New method for calculating helicity amplitudes of jet–like QED processes

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C. Carimalo, A. Schiller, V.G. Serbo

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and
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OUTLOOK:

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For simplicity we consider here
mainly bremsstrahlung processes

1. Introduction

Colliders

Accelerators with high-energy colliding e^+e^-, γe, $\gamma\gamma$ and $\mu^+\mu^-$ beams.

QED processes with σ which **do not drop** with increasing energy.

M_{fi} for polarized $|i\rangle$ and $|f\rangle$ states.

Jet-like processes

These reactions have the form of a two–jet process with the exchange of a virtual photon γ^* in the t-channel (Fig. 1):

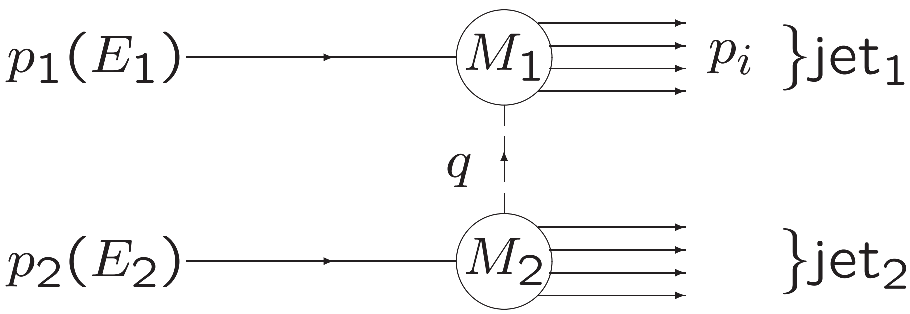

Fig. 1. Generic block diagram of the two–jet process $ee \to$ jet$_1$ jet$_2$

The subject of our consideration is the jet–like process of Fig.1 at high energies

$$s = 2p_1p_2 = 4E_1E_2 \gg m_i^2 \tag{1}$$

for arbitrary helicities of leptons $\lambda_i = \pm 1/2$ and photons $\Lambda_i = \pm 1$.

The emission and scattering angles θ_i are:

$$\frac{m_i}{E_i} \lesssim \theta_i \ll 1\,, \quad m_i \lesssim |\mathbf{p}_{i\perp}| \ll E_i\,. \tag{2}$$

Importance of jet-like QED processes

Figs. 2–12.

The discussed processes are important:

1) As monitoring processes;

2) Due to their large cross sections those reactions contribute as significant background;

3) The bremsstrahlung process is the leading beam loss mechanism (H. Burkhardt; Y. Funakoshi).

4) The methods to calculate the helicity amplitudes of these processes can be easily translated to several semihard QCD processes such as $\gamma\gamma \to q\bar{q}Q\bar{Q}$ and $\gamma\gamma \to MM'$, $\gamma\gamma \to Mq\bar{q}$ (I.F. Ginzburg, S.L. Panfil, V.G. Serbo).

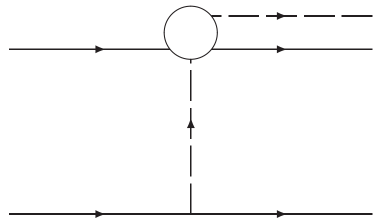

Fig. 2. Single bremsstrahlung in ee collisions: $ee \to ee\gamma$

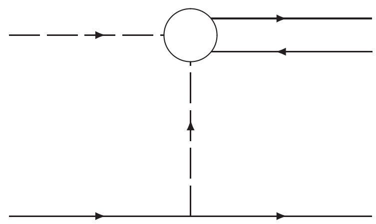

Fig. 3.Single lepton pair production in γe collisions: $\gamma e \to l^+ l^- e$

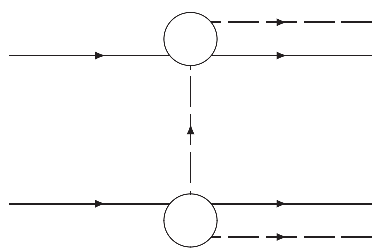

Fig. 4. Double bremsstrahlung with single photons along each initial lepton direction: $ee \to ee\gamma\gamma$

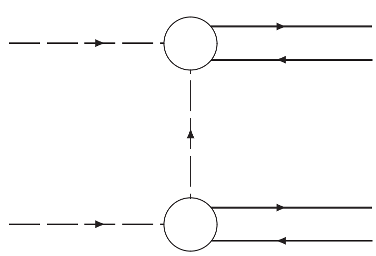

Fig. 5. Double lepton pair production in $\gamma\gamma$ collisions: $\gamma\gamma \to e^+ e^- l^+ l^-$.

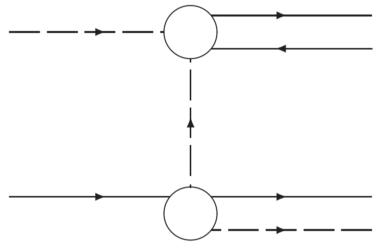

Fig. 6. Process $\gamma e \to l^+ l^- e\gamma$ with a final photon along the initial lepton direction.

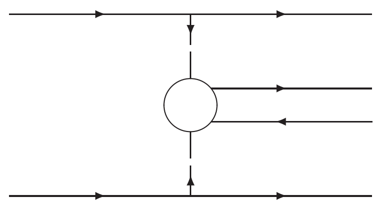

Fig. 7. Two–photon pair production in ee collisions: $ee \to ee\, l^- l^+$.

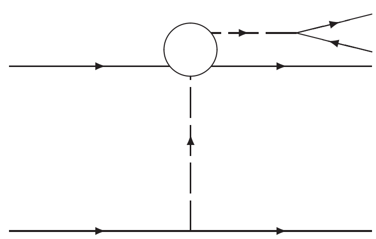

Fig. 8. Bremsstrahlung pair production in ee collisions: $ee \to ee\, l^+ l^-$.

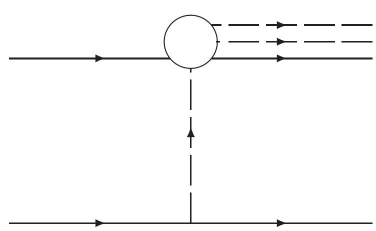

Fig. 9. Double bremsstrahlung $ee \to ee\gamma\gamma$ with two photons along the direction of one initial lepton.

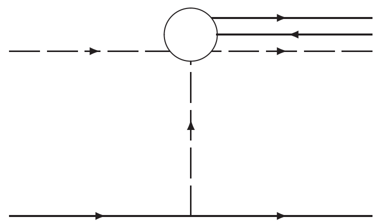

Fig. 10. Process $\gamma e \to \gamma l^+ l^- e$ with the both final photon and lepton pair along the direction of the initial photon.

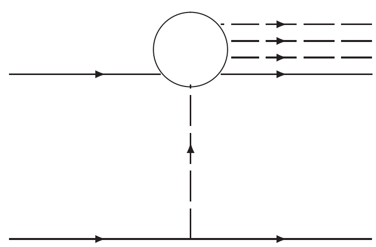

Fig. 11. Triple bremsstrahlung $ee \to ee\gamma\gamma\gamma$ with three photons along the direction of one initial lepton.

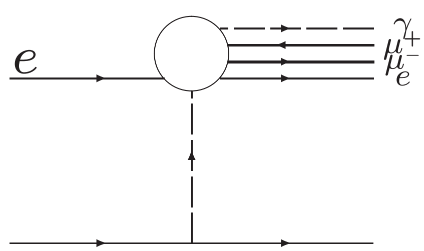

Fig. 12. The block diagram for the process $ee \to ee\mu^+\mu^-\gamma$

Some references

Particular problems related to these processes were discussed in a number of original papers starting from the brilliant paper of G. Racah, Nuovo Cim. **14** (1937) 93 and in reviews:

V.M. Budnev, I.F. Ginzburg, G.V. Meledin, V.G. Serbo, Phys. Rep. **15** C (1975) 181

V.N. Baier, V.S. Fadin,V.A. Khoze and E.A. Kuraev, Phys. Rep. **78** (1981) 293

G.L. Kotkin, A. Schiller, V.G. Serbo, Int. J. Mod. Phys. A **7** (1992) 4707.

But only recently (see E.A. Kuraev, A. Schiller, V.G. Serbo, B.G. Shaikhatdenov, Nucl. Phys. B **570** (2000) 359 and references therein) the highly accurate analytical calculation of the helicity amplitudes of all jet–like processes up to e^4 (shown in Figs. 2–10) has been completed.

In the above–mentioned original papers different approaches have been used. Here we develop a **new simple and effective** method to calculate jet–like processes

The form of "the final result"

At high energies the region of scattering angles (2) gives the dominant contribution to the cross sections of all QED jet–like processes. In this region we obtain all helicity amplitudes with high accuracy, omitting only terms of the order of

$$\frac{m_i^2}{E_i^2}\ , \quad \theta_i^2\ , \quad \theta_i \cdot \frac{m_i}{E_i} \tag{3}$$

or smaller.

The amplitude M_{fi} has a **simple factorized form**

$$M_{fi} = \frac{s}{q^2} J_1 J_2 \tag{4}$$

where the **impact factors** J_1 and J_2 **do not depend** on s.

We give **analytical expressions** for J_1 and J_2.

They are not only **compact but are also very convenient** for numerical calculations, since **large compensating terms are already canceled**.

It is well known that this problem of large compensating terms is very difficult to manage in all computer packages like CompHEP.

The discussed approximation differs considerably from the known approach of the CALCUL group and others (in which terms of the order of $m_i/|\mathbf{p}_{i\perp}|$ are neglected).

Three basic ideas

(i) A convenient decomposition of all 4–momenta into large and small components (using the so–called Sudakov or light–cone variables);

(ii) Gauge invariance of the amplitudes is used in order to combine large terms into finite expressions;

(iii) Calculations are significantly simplified in replacing the numerators of lepton propagators by vertices involving real leptons.

All these ideas are not new. In particular, the last one is the basis of the equivalent–electron approximation
P. Kessler, Nuovo Cim. **17** (1960) 809;
V.M. Baier, V.S. Fadin, V.A. Khoze, Nucl. Phys. B **65** (1973) 381
and has been used to calculate some QCD amplitudes with massless quarks
G.R. Farrar, F. Neri, Phys. Lett **130** B (1983) 109.
However, the **combination** of these ideas leads to a **very efficient** way in calculating the amplitudes of interest just in the jet kinematics.

2. Method of calculation

Sudakov or light-cone variables

We use light-like 4–vectors P_1 and P_2:

$$\begin{aligned} P_1 &= p_1 - \frac{m_1^2}{s}\, p_2, \;\; P_2 = p_2 - \frac{m_2^2}{s} p_1, \\ P_1^2 &= P_2^2 = 0\,, \;\; s = 2p_1p_2, \end{aligned} \tag{5}$$

and decompose any 4–vector A as

$$\begin{aligned} & A = x_A P_1 + y_A P_2 + A_\perp\,, \;\; A^2 = s x_A y_A + A_\perp^2\,, \\ & x_A = \frac{2AP_2}{s}\,, \;\; y_A = \frac{2AP_1}{s}\,. \end{aligned} \tag{6}$$

The parameters x_A and y_A are the so-called *Sudakov variables* (they often are referred also as *light-cone variables*). In the used reference frame

$$A_\perp = (0,\, A_x,\, A_y,\, 0) = (0,\, \mathbf{A}_\perp,\, 0)\,, \quad A_\perp^2 = -\mathbf{A}_\perp^2\,.$$

The 4–vectors p_i of particles from the first jet have large components along P_1 and small ones along P_2. Therefore, in the limit $s \to \infty$ the parameters

$$x_i = \frac{2p_i P_2}{s} = \frac{E_i}{E_1}, \;\; i \in \text{ jet}_1 \qquad (7)$$

are finite, whereas

$$y_i = \frac{2p_i P_1}{s} = \frac{m_i^2 + \mathbf{p}_{i\perp}^2}{s x_i}, \;\; i \in \;\text{ jet}_1 \qquad (8)$$

are small.

The Sudakov variable x_i is the fraction of energy of the first incoming particle carried by the i-th final particle.

The same is true for a 4–vector p_j of particles from the second jet with replacement $x_j \leftrightarrow y_j$.

The Sudakov parameters x_q and y_q for the virtual photon are small.

Photon polarization vector

Let $e \equiv e^{(\Lambda)}(k)$ be the polarization 4-vector of the final photon in the first jet. Using gauge invariance, this vector can be presented in the form

$$e = y_e P_2 + e_\perp \tag{9}$$

with

$$y_e = \frac{-2k_\perp e_\perp}{s x_k}, \;\; x_k = \frac{2kP_2}{s}\,. \tag{10}$$

and

$$e_\perp \equiv e_\perp^{(\Lambda)} = -\frac{\Lambda}{\sqrt{2}}(0,\, 1,\, \mathrm{i}\Lambda,\, 0) = -e_\perp^{(-\Lambda)\,*}. \tag{11}$$

Therefore, $e_\perp$ **does not depend** on the 4–momentum of the photon k contrary to the polarization vector e itself.

Factorization of amplitudes

The amplitude of Fig. 1 can be written as

$$M_{fi} = M_1^\mu \, \frac{g_{\mu\nu}}{q^2} \, M_2^\nu, \tag{12}$$

where M_1^μ and M_2^ν are the amplitudes of the upper and lower block.

The metric tensor $g_{\mu\nu}$ can be presented in the form

$$g^{\mu\nu} \;=\; \frac{2}{s}\left(P_2^\mu P_1^\nu + P_1^\mu P_2^\nu\right) + g_\perp^{\mu\nu}\,, \tag{13}$$

therefore,

$$\begin{aligned} M_{fi} \;&=\; \frac{2}{sq^2}\left(M_1^\mu P_{2\mu}\right)\left(M_2^\nu P_{1\nu}\right) + \\ &+\; \frac{2}{sq^2}\left(M_1^\mu P_{1\mu}\right)\left(M_2^\nu P_{2\nu}\right) + M_1^\mu \frac{g_{\perp\,\mu\nu}}{q^2} M_2^\nu\,. \end{aligned} \tag{14}$$

Since p_1 and p_i have large components along P_1 and small components along P_2 and q has small components both along P_1 and along P_2, one obtains the estimates

$$M_1^\mu P_{1\mu} \propto s^0\,, \;\; M_1^\mu P_{2\mu} \propto s \tag{15}$$

and analogously

$$M_2^\nu P_{1\nu} \propto s\,, \;\; M_2^\nu P_{2\nu} \propto s^0\,, \tag{16}$$

therefore, only the first term in Eq. (14) can give a contribution proportional to s.

As a result,

$$M_{fi} \;\; = \;\; \frac{s}{q^2}\, J_1\, J_2\,, \tag{17}$$

$$J_1 = \frac{\sqrt{2}}{s}\, M_1^\mu\, P_{2\mu}\,, \qquad\qquad J_2 = \frac{\sqrt{2}}{s}\, M_2^\nu\, P_{1\nu}\,.$$

The impact factor J_1 depends on x_i, $\mathbf{p}_{i\perp}$ with $i \in$ jet$_1$ and on the helicities of the first particle and of the particles in the first jet.

Another form of impact factors

Due to gauge invariance,

$$q_\mu M_1^\mu = (x_q P_1 + y_q P_2 + q_\perp)_\mu \, M_1^\mu = 0\,. \qquad (18)$$

Taking into account that x_q is small, one finds

$$M_1^\mu \, P_{2\mu} = -M_1^\mu \, \frac{q_{\perp\,\mu}}{y_q}\,. \qquad (19)$$

and

$$J_1 = -\frac{\sqrt{2}}{s y_q} \, M_1^\mu \, q_{\perp\,\mu}\,, \quad J_2 = -\frac{\sqrt{2}}{s x_q} \, M_2^\nu \, q_{\perp\,\nu}\,. \qquad (20)$$

This representation shows that at small transverse momentum of the exchanged photon

$$J_{1,2} \propto |\mathbf{q}_\perp| \ \text{ at } \ \mathbf{q}_\perp \to 0\,. \qquad (21)$$

In our further analysis we will **combine** various contributions of the impact factor into expressions which clearly exhibit such a behavior.

Vertices instead of spinor lines

Let us consider a virtual electron in the amplitude M_1 with $p = (E, \mathbf{p})$, $E > 0$ and virtuality $p^2 - m^2$. Due to jet kinematics, $|p^2 - m^2| \ll E^2$. We introduce an **artificial energy**

$$E_p = \sqrt{m^2 + \mathbf{p}^2}$$

and the bispinors $u_{\mathbf{p}}^{(\lambda)}$ and $v_{\mathbf{p}}^{(\lambda)}$ corresponding to **a real electron and a real positron** with 3–momentum $\mathbf{p}$ and energy E_p. In the high-energy limit

$$\frac{E - E_p}{E + E_p} = \frac{p^2 - m^2}{(E + E_p)^2} \approx \frac{p^2 - m^2}{4E^2}\,. \tag{22}$$

There is an exact identity:

$$\hat{p} + m = \frac{E + E_p}{2E_p}\, u_{\mathbf{p}}^{(\lambda)} \bar{u}_{\mathbf{p}}^{(\lambda)} + \frac{E - E_p}{2E_p}\, v_{-\mathbf{p}}^{(\lambda)} \bar{v}_{-\mathbf{p}}^{(\lambda)} \tag{23}$$

or

$$\hat{p} + m \approx u_{\mathbf{p}}^{(\lambda)} \bar{u}_{\mathbf{p}}^{(\lambda)} + \frac{p^2 - m^2}{4E^2}\, v_{-\mathbf{p}}^{(\lambda)} \bar{v}_{-\mathbf{p}}^{(\lambda)}\,. \tag{24}$$

Using these eqations for all virtual electrons, we are able to substitute the numerators of all spinor propagators by vertices involving real electrons and real positrons. These generalized vertices **are finite** in the limit $s \to \infty$.

On the contrary, a numerator like $\hat{p}+m$ is a sum of a finite term $\hat{p}_\perp + m$ and an unpleasant combination $E\gamma^0 - p_z\gamma_z$ of large terms that requires special care. Therefore, those replacements **significantly simplify** all calculations.

More detail consideration shows that:

1) If an electron line with numerator $\hat{p}+m$ connects vertices with the emission of one real and one virtual photon,

the simple substitution rule takes place:

$$\widehat{p} + m \to u_{\mathbf{p}}^{(\lambda)} \overline{u}_{\mathbf{p}}^{(\lambda)} \,. \tag{25}$$

2) For emission by electron, the vertices of the types

$$\overline{v}_{-\mathbf{p}'}^{(\lambda')} \, \widehat{P}_2 \, v_{-\mathbf{p}}^{(\lambda)} \tag{26}$$

and

$$\overline{v}_{-\mathbf{p}'}^{(\lambda')} \, \widehat{e}^{*} \, v_{-\mathbf{p}}^{(\lambda)} \tag{27}$$

are absent.

3) It is easy to check that

$$\overline{u}_{\mathbf{p}'}^{(\lambda')} \, \widehat{e}^{*} \, v_{-\mathbf{p}}^{(\lambda)} = -\overline{v}_{-\mathbf{p}'}^{(\lambda')} \, \widehat{e}^{*} \, u_{\mathbf{p}}^{(\lambda)} \,.$$

3. Vertices for bremsstrahlung processes

To calculate the impact factors involving the emission of real photons we need only two types of vertices:

1) those for the transition $e(p) \to e(p') + \gamma(k)$ where $\gamma(k)$ is a real photon with helicity Λ

$$\begin{aligned} V(p,\,k) &\equiv V^{\Lambda}_{\lambda\lambda'}(p,\,k) = \bar{u}^{(\lambda')}_{\mathbf{p}'}\,\hat{e}^{(\Lambda)\,*}\,u^{(\lambda)}_{\mathbf{p}}\,, \\ \tilde{V}(p,\,k) &\equiv \tilde{V}^{\Lambda}_{\lambda\lambda'}(p,\,k) = \\ &= \bar{u}^{(\lambda')}_{\mathbf{p}'}\,\hat{e}^{(\Lambda)\,*}\,v^{(\lambda)}_{-\mathbf{p}} = -\bar{v}^{(\lambda')}_{-\mathbf{p}'}\,\hat{e}^{(\Lambda)\,*}\,u^{(\lambda)}_{\mathbf{p}}\,. \end{aligned}$$

The result of calculation

$$\begin{aligned} V(p,\,k) &= \Big[\delta_{\lambda\lambda'}\,2\,\left(e^{(\Lambda)\,*}p\right)\left(1 - x\,\delta_{\Lambda,-2\lambda}\right) + \\ &+ \delta_{\lambda,-\lambda'}\,\delta_{\Lambda,2\lambda}\,\sqrt{2}\,mx\Big]\,\Phi\,, \\ \tilde{V}(p,\,k) &= 2\sqrt{2}\,E'\,\delta_{\lambda,-\lambda'}\,\delta_{\Lambda,2\lambda}\,\Phi \end{aligned}$$

where

$$x = \frac{\omega}{E}\,, \quad \Phi = \sqrt{\frac{E}{E'}}\,\mathrm{e}^{\mathrm{i}(\lambda'\varphi' - \lambda\varphi)}\,. \qquad (28)$$

It is useful to remind here that for the polarization vectors used here one has

$$ep = e_\perp \left(p_\perp - \frac{k_\perp}{x}\right) . \tag{29}$$

2) the vertex for the transition
$e(p) + \gamma^*(q) \to e(p')$

where $\gamma^*(q)$ is a virtual photon with energy fraction $x_q = 0$ (within our accuracy):

$$V(p) \equiv V_{\lambda\lambda'}(p) = \frac{\sqrt{2}}{s}\, \bar{u}_{\mathbf{p}'}^{(\lambda')}\, \hat{P}_2\, u_{\mathbf{p}}^{(\lambda)} = \sqrt{2}\, \frac{E}{E_1}\, \delta_{\lambda\lambda'}\, \Phi\, .$$

Properties of vertices

1) Vertices with a maximal change of helicity,

$$\max|\Delta\lambda| = \max|\Lambda + \lambda' - \lambda| = 2\,, \tag{30}$$

are absent.

2) If the produced photon becomes very hard ($\omega \to E$) the initial electron "transmits" its helicity to the photon:

$$V(p,k) \propto \delta_{\Lambda,2\lambda}\,, \quad \tilde{V}(p,k) \to 0 \quad \text{for} \quad x \to 1\,. \tag{31}$$

3) If the final electron becomes hard ($E' \to E$, soft photon limit, $x \ll 1$), the initial electron "transmits" its helicity to the final electron: in that limit the vertex

$$V(p,k) = -\frac{2}{x}\,(e^*_\perp\, k_\perp)\;\delta_{\lambda\lambda'} \tag{32}$$

dominates, which corresponds to the approximation of a classical current.

4) For HNC vertices a strong correlation between the helicities of the initial electron and the photon exists:

$$\Lambda = 2\lambda \quad \text{if} \quad \lambda' = -\lambda\,. \tag{33}$$

4. Impact factor for the single bremsstrahlung

$$e(p_1) + \gamma^*(q) \to e(p_3) + \gamma(k)$$

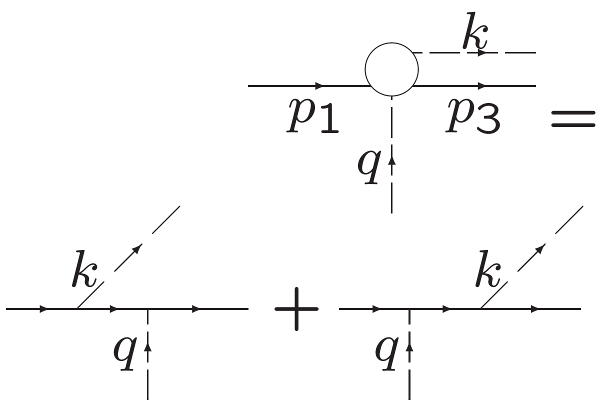

Fig. 13. Amplitude for the virtual Compton scattering

$$J_1(e_{\lambda_1} + \gamma^* \to e_{\lambda_3} + \gamma_\Lambda) = 4\pi\alpha \left(\frac{N_1}{2p_1k} - \frac{N_3}{2p_3k} \right)$$

with

$$\begin{aligned} N_1 &= \bar{u}_3 \frac{\sqrt{2}\hat{P}_2}{s} (\hat{p}_1 - \hat{k} + m)\hat{e}^* u_1 \,, \\ N_3 &= \bar{u}_3 \hat{e}^* (\hat{p}_3 + \hat{k} + m) \frac{\sqrt{2}\hat{P}_2}{s} u_1 \,. \end{aligned}$$

Then

$$J_1 = \sqrt{2}\, 4\pi\alpha \, \left[\frac{1-x}{2p_1 k} V(p_1, k) - \frac{1}{2p_3 k} V(p_3 + k, k) \right] \Phi$$

where

$$\begin{aligned} V(p_1,\ k) &= \delta_{\lambda_1 \lambda_3} 2 \left(e^{(\Lambda)\,*} p_1 \right) \left(1 - x\, \delta_{\Lambda, -2\lambda_1} \right) + \\ &+ \delta_{\lambda_1, -\lambda_3} \delta_{\Lambda, 2\lambda_1} \sqrt{2}\, mx\,, \\ V(p_3 + k,\ k) &= \delta_{\lambda_1 \lambda_3} 2 \left(e^{(\Lambda)\,*} p_3 \right) \left(1 - x\, \delta_{\Lambda, -2\lambda_1} \right) + \\ &+ \delta_{\lambda_1, -\lambda_3} \delta_{\Lambda, 2\lambda_1} \sqrt{2}\, mx \end{aligned}$$

and the Φ factor

$$\Phi = \frac{1}{\sqrt{1-x}} \, \mathrm{e}^{\mathrm{i}(\lambda_3 \varphi_3 - \lambda_1 \varphi_1)} \,.$$

The impact factor $J_1 \propto q_\perp$. Indeed, if we use equation

$$\begin{aligned} V(p_3 + k, k) &= V(p_1 + q, k) = V(p_1, k) + \\ &+ 2 \left(q_\perp \, e_\perp^{(\Lambda)\,*} \right) \left(1 - x\, \delta_{\Lambda, -2\lambda_1} \right) \delta_{\lambda_1 \lambda_3} \,, \end{aligned}$$

that leads to

$$J_1 = \sqrt{2}\, 4\pi\alpha \, [A_1 \, V(p_1, k) + q_\perp B_1] \; \Phi \, ,$$

$$A_1 = \frac{1-x}{2p_1 k} - \frac{1}{2p_3 k} \, ,$$

$$B_1 = -\frac{e_\perp^{(\Lambda)\,*}}{p_3 k} \left(1 - x\, \delta_{\Lambda, -2\lambda_1}\right) \, \delta_{\lambda_1 \lambda_3} \, .$$

It is **a simple and compact expression for all 8 helicity states** written in such a form that all individual large (compared to $q_\perp$) contributions are canceled.

Indeed, the last term in J_1 is directly proportional to $q_\perp$. Since

$$\begin{aligned} 2p_1 k &= xa \, , \quad a = m^2 + \frac{\mathbf{k}_\perp^2}{x^2} \, , \\ 2p_3 k &= \frac{x}{1-x} b \, , \quad b = m^2 + \left(\mathbf{q}_\perp - \frac{\mathbf{k}_\perp}{x}\right)^2 \, , \end{aligned}$$

we immediately obtain

$$A_1 = \frac{1-x}{x} \left(\frac{1}{a} - \frac{1}{b}\right) \propto q_\perp \, .$$

5. Impact factor for the double bremsstrahlung

$$e(p_1) + \gamma^*(q) \to$$
$$e(p_3) + \gamma(k_1) + \gamma(k_2)$$

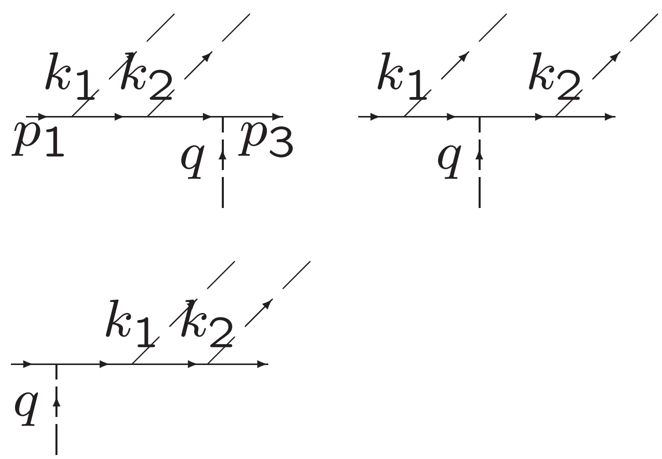

Fig.14. Feynman diagrams for the impact factor related to the double bremsstrahlung, diagrams with $k_1 \leftrightarrow k_2$ photon exchange have to be added

Notation

This impact factor corresponds to six diagrams:

$$J_1 = \sqrt{2}\,(4\pi\alpha)^{3/2}\,X_3\,\mathcal{M}^{\Lambda_1\,\Lambda_2}_{\lambda_1\,\lambda_3}\,(x_1, x_2, k_{1\perp}, k_{2\perp}, p_{3\perp})\,\Phi$$

with

$$\Phi = \frac{1}{\sqrt{X_3}}\,\mathrm{e}^{\mathrm{i}(\lambda_3\varphi_3 - \lambda_1\varphi_1)}\,.$$

J_1 and $\mathcal{M}$ do not depend on s but on the energy fractions

$$x_{1,2} = \omega_{1,2}/E_1,\ \ X_3 = E_3/E_1\,,\ \ \ x_1 + x_2 + X_3 = 1$$

and on the transverse momenta of the final particles in the combinations:

$$q_\perp = k_{1\perp} + k_{2\perp} + p_{3\perp}\,, \qquad r_j = (X_3 k_j - x_j p_3)_\perp$$

and

$$K_j = k_{jx} + \mathrm{i}\,k_{jy},\ \ Q = q_x + \mathrm{i}\,q_y,\ \ R_j = r_{jx} + \mathrm{i}\,r_{jy}\,.$$

The denominators of the propagators :

$$\begin{aligned}
a_j &\equiv -(p_1-k_j)^2+m^2=\frac{1}{x_j}(m^2x_i^2+\mathbf{k}_{j\perp}^2)\,, \\
b_j &\equiv (p_3+k_j)^2-m^2=\frac{1}{x_jX_3}(m^2x_j^2+\mathbf{r}_{j\perp}^2)\,, \\
a_{12}=a_{21} &\equiv -(p_1-k_1-k_2)^2+m^2= \\
&= a_1+a_2-\frac{1}{x_1x_2}\left(x_1\mathbf{k}_{2\perp}-x_2\mathbf{k}_{1\perp}\right)^2\ , \\
b_{12}=b_{21} &\equiv (p_3+k_1+k_2)^2-m^2= \\
&= b_1+b_2+\frac{1}{x_1x_2}\left(x_1\mathbf{k}_{2\perp}-x_2\mathbf{k}_{1\perp}\right)^2\ .
\end{aligned}$$

General formula

$$\mathcal{M}=(1+\mathcal{P}_{12})\ M,$$

where

$$\mathcal{P}_{12}f(k_1,e_1;k_2,e_2)=f(k_2,e_2;k_1,e_1)\,, \qquad \mathcal{P}_{12}^2=1.$$

and

$$
\begin{aligned}
X_3 M &= \frac{X_3}{a_1 a_{12}} V(p_1,k_1)\, V(p_1-k_1,k_2) - \\
&- \frac{1-x_1}{a_1 b_2} V(p_1,k_1)\, V(p_1-k_1+q,k_2) + \\
&+ \frac{1}{b_{12} b_2} V(p_1+q,k_1)\, V(p_1-k_1+q,k_2) - \\
&+ \frac{X_3}{a_{12}} \frac{\tilde V(p_1,k_1)\, \tilde V(p_1-k_1,k_2)}{4(E_1-\omega_1)^2} + \\
&- \frac{1}{b_{12}} \frac{\tilde V(p_1+q,k_1)\, \tilde V(p_1-k_1+q,k_2)}{4(E_3+\omega_2)^2} .
\end{aligned}
$$

Now we transform J_1 to a form which clearly exhibits the proportionality $J_1 \propto q_\perp$:

$$
\begin{aligned}
X_3 M^{\Lambda_1\Lambda_2}_{\lambda_1\lambda_3} &= A_2\, V^{\Lambda_1}_{\lambda_1\lambda}(p_1,k_1)\, V^{\Lambda_2}_{\lambda\lambda_3}(p_1-k_1+q,k_2) + \\
&+ q_\perp B_2{}^{\Lambda_1\Lambda_2}_{\lambda_1\lambda_3} + \\
&- \tilde A_2 \frac{\tilde V^{\Lambda_1}_{\lambda_1\lambda}(p_1,k_1)\, \tilde V^{\Lambda_2}_{\lambda\lambda_3}(p_1-k_1,k_2)}{4E_1^2(1-x_1)^2}
\end{aligned}
$$

with the scalars

$$
A_2 = \frac{X_3}{a_1 a_{12}} - \frac{1-x_1}{a_1 b_2} + \frac{1}{b_{12} b_2}\,, \quad \tilde A_2 = -\frac{X_3}{a_{12}} + \frac{1}{b_{12}}
$$

and the transverse 4–vector B_2

$$
\begin{aligned}
B_2{}^{\Lambda_1\Lambda_2}_{\lambda_1\,\lambda_3} &= -X_3\,\frac{2e_\perp^{(\Lambda_2)\,*}}{a_1a_{12}}\,V^{\Lambda_1}_{\lambda_1\lambda_3}(p_1,k_1) \\
&\times\left(1-\frac{x_2}{1-x_1}\,\delta_{\Lambda_2,-2\lambda_3}\right)+ \\
&+\frac{2e_\perp^{(\Lambda_1)\,*}}{b_{12}b_2}\,V^{\Lambda_2}_{\lambda_1\lambda_3}(p_1-k_1+q,k_2)\,\left(1-x_1\,\delta_{\Lambda_1,-2\lambda_1}\right)\,.
\end{aligned}
\tag{34}
$$

It is not difficult to check that the quantities A_2 and $\tilde{A}_2$ vanish in the limit of small $q_\perp$:

$$
A_2 \propto q_\perp\,,\;\;\tilde{A}_2 \propto q_\perp\,,
\tag{35}
$$

whereas B_2 is finite in this limit.

This equation represents a **very simple and compact expression for all 16 helicity states**, where all individual large (compared to $q_\perp$) contributions have been rearranged into **finite expressions**.

Explicit expressions

To find the amplitudes with given initial and final helicities, **it is sufficient** to substitute in the above equations the expressions for vertices:

$$
\begin{aligned}
M^{++}_{++} &= 2\left\{A_2\frac{K_1^*R_2^*}{x_1x_2X_3}+\frac{K_1^*Q^*}{x_1a_1a_{12}}-\frac{Q^*R_2^*}{x_2X_3b_{12}b_2}\right\}, \\
M^{--}_{++} &= X_3\left(M^{++}_{++}\right)^*, \\
M^{-+}_{++} &= -2(1-x_1) \\
&\times \left(A_2\frac{K_1R_2^*}{x_1x_2X_3}+\frac{K_1Q^*}{x_1a_1a_{12}}-\frac{QR_2^*}{x_2X_3b_{12}b_2}\right), \\
M^{+-}_{++} &= \frac{X_3}{(1-x_1)^2}\left(M^{-+}_{++}\right)^*+ \\
&+ \frac{2}{1-x_1}\left(m^2A_2\frac{x_1x_2}{X_3}-\tilde{A}_2\right), \\
M^{+-}_{+-} &= 2mx_1\left(A_2\frac{R_2}{x_2X_3}+\frac{Q}{a_1a_{12}}\right), \\
M^{-+}_{+-} &= 2m\frac{x_2}{X_3}\left(A_2\frac{K_1}{x_1}-\frac{Q}{b_{12}b_2}\right), \\
M^{--}_{+-} &= 0, \\
M^{++}_{+-} &= -\frac{1}{1-x_1}\left(X_3\,M^{+-}_{+-}+M^{-+}_{+-}\right)^*.
\end{aligned}
$$

6. Impact factor for the multiple bremsstrahlung

The generalization of the results obtained for the single and double bremsstrahlung to the bremsstrahlung of n photons can be done straightforwardly.

To demonstrate this, we consider the case $n = 3$
......

7. Some general properties of bremsstrahlung impact factors

These properties are directly related to the corresponding properties of vertices.

1) Bremsstrahlung amplitudes or impact factors with **a maximal change of helicities are absent** since in this case at least one transition vertex has to appear with a maximal change of its helicities ($= 2$) as well.

2) If one of the final particles in the jet becomes hard ($\omega_i \to E_1$ or $E_3 \to E_1$) the sign of the helicity of the initial lepton coincides with that of the helicity of the hard final particle.

3) In HNC amplitudes the sign of the helicity of at least one final photon has to coincide with the sign of the initial lepton helicity

$$\mathcal{M}^{\Lambda_1 \cdots \Lambda_n}_{\lambda_1 \, -\lambda_1} \propto \delta_{\Lambda_i, 2\lambda_1} \, .$$

8. Summary

1. In the present paper we have formulated **a new effective method** to calculate all helicity amplitudes for bremsstrahlung jet–like QED processes at tree level.

The main advantage of our method consists in using **simple universal "building blocks"** — transition vertices with **real** leptons. Those vertices replace efficiently the spinor structure involving leptons of small virtuality in the impact factors, making the calculations short and transparent for any final helicity state.

2. In the case of bremsstrahlung we have found that **only three** nonzero transition vertices are required. The properties of the vertices determine all nontrivial general properties of the helicity amplitudes.

3. By construction, the impact factors are **finite in the high energy limit** and depend only on energy fractions and transverse momenta of particles in the final jet, and on helicities of all real photons and leptons.

4. We have calculated the impact factors for single, double and triple bremsstrahlung, following the same procedure:

In a **first step**, we use the allowed vertices to write down the corresponding impact factors.

In the **next step**, we use gauge invariance with respect to the virtual photon of 4–momentum q and rearrange impact factors into a form in which all individual large (compared to $q_\perp$) contributions have been canceled.

5. Those rules together with the found impact factors allows us to give **a complete analytic and compact description of all helicity amplitudes** in $e^-e^\pm$ scattering with the emission of up to three photons in one lepton direction, where in the last case $2^5 \times 2^5 = 1024$ different helicity amplitudes are involved.

6. Since by construction individual large contributions (compared to $q_\perp$) have been rearranged into finite expressions, the expressions obtained for the amplitudes **are very convenient for numerical** calculations of various cross sections.