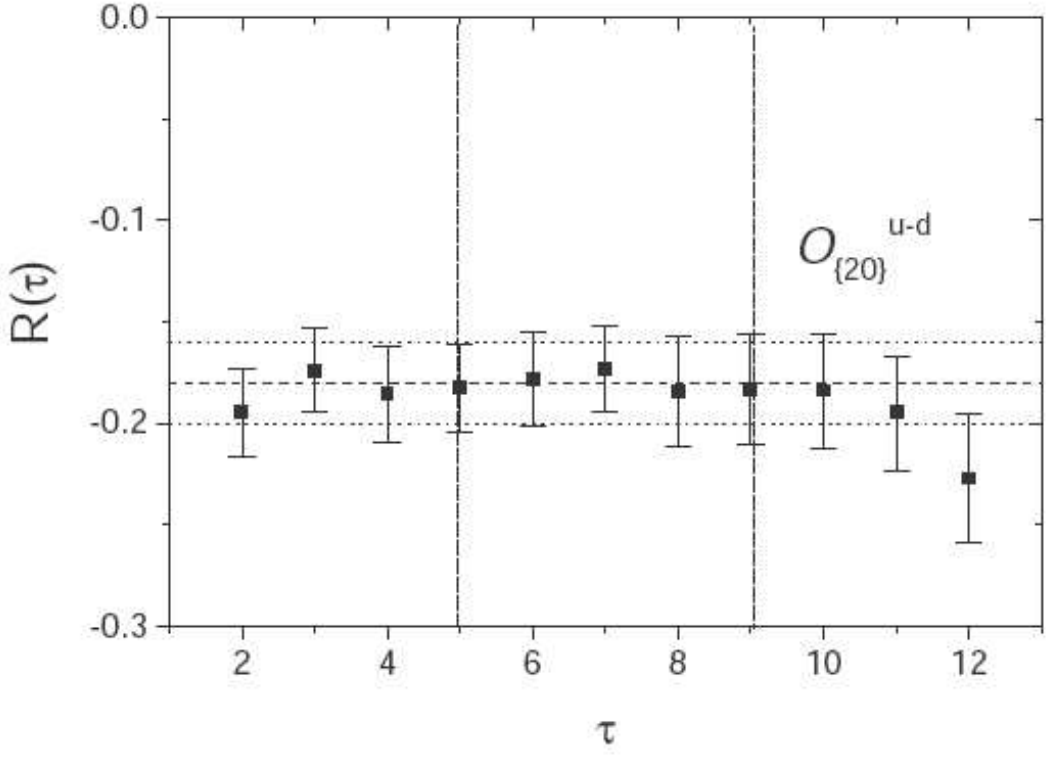
$$\left[C^{2pt}(t,\vec{p'})C^{2pt}(t,\vec{p})\right]^{\frac{1}{2}} \left[\frac{C^{2pt}(\tau,\vec{p})C^{2pt}(t-\tau,\vec{p'})}{C^{2pt}(\tau,\vec{p'})C^{2pt}(t-\tau,\vec{p})}\right]^{\frac{1}{2}}$$

$$\approx A(\vec{p'})e^{-E_{p'}(t-\tau)}e^{-E_p\tau}A(\vec{p})$$

Exactly the factor needed to extract the matrix element.

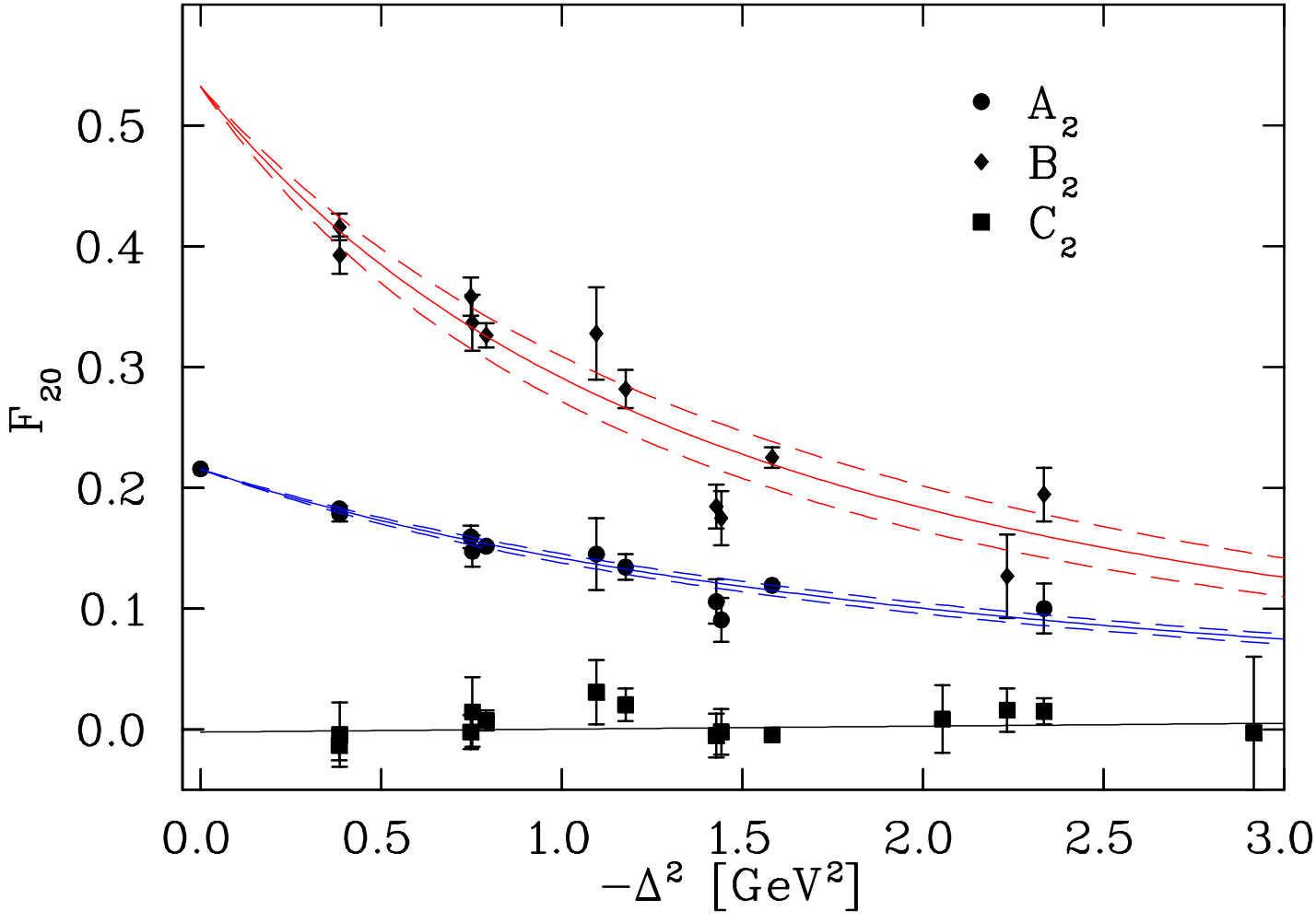
Plateau

C^{2pt} factors should cancel t and τ dependence, leaving a plateau.

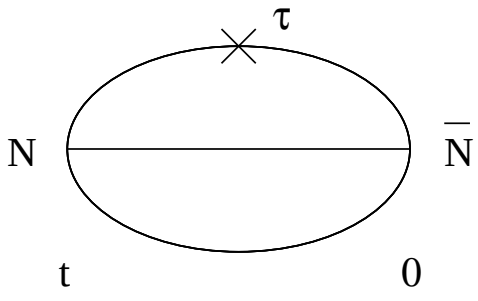


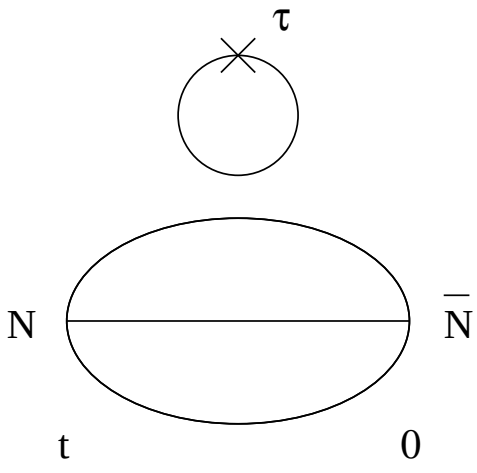
hep-lat/0304018 LHPC and SESAM

Three-point functions



Three-point functions





Two terms. “Quark line disconnected” only contributes for flavour singlet operators. Hard to measure. Under renormalisation mixes with gluonic operators (which are also hard to measure). Usually we make our lives easy by concentrating on non-singlet quantities.

Operator Product Expansion (OPE)

We can relate the moments of structure functions (both polarised and unpolarised) to the expectation values of matrix elements via the OPE.

$$\int_0^1 dx\; x^{n-2} F_2(x, Q^2) = \sum_f c^f_{2,n}(Q^2, \mu^2) v^f_n(\mu^2)$$

The Wilson coefficients c^f can be found perturbatively if Q^2, μ^2 are large enough. Lattice QCD offers a method of finding the reduced matrix elements v^f.

Typical operator

$$\left(\frac{i}{2}\right)^n \bar{\psi}\gamma_{\mu_1}\gamma_5 \overleftrightarrow{D}_{\mu_2} \cdots \overleftrightarrow{D}_{\mu_n}\psi$$

symmetrised and made traceless.

Add momentum transfer: Get form factors and generalised parton distributions.

Covariant derivatives replaced by finite differences to produce lattice operators.

This is a source of discretisation error, $a^2\Delta^2$.

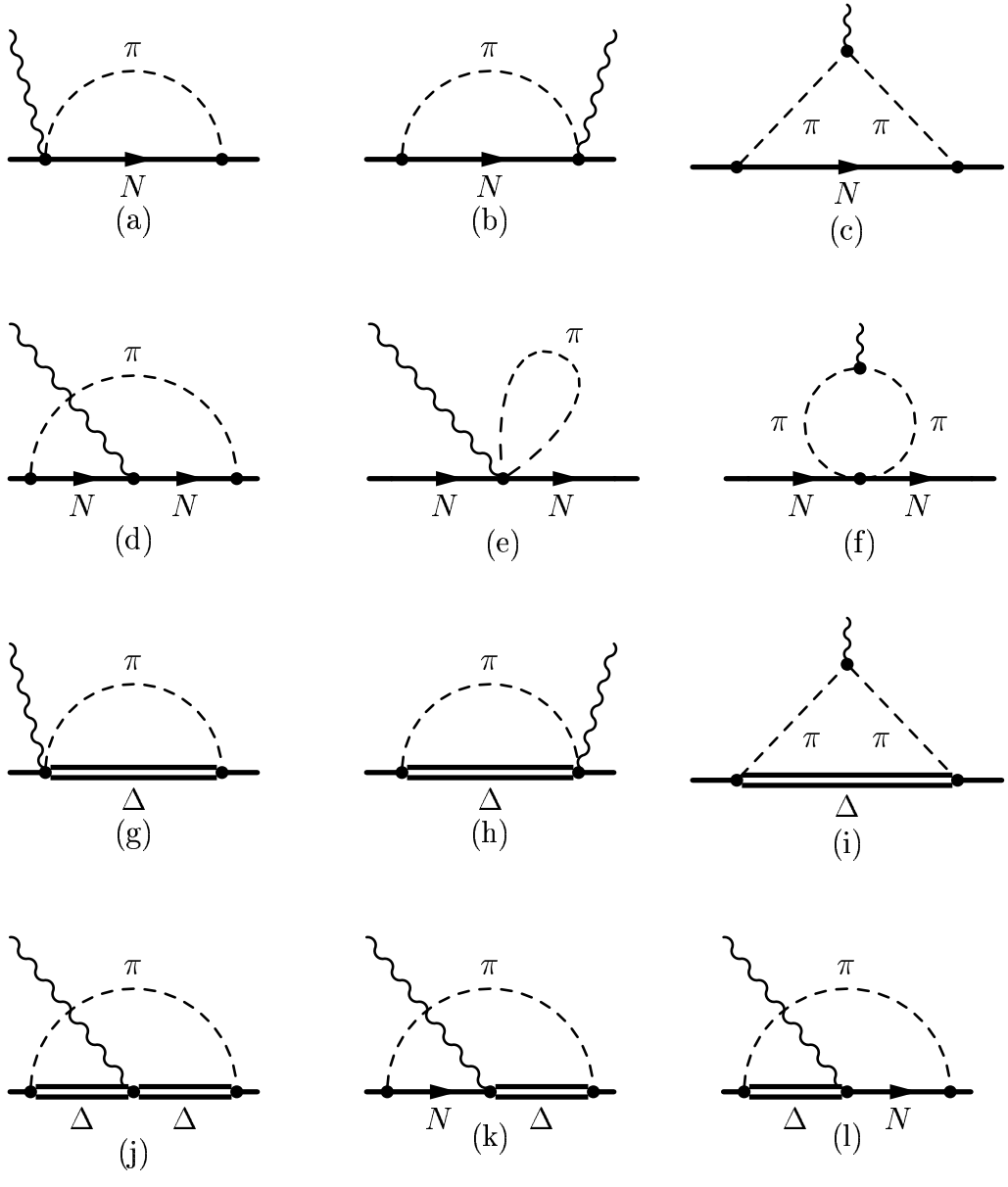
Systematic Issues

- Lattice spacing a. Extrapolate towards $a \to 0$.
- Lattice volume: Much bigger than proton.
- Quark mass: At present lattice quarks are too heavy (computer cost).
- Renormalisation (improve on pert theory?)

Extrapolating in m_q

To keep down the cost of calculation, and to minimise the effects of calculating on a finite size lattice, calculations have to be done at a rather large quark mass ($m_q \approx m_s$).
We then have to extrapolate to the physical m_u, m_d.
How?
Simplest hypothesis: Everything is linear in m_q (or equivalently m_π^2).
This usually fits the data reasonably well.
More sophisticated idea - use formulae from chiral perturbation theory (or quenched chiral perturbation theory). Negele hep-lat/0107010; Detmold et al. hep-lat/0108002

Extrapolating in m_q



Extrapolating in m_q

How far should one-loop results apply? (Folklore - up until $m_{ps} = m_K$?)
Variants of chiral perturbation theory.
Typical Issues: Cut offs. Inclusion of Δ baryon. Multitude of coupling constants.
Someday we will bypass this by calculating with realistic quark masses.

Conclusions

- Lattice methods allow first principles calculation of hadronic matrix elements.
- Disconnected contributions hard to measure.
- Extrapolation still needed to reach physical m_u and m_d.

Time for easy questions

Time for easy questions

(Save hard questions for the next speaker).