Bhabha Scattering at NNLO

Andrea Ferroglia

Universität Zürich

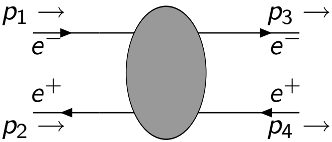
PHIPSI08 - Frascati, 9 April '08

Outline

1. Why QED Corrections to Bhabha Scattering?
2. NNLO Electron Loop Corrections
3. NNLO Photonic Corrections
4. NNLO Heavy Flavor Loop Corrections
5. Conclusions

Bhabha Scattering and Luminosity-I

$e^+e^- \longrightarrow e^+e^-$



$$s \equiv -P^2 = -(p_1+p_2)^2 = 4E^2 > 4m^2 \quad t \equiv -Q^2 = -(p_1-p_3)^2 = -4(E^2-m^2)\sin^2\frac{\theta}{2} < 0$$

- Effective tool for the Luminosity measurement @ e^+e^- colliders

$$\sigma_{\text{exp}} \equiv \frac{N}{L} \qquad L = \frac{N}{\sigma_{\text{bh-th}}}$$

Bhabha Scattering and Luminosity-II

- In the region employed for L measurements the Bhabha scattering cross section is large, QED dominated, and measured with very high precision (1 permille at KLOE/DAFNE)
- SABH is employed at LEP and ILC, while LABH is employed at colliders operating at $\sqrt{s} = 1 - 10\text{GeV}$
- Realistic simulation of Bhabha events are performed by sophisticated MC generators which take into account the detector geometry, experimental cuts, theoretical input (fixed order calculations, resummation)

Bhabha Scattering and Luminosity-II

- In the region employed for L measurements the Bhabha scattering cross section is large, QED dominated, and measured with very high precision (1 permille at KLOE/DAFNE)
- SABH is employed at LEP and ILC, while LABH is employed at colliders operating at $\sqrt{s} = 1 - 10\text{GeV}$
- Realistic simulation of Bhabha events are performed by sophisticated MC generators which take into account the detector geometry, experimental cuts, theoretical input (fixed order calculations, resummation)

The accuracy of the theoretical evaluation of the Bhabha scattering cross section directly affects the luminosity determination

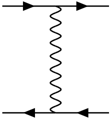
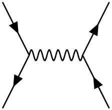
Bhabha Scattering and Luminosity-II

- In the region employed for L measurements the Bhabha scattering cross section is large, QED dominated, and measured with very high precision (1 permille at KLOE/DAFNE)
- SABH is employed at LEP and ILC, while LABH is employed at colliders operating at $\sqrt{s} = 1 - 10\text{GeV}$
- Realistic simulation of Bhabha events are performed by sophisticated MC generators which take into account the detector geometry, experimental cuts, theoretical input (fixed order calculations, resummation)

The accuracy of the theoretical evaluation of the Bhabha scattering cross section directly affects the luminosity determination

$\Longrightarrow$ study of radiative corrections to Bhabha scattering

Warning

$\mathcal{M} =$  $-$ 

- In this talk we consider the QED process only
- We consider differential cross-sections summed over the spins of the final state particles and averaged over the spin of the initial ones

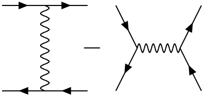
$$
\begin{aligned}
\frac{d\sigma_0(s,t)}{d\Omega} &= \frac{\alpha^2}{s}\left\{ \frac{1}{s^2}\left[st + \frac{s^2}{2} + (t-2m^2)^2\right] + \frac{1}{t^2}\left[st + \frac{t^2}{2} + (s-2m^2)^2\right] \right. \\
&\left. \quad + \frac{1}{st}\left[(s+t)^2 - 4m^4\right] \right\}
\end{aligned}
$$

$$\frac{d\sigma(s,t)}{d\Omega} = \frac{d\sigma_0(s,t)}{d\Omega} + \left(\frac{\alpha}{\pi}\right)\frac{d\sigma_1(s,t)}{d\Omega} + \left(\frac{\alpha}{\pi}\right)^2\frac{d\sigma_2(s,t)}{d\Omega} + \mathcal{O}\left((\alpha/\pi)^3\right)$$

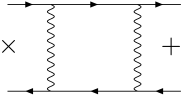
The $\mathcal{O}(\alpha^3)$ virtual corrections (one-loop $\times$ tree-level) are well known (in the full SM), no problem in keeping $m_e \neq 0$

M. Consoli (1979),
M. Böhm, A. Denner, and W. Hollik (1988),
M. Greco (1988),...

$$\left(\frac{\alpha}{\pi}\right)\frac{d\sigma_1^V(s,t)}{d\Omega} = \frac{s}{16}\sum_{\text{spin}}\Bigg\{\Bigg($$



$$\Bigg)^* \times$$

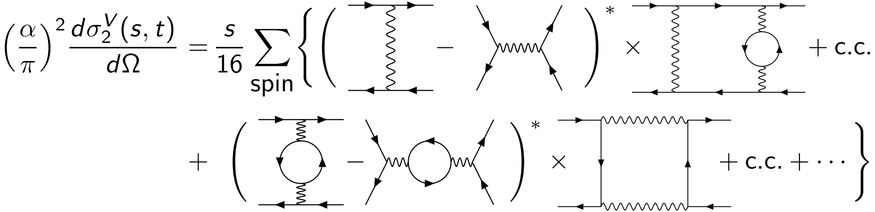


$$+\,\text{c.c.} + \cdots\Bigg\}$$

Virtual Corrections to the Cross Section -II

Order α^4 *QED* corrections:

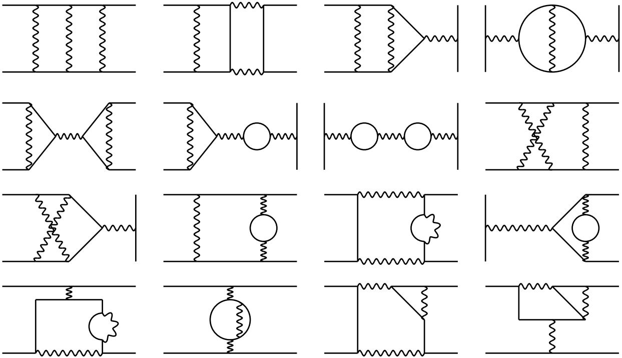
- Contributions from two-loop $\times$ tree-level and one-loop $\times$ one-loop
- Can be divided in three sets,
  i) with a closed electron loop,
  ii) closed heavy(er) flavor loop, and
  iii) photonic (without fermion loops)



Radiative corrections in $m_e = 0$ approximation

The virtual corrections where first obtained in the massless electron approximation

Z. Bern, L Dixon, and A. Ghinculov ('00)

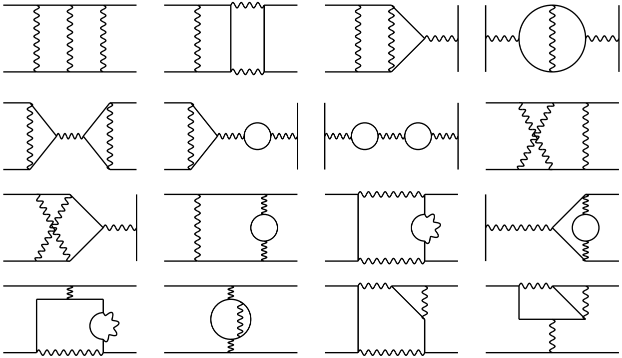


Radiative corrections in $m_e = 0$ approximation

The virtual corrections where first obtained in the massless electron approximation

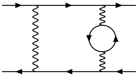
Z. Bern, L Dixon, and A. Ghinculov ('00)

However, in order to interface the fixed order calculation with the existing MC, it is necessary to keep the electron mass as a collinear regulator

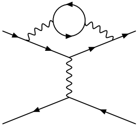


α^4 Corrections With an Electron Loop

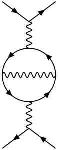
All the two-loop graphs including a closed electron loop can be calculated also keeping $m_e \neq 0$ and without relying on any approximation or expansion



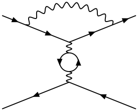
$\otimes$ Born Amplitude



$\otimes$ Born Amplitude



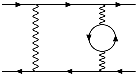
$\otimes$ Born Amplitude



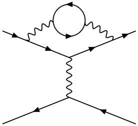
$\otimes$ Born Amplitude

α^4 Corrections With an Electron Loop

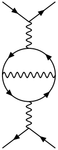
All the two-loop graphs including a closed electron loop can be calculated also keeping $m_e \neq 0$ and without relying on any approximation or expansion



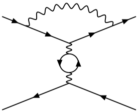
$\otimes$ Born Amplitude



$\otimes$ Born Amplitude



$\otimes$ Born Amplitude



$\otimes$ Born Amplitude

- The relevant integrals can be reduced to combination of a relatively small set of Master Integrals employing the Laporta algorithm
- The MIs (including the ones for the box) can be evaluated employing the differential equation method

R. Bonciani *et al.*('03-'04)

$\mathcal{O}(\alpha^4(N_F = 1))$ Virtual Corrections

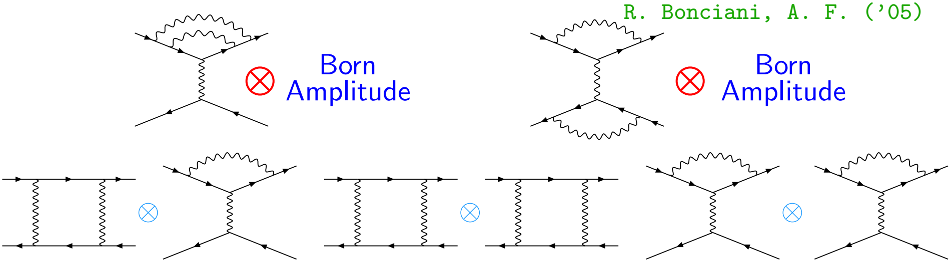
In the $\mathcal{O}(\alpha^4(N_F = 1))$ corrections to the CS

- both UV and IR divergences are regularized within the DIM REG scheme
- the UV renormalization is carried out in the on-shell scheme
- the graphs are at first calculated in the non physical region $s < 0$ and then analytically continued to the physical region $s > 4m_e^2$
- the cross section can be expressed in terms of HPLs and 2dHPLs with arguments

$$x = \frac{\sqrt{s} - \sqrt{s - 4m_e^2}}{\sqrt{s} + \sqrt{s - 4m_e^2}} \quad y = \frac{\sqrt{4m_e^2 - t} - \sqrt{-t}}{\sqrt{-t} + \sqrt{4m_e^2 - t}} \quad z = \frac{\sqrt{4m_e^2 - u} - \sqrt{-u}}{\sqrt{-u} + \sqrt{4m_e^2 - u}}$$

- The residual IR poles are eliminated by adding the contribution of the soft photon radiation

With the same techniques employed in obtaining the $\mathcal{O}(\alpha^4(N_F = 1))$ non-approximated differential CS, it is possible to calculate the photonic virtual corrections (and related soft photon emission) to the CS at order $\mathcal{O}(\alpha^4)$, except for the ones arising from the the two loop photonic boxes



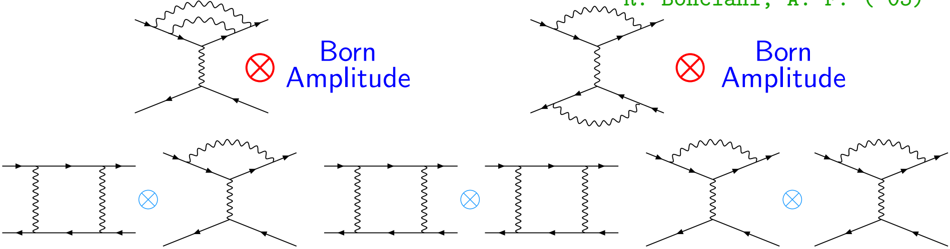
The one- and two-loop Dirac form factors in the t-channel are sufficient to determine completely the small angle cross section

$$\frac{d\sigma_2}{d\sigma_0} = 6\left(F_1^{(1l)}(t)\right)^2 + 4F_1^{(2l)}(t)$$

$\mathcal{O}(\alpha^4)$ Photonic Corrections

With the same techniques employed in obtaining the $\mathcal{O}(\alpha^4(N_F = 1))$ non-approximated differential CS, it is possible to calculate the photonic virtual corrections (and related soft photon emission) to the CS at order $\mathcal{O}(\alpha^4)$, except for the ones arising from the the two loop photonic boxes

R. Bonciani, A. F. ('05)



The one- and two-loop Dirac form factors in the t-channel are sufficient to determine completely the small angle cross section

$$\frac{d\sigma_2}{d\sigma_0} = 6({F_1}^{(1l)}(t))^2 + 4{F_1}^{(2l)}(t)$$

α^4 Photonic Corrections-m_e^2/s Expansion

- Building on the BDG result and on works by A. B. Arbuzov *et al.*, B. Tausk, N. Glover, and J. J. van der Bij ('01) obtained the terms proportional to $L = \ln m_e^2/s$ of the full (virtual + soft) photonic CS (i. e. graphs including a closed electron loop have been neglected)
- A. Penin ('05) obtained also the constant terms of the photonic CS in the m_e^2/s expansion

Therefore, in the expansion

$$\frac{d\sigma_2}{d\sigma_0} = \delta_2^{(2)} \ln^2\left(\frac{s}{m_e^2}\right) + \delta_2^{(1)} \ln\left(\frac{s}{m_e^2}\right) + \delta_2^{(0)} + \mathcal{O}\left(\frac{m_e^2}{s}\right)$$

$\delta_2^{(2)}$, $\delta_2^{(1)}$, and $\delta_2^{(0)}$ are known

- Several partial cross-checks of this results were possible by comparing it with the $m_e^2/s \to 0$ limit of the exact result for the photonic vertex and one-loop by one-loop corrections

Mass from Massless-I

As can be seen from Penin's result, when neglecting positive powers of the electron mass, the problem is to change the regularization scheme for the collinear singularities:

Is it possible to calculate graphs employing DIM REG to regulate both IR and collinear singularities and then translate a posteriori the collinear poles into collinear logs?

Mass from Massless-I

As can be seen from Penin's result, when neglecting positive powers of the electron mass, the problem is to change the regularization scheme for the collinear singularities:

> Is it possible to calculate graphs employing DIM REG to regulate both IR and collinear singularities and then translate a posteriori the collinear poles into collinear logs?

For a generic QED/QCD process, with no closed fermion loops

$$\mathcal{M}^{(m\neq 0)} = \prod_{i\in\{\text{all legs}\}} Z_i^{\frac{1}{2}}(m,\varepsilon)\mathcal{M}^{(m=0)}$$

where Z is defined through the Dirac form factor

$$F^{(m\neq 0)}(Q^2) = Z(m,\varepsilon)\, F^{(m=0)}(Q^2) + \mathcal{O}(m^2/Q^2)$$

A. Mitov and S. Moch ('06)

Mass from Massless-II

with a similar technique applied to Bhabha scattering it was possible to calculate all the NNLO corrections in the limit $s, |t|, |u| \gg m_f^2 \gg m_e^2$

T. Becher and K. Melnikov ('07)

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{s}\left(\frac{1-r+r^2}{r}\right)\left[1+\frac{\alpha}{\pi}\delta_1+\left(\frac{\alpha}{\pi}\right)^2\delta_2\right]$$

$(r = 1/2(1-\cos\theta))$

$$\delta_2 = \delta_2^{\text{photonic}} + \delta_2^{\text{electron loop}} + \delta_2^{\text{heavy flavor loop}}$$

- photonic corrections in agreement with A. Penin ('05)
- electron loop corrections in agreement with R. Bonciani *et al* ('04)
- "heavy flavor" loop corrections in agreement with S. Actis *et al* ('07)

Mass from Massless-II

with a similar technique applied to Bhabha scattering it was possible to calculate all the NNLO corrections in the limit $s, |t|, |u| \gg m_f^2 \gg m_e^2$

T. Becher and K. Melnikov ('07)

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{s}\left(\frac{1-r+r^2}{r}\right)\left[1+\frac{\alpha}{\pi}\delta_1+\left(\frac{\alpha}{\pi}\right)^2\delta_2\right]$$

$$(r = 1/2(1-\cos\theta))$$

$$\delta_2 = \delta_2^{\text{photonic}} + \delta_2^{\text{electron loop}} + \delta_2^{\text{heavy flavor loop}}$$

- photonic corrections in agreement with A. Penin ('05)
- electron loop corrections in agreement with R. Bonciani *et al* ('04)
- "heavy flavor" loop corrections in agreement with S. Actis *et al* ('07)

Mass from Massless-II

with a similar technique applied to Bhabha scattering it was possible to calculate all the NNLO corrections in the limit $s, |t|, |u| \gg m_f^2 \gg m_e^2$

T. Becher and K. Melnikov ('07)

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{s}\left(\frac{1-r+r^2}{r}\right)\left[1+\frac{\alpha}{\pi}\delta_1+\left(\frac{\alpha}{\pi}\right)^2\delta_2\right]$$

$(r = 1/2(1-\cos\theta))$

$$\delta_2 = \delta_2^{\text{photonic}} + \delta_2^{\text{electron loop}} + \delta_2^{\text{heavy flavor loop}}$$

- photonic corrections in agreement with A. Penin ('05)
- electron loop corrections in agreement with R. Bonciani *et al* ('04)
- "heavy flavor" loop corrections in agreement with S. Actis *et al* ('07)

Beyond $s \gg m_f^2$

In any realistic case the approximation $s, |t|, |u| \gg m_e^2$ is more than enough
However, in the case of corrections with a closed heavy fermion loop, it is not always true that $s, |t|, |u| \gg m_f^2$

Beyond $s \gg m_f^2$

In any realistic case the approximation $s, |t|, |u| \gg m_e^2$ is more than enough
However, in the case of corrections with a closed heavy fermion loop, it is not always true that $s, |t|, |u| \gg m_f^2$

for example

- τ loop at KLOE, where $\sqrt{s} = 1\,\text{GeV} < m_\tau$
- top quark loop at ILC, where $\sqrt{s} \approx 500\,\text{GeV}$ and $m_t^2/t, m_t^2/u < 1$ just in the angular region $40^o < \theta < 140^o$

BEYOND $s \gg m_f^2$

In any realistic case the approximation $s, |t|, |u| \gg m_e^2$ is more than enough However, in the case of corrections with a closed heavy fermion loop, it is not always true that $s, |t|, |u| \gg m_f^2$

for example

- τ loop at KLOE, where $\sqrt{s} = 1\,\text{GeV} < m_\tau$
- top quark loop at ILC, where $\sqrt{s} \approx 500\,\text{GeV}$ and $m_t^2/t, m_t^2/u < 1$ just in the angular region $40^o < \theta < 140^o$

It would be nice to calculate the NNLO corrections including an heavy fermion loop retaining the exact dependence on m_f

$$s, |t|, |u|, m_f^2 \gg m_e^2$$

this is a non trivial problem involving four-scale two-loop boxes ...

$\Longrightarrow$

Structure of the Collinear Poles

What is the collinear structure of these corrections?

$$\delta_2 = \delta_2^C(s,t,m_f^2) \ln\left(\frac{s}{m_e^2}\right) + \delta_2^R(s,t,m_f^2) + \mathcal{O}\left(\frac{m_e^2}{s}\right)$$

there is just a single collinear logarithm

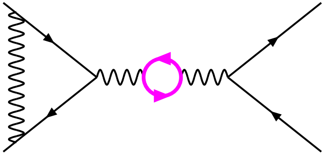
Structure of the Collinear Poles

What is the collinear structure of these corrections?

$$\delta_2 = \delta_2^C(s,t,m_f^2)\ln\left(\frac{s}{m_e^2}\right) + \delta_2^R(s,t,m_f^2) + \mathcal{O}\left(\frac{m_e^2}{s}\right)$$

there is just a single collinear logarithm

It is possible to show that the collinear logarithm arises from trivial reducible graphs only



The Calculation of the Boxes

The sum of the one-particle irreducible diagrams has a regular behavior in the small electron mass m_e

The sum of the one-particle irreducible diagrams has a regular behavior in the small electron mass m_e

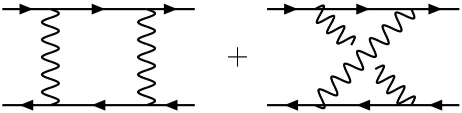
This means that we can set $m_e = 0$ from the beginning, getting rid of one scale

The Calculation of the Boxes

The sum of the one-particle irreducible diagrams has a regular behavior in the small electron mass m_e

This means that we can set $m_e = 0$ from the beginning, getting rid of one scale

In the Feynman gauge

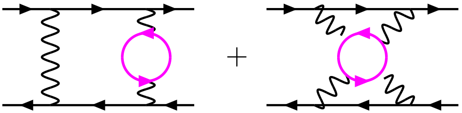
 = Free of collinear poles

$$u = -s - t$$

The sum of the one-particle irreducible diagrams has a regular behavior in the small electron mass m_e

This means that we can set $m_e = 0$ from the beginning, getting rid of one scale

In the Feynman gauge



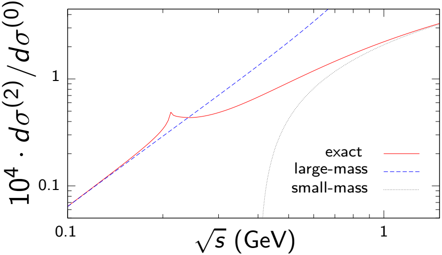
$= $ Free of collinear poles

$$u = -s - t$$

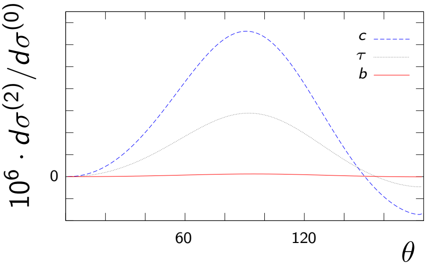
After UV renormalization, the only remaining poles are the IR (soft) ones

R. Bonciani, AF, and A. Penin ('07-'08)

- It was possible to calculate the boxes for $m_e = 0$ and generic $s, |t|, |u|, m_f^2 \gg m_e^2$, therefore effectively eliminating one mass scale from the most challenging part of the calculation
- We employed IBPs and Differential Eq. Method
- The result can be expressed in terms of HPL and a few GHPLs of a new class. The latter can be expressed in closed form in terms of polylogs
- by expanding the exact result it was possible to recover the result of Actis *et al* and Becher Melnikov
- With the exact dependence of the cross section on m_f we can get numbers for the τ loop at intermediate energies and top loop at ILC energies



Two-loop corrections to the Bhabha scattering differential cross section at $\theta = 60^\circ$ due to a closed muon loop



Two-loop corrections to the Bhabha scattering differential cross section at $\sqrt{s} = 1$ GeV due to a closed τ-lepton , c-quark and b-quark loop for $m_c = 1.4$ GeV and $m_b = 4.7$ GeV.

The large logs depending on the IR cut-off ω are excluded from the numerical analysis

Numerical Analysis

The actual impact of the two-loop virtual corrections on the theoretical predictions can be determined only after the corrections are implemented into MC event generators

However, for $\sqrt{s} = 1\,\mathrm{GeV}$ it is possible to conclude that

- the heavy flavor corrections are dominated by the collinear logs $\ln\left(s/m_e^2\right)$
- the muon loop corrections reach 0.45 permille of the tree level at $\theta \sim 140$
- the tau, b, and c loop corrections are one order of magnitude smaller that the muon corrections
- The contribution of the light quarks u, d, s must be treated with the dispersion relation approach

S. Actis *et al.* ('07)

- It is possible to estimate the results obtained by means of dispersion relations by employing an effective light quark mass $m_{eff} \sim 180\,\mathrm{MeV}$, we find agreement with the dispersion relation calculation within 20%

Numerical Analysis

The actual impact of the two-loop virtual corrections on the theoretical predictions can be determined only after the corrections are implemented into MC event generators

However, for $\sqrt{s} = 1\,\text{GeV}$ it is possible to conclude that

- the heavy flavor corrections are dominated by the collinear logs $\ln\left(s/m_e^2\right)$
- the muon loop corrections reach 0.45 permille of the tree level at $\theta \sim 140$
- the tau, b, and c loop corrections are one order of magnitude smaller that the muon corrections
- The contribution of the light quarks u, d, s must be treated with the dispersion relation approach

S. Actis *et al.* ('07)

- It is possible to estimate the results obtained by means of dispersion relations by employing an effective light quark mass $m_{eff} \sim 180\,\text{MeV}$, we find agreement with the dispersion relation calculation within 20%

Summary & Conclusions

- A precise knowledge of the Bhabha scattering cross section (both at small and large angle) is crucial in order to determine the luminosity at e^+e^- colliders
- Photonic and electron loop corrections have been known for some time. Recently, we calculate also the heavy flavor NNLO corrections. This was done with a technique that effectively eliminates one mass scale from the most challenging part of the calculation and provides result in closed analytical form
- The calculation of the virtual NNLO QED corrections is basically complete, but there is still work to be done: ex. one-loop hard photon emission
- These fixed order results must be included/interfaced with MC generators

Backup Slides

Penin's technique (in a Nutshell)

- Consider the amplitude of the two loop virtual corrections to the cross-section in which collinear and IR divergencies are regularized by m_e and λ: $\mathcal{A}^{(2)}(m_e, \lambda)$
- Build an auxiliary amplitude $\overline{\mathcal{A}}^{(2)}(m_e, \lambda)$ with the same IR singularities of the $\mathcal{A}^{(2)}(m_e, \lambda)$ but sufficiently simple to be evaluated in the small mass expansion
- The quantity $\delta\mathcal{A}^{(2)} = \mathcal{A}^{(2)} - \overline{\mathcal{A}}^{(2)}$ has a finite limit when m_e and λ tend to zero
- $\delta\mathcal{A}^{(2)}$ is regularization scheme independent and it can be reconstructed from the known results for the virtual corrections calculated by setting $m_e = \lambda = 0$ from the start

Penin's technique (in a Nutshell)

- Consider the amplitude of the two loop virtual corrections to the cross-section in which collinear and IR divergencies are regularized by m_e and λ: $\mathcal{A}^{(2)}(m_e, \lambda)$
- Build an auxiliary amplitude $\overline{\mathcal{A}}^{(2)}(m_e, \lambda)$ with the same IR singularities of the $\mathcal{A}^{(2)}(m_e, \lambda)$ but sufficiently simple to be evaluated in the small mass expansion
- The quantity $\delta\mathcal{A}^{(2)} = \mathcal{A}^{(2)} - \overline{\mathcal{A}}^{(2)}$ has a finite limit when m_e and λ tend to zero
- $\delta\mathcal{A}^{(2)}$ is regularization scheme independent and it can be reconstructed from the known results for the virtual corrections calculated by setting $m_e = \lambda = 0$ from the start

Finally

$$\mathcal{A}^{(2)} = \overline{\mathcal{A}}^{(2)}(m_e, \lambda) + \delta\mathcal{A}^{(2)} + \mathcal{O}(m_e, \lambda)$$

Penin's technique (in a Nutshell)

- Consider the amplitude of the two loop virtual corrections to the cross-section in which collinear and IR divergencies are regularized by m_e and λ: $\mathcal{A}^{(2)}(m_e, \lambda)$
- Build an auxiliary amplitude $\overline{\mathcal{A}}^{(2)}(m_e, \lambda)$ with the same IR singularities of the $\mathcal{A}^{(2)}(m_e, \lambda)$ but sufficiently simple to be evaluated in the small mass expansion
- The quantity $\delta\mathcal{A}^{(2)} = \mathcal{A}^{(2)} - \overline{\mathcal{A}}^{(2)}$ has a finite limit when m_e and λ tend to zero
- $\delta\mathcal{A}^{(2)}$ is regularization scheme independent and it can be reconstructed from the known results for the virtual corrections calculated by setting $m_e = \lambda = 0$ from the start

Finally

$$\mathcal{A}^{(2)} = \overline{\mathcal{A}}^{(2)}(m_e, \lambda) + \delta\mathcal{A}^{(2)} + \mathcal{O}(m_e, \lambda)$$

$\Longrightarrow$The method cannot be applied to the $\alpha^4(N_F = 1)$ corrections

Collinear Poles & Gauge Invariant Sets

- In a physical (Coulomb or axial) gauge, the collinear divergencies factorize and can be reabsorbed in the external field renormalization

J. Frenkel and J. C. Taylor ('76)

Collinear Poles & Gauge Invariant Sets

- In a physical (Coulomb or axial) gauge, the collinear divergencies factorize and can be reabsorbed in the external field renormalization

  J. Frenkel and J. C. Taylor ('76)

- In Bhabha scattering, the box (and the photon self-energy) diagrams form gauge-independent sets

Collinear Poles & Gauge Invariant Sets

- In a physical (Coulomb or axial) gauge, the collinear divergencies factorize and can be reabsorbed in the external field renormalization

  J. Frenkel and J. C. Taylor ('76)

- In Bhabha scattering, the box (and the photon self-energy) diagrams form gauge-independent sets

therefore the boxes must form a collinear safe set in any gauge

Collinear Poles & Gauge Invariant Sets

- In a physical (Coulomb or axial) gauge, the collinear divergencies factorize and can be reabsorbed in the external field renormalization

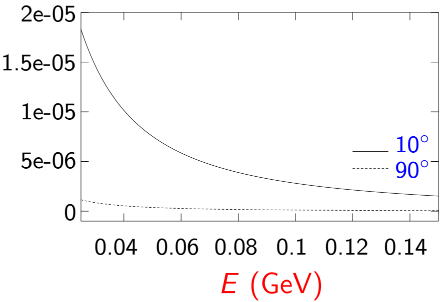
  J. Frenkel and J. C. Taylor ('76)

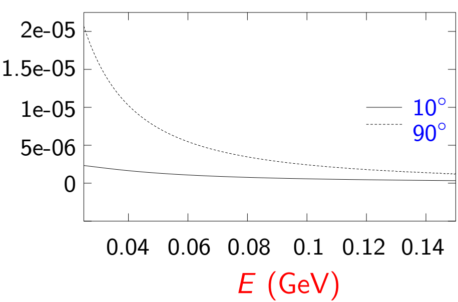
- In Bhabha scattering, the box (and the photon self-energy) diagrams form gauge-independent sets

therefore the boxes must form a collinear safe set in any gauge

In the Feynman gauge we employ in the calculation, single box diagrams show collinear divergencies that cancel in the sum over all the box diagrams

(Ir)Relevance on the Terms $\propto m_e^2$

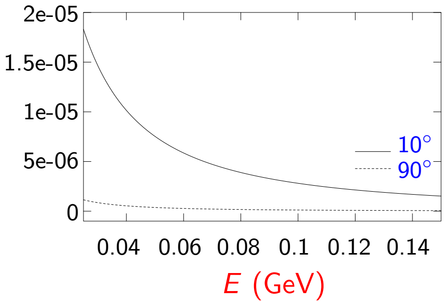


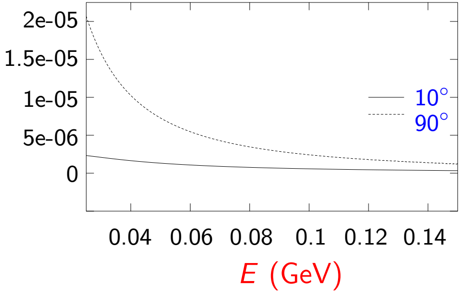


$$D_{\texttt{Vert}} = \left(\frac{\alpha}{\pi}\right)^2 \frac{\left| \left. \frac{d\sigma_2^{(\texttt{Vert})}}{d\Omega} - \frac{d\sigma_2^{(\texttt{Vert})}}{d\Omega} \right|_L \right|}{\frac{d\sigma_0}{d\Omega} + \left(\frac{\alpha}{\pi}\right) \frac{d\sigma_1}{d\Omega}}$$

$$D_{\texttt{BoxBox}} = \left(\frac{\alpha}{\pi}\right)^2 \frac{\left| \left. \frac{d\sigma_2^{(\texttt{Vert})}}{d\Omega} - \frac{d\sigma_2^{(\texttt{Vert})}}{d\Omega} \right|_L \right|}{\frac{d\sigma_0}{d\Omega} + \left(\frac{\alpha}{\pi}\right) \frac{d\sigma_1}{d\Omega}}$$

(Ir)Relevance on the Terms $\propto m_e^2$





$$D_{\texttt{Vert}} = \left(\frac{\alpha}{\pi}\right)^2 \frac{\left|\left.\frac{d\sigma_2^{(\texttt{Vert})}}{d\Omega} - \frac{d\sigma_2^{(\texttt{Vert})}}{d\Omega}\right|_L\right|}{\frac{d\sigma_0}{d\Omega} + \left(\frac{\alpha}{\pi}\right)\frac{d\sigma_1}{d\Omega}} \qquad D_{\texttt{BoxBox}} = \left(\frac{\alpha}{\pi}\right)^2 \frac{\left|\left.\frac{d\sigma_2^{(\texttt{Vert})}}{d\Omega} - \frac{d\sigma_2^{(\texttt{Vert})}}{d\Omega}\right|_L\right|}{\frac{d\sigma_0}{d\Omega} + \left(\frac{\alpha}{\pi}\right)\frac{d\sigma_1}{d\Omega}}$$

the terms proportional to m_e become negligible for values of E that are very small with respect to the ones encountered in e^+e^- experiments

NNLO Heavy Flavor CS at $\sqrt{s} = 1$ GeV

$\sqrt{s} = 1$ GeV

θ	e (10^{-4})	μ (10^{-4})	c (10^{-4})	τ (10^{-4})	b (10^{-4})
50°	17.341004	1.7972877	0.0622677	0.0264013	0.0010328
60°	18.407836	2.2267654	0.0861876	0.0367058	0.0014184
70°	19.438718	2.6504950	0.1086126	0.0465329	0.0018907
80°	20.465455	3.0655973	0.1253094	0.0540991	0.0022442
90°	21.463240	3.4581845	0.1321857	0.0576348	0.0024428
100°	22.366427	3.8070041	0.1268594	0.0560581	0.0024304
110°	23.099679	4.0922189	0.1098317	0.0495028	0.0022024
120°	23.605216	4.3030725	0.0843311	0.0392810	0.0018086
130°	23.847394	4.4392717	0.0549436	0.0273145	0.0013297

Table: *The second-order electron, muon, c-quark, τ-lepton, and b-quark QED contributions to the Bhabha scattering differential cross section at $\sqrt{s} = 1$ GeV in units of 10^{-4} of the Born cross section.*