Heavy quark masses from lattice QCD

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PHIPSI 08
International Workshop on $e^+e^-$ collisions from Phi to Psi
I.N.F.N., Laboratori Nazionali di Frascati, Italy
7. – 10. April 2008
Based on a lot of past, recent and ongoing work of the \textit{ALPHA Collaboration}, in particular

- \textbf{Precision computation of the strange quark's mass in quenched QCD}

- \textbf{A precise determination of the charm quark's mass in quenched QCD}

- \textbf{The $D_s$-meson decay constant in the continuum limit of quenched QCD}
  J. H. and A. Jüttner, in preparation

- \textbf{Computation of the strong coupling in QCD with two dynamical flavours}

- \textbf{Non-perturbative quark mass renormalization in two-flavor QCD}
  M. Della Morte, R. Hoffmann, F. Knechtli, J. Rolf, R. Sommer, I. Wetzorke and U. Wolff,

- \textbf{Non-perturbative heavy quark effective theory}

- \textbf{Effective heavy-light meson energies in small-volume quenched QCD}

- \textbf{Heavy Quark Effective Theory computation of the mass of the bottom quark}

- \textbf{Towards a non-perturbative matching of HQET and QCD with dynamical light quarks}

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- \textbf{Towards a non-perturbative matching of HQET and QCD with dynamical light quarks}
Outline

1. Computing quark masses with lattice QCD
   - Generic strategy & Illustration
   - The charm quark’s mass

2. The bottom quark’s mass
   - Non-perturbative Heavy Quark Effective Theory
   - Application: Computation of $M_b$
   - Status in two-flavour QCD

3. Challenges & Outlook
Lattice QCD
‘Ab initio’ approach to determine phenomenologically relevant key parameters

\[
\mathcal{L}_{\text{QCD}} [g_0, m_f] = - \frac{1}{2g_0^2} \text{Tr} \{ F_{\mu\nu} F_{\mu\nu} \} + \sum_{f=u,d,s,...} \bar{\psi}_f \{ \gamma_\mu (\partial_\mu + g_0 A_\mu) + m_f \} \psi_f
\]

Experiment

\[
\begin{bmatrix}
F_\pi \\
m_\pi \\
m_K \\
m_D \\
m_B
\end{bmatrix}
\]

QCD parameters (RGIs)

\[
\begin{bmatrix}
\Lambda_{\text{QCD}} \\
M_u, M_d \\
M_s \\
M_c \\
M_b
\end{bmatrix}
\]

Predictions

\[
\begin{bmatrix}
F_D \\
F_B \\
B_K, B_B \\
\xi \\
\ldots
\end{bmatrix}
\]

\[\mathcal{L}_{\text{QCD}} [g_0, m_f] \rightarrow \text{means formulation of QCD on a Euclidean lattice with:}\]

- Gauge invariance
- Locality
- Unitarity

- Applicable for all scales
- Technical issues/obstacles
  - Continuum limit & Renormalization
  - Computing resources of > 1 TFlop/s
Lattice QCD
‘Ab initio’ approach to determine phenomenologically relevant key parameters

\[ \mathcal{L}_{\text{QCD}}[g_0, m_f] = -\frac{1}{2g_0^2} \text{Tr} \{ F_{\mu\nu} F_{\mu\nu} \} + \sum_{f=u,d,s,...} \bar{\psi}_f \{ \gamma_\mu (\partial_\mu + g_0 A_\mu) + m_f \} \psi_f \]

Experiment

[\begin{bmatrix} F_\pi \\ m_\pi \\ m_K \\ m_D \\ m_B \end{bmatrix}] = \mathcal{L}_{\text{QCD}}[g_0, m_f]

QCD parameters (RGIs)

[\begin{bmatrix} \Lambda_{\text{QCD}} \\ M_u, M_d \\ M_s \\ M_c \\ M_b \end{bmatrix}]

Predictions

[\begin{bmatrix} F_D \\ F_B \\ B_K, B_B \\ \xi \\ \cdots \end{bmatrix}]

\[ \mathcal{L}_{\text{QCD}}[g_0, m_f] \] means formulation of QCD on a Euclidean lattice with:

- Gauge invariance
- Locality
- Unitarity

- Applicable for all scales
- Technical issues/obstacles
  - Continuum limit & Renormalization
  - \( O(1/\sqrt{t_{\text{CPU}}}) \) statistical errors
\( \mathcal{U}_\mu(x) = e^{i a g_0 A_\mu(x)} \)  
\( \psi(x) \)

- **Lattice cutoff**  
  \( a^{-1} \sim \Lambda_{UV} \)

- **Finite volume**  
  \( L^3 \times T \)

- **Lattice action**
  \[
  S[U, \overline{\psi}, \psi] = S_G[U] + S_F[U, \overline{\psi}, \psi]
  \]

  \[
  S_G = \frac{1}{g_0^2} \sum_p \text{Tr} \{1 - U(p)\}
  \]

  \[
  S_F = a^4 \sum_x \overline{\psi}(x) D[U] \psi(x)
  \]

**EVs:** represented as path integrals

\[
Z = \int D[U] D[\overline{\psi}, \psi] e^{-S[U, \overline{\psi}, \psi]} = \int D[U] \prod_f \text{det} (\not{D} + m_f) e^{-S_G[U]}
\]

\[
\langle O \rangle = \frac{1}{Z} \int \prod_{x, \mu} dU_\mu(x) O \prod_f \text{det} (\not{D} + m_f) e^{-S_G[U]} \equiv \text{thermal average}
\]

**Stochastic evaluation with Monte Carlo methods**

\[\rightarrow \text{Observables } \langle O \rangle = \frac{1}{N} \sum_{n=1}^{N} O_n \pm \Delta_O \text{ from numerical simulations}\]
\[ U_\mu(x) = e^{i a g_0 A_\mu(x)} \]

- **Lattice cutoff** \( a^{-1} \sim \Lambda_{\text{UV}} \)
- **Finite volume** \( L^3 \times T \)
- **Lattice action**

\[
S[U, \overline{\psi}, \psi] = S_G[U] + S_F[U, \overline{\psi}, \psi]
\]

\[
S_G = \frac{1}{g_0^2} \sum_p \text{Tr} \{1 - U(p)\}
\]

\[
S_F = a^4 \sum_x \overline{\psi}(x)D[U]\psi(x)
\]

**EVs:** represented as path integrals

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**Towards realistic QCD simulations — Systematics**

1. **dynamical** fermion effects \( \leftrightarrow \) quenched approximation: \( \det(D + m_f) = 1 \)
2. lattice artefacts cause discretization errors \( \rightarrow \) \( \langle O \rangle^{\text{lat}} = \langle O \rangle^{\text{cont}} + O(a^p) \)
3. quark mass restrictions \( \rightarrow a \ll m_q^{-1} \ll L : \) extrapolate by ChPT & HQET
4. energy range restrictions \( \rightarrow \mu < a^{-1} \lesssim 4 \text{ GeV} \) if (typically) \( a \gtrsim 0.05 \text{ fm} \)
Computing quark masses with lattice QCD

- Generic strategy & Illustration
- The charm quark’s mass
Scale problem and generic strategy

Multiple scale problems always difficult for a numerical/lattice treatment

\[ \frac{1}{L} \ll \mu, E, |p| \ll \frac{1}{a} \]

\[ \mu_{\text{pert}} \gg \Lambda_{\text{QCD}} \]
Scale problem and generic strategy

Multiple scale problems always difficult for a numerical/lattice treatment

\[ \mu \rightarrow \infty \]

\[ m_D, m_p, m_\pi \]

\[ m_\tau, m_Z \]

intermediate = Schrödinger functional, finite-volume scheme
Scale dependence of QCD parameters

- Renormalization group (RG) equations, $\bar{g} \equiv \bar{g}(\mu)$

$$\mu \frac{\partial \bar{g}}{\partial \mu} = \beta(\bar{g}) \quad \bar{g} \to 0 \quad -\bar{g}^3 \left\{ b_0 + b_1 \bar{g}^2 + b_2 \bar{g}^4 + \ldots \right\}$$

$$\mu \frac{\partial \bar{m}}{\partial \mu} = \tau(\bar{g}) \bar{m} \quad \tau(\bar{g}) \to 0 \quad -\bar{g}^2 \left\{ d_0 + d_1 \bar{g}^2 + \ldots \right\}$$

- Solution leads to exact equations in a mass-independent scheme

$$\Lambda \equiv \mu \left( b_0 \bar{g}^2 \right)^{-b_1/(2b_0^2)} e^{-1/(2b_0 \bar{g}^2)} \exp \left\{ -\int_0^{\bar{g}} \! dg \left[ \frac{1}{\beta(g)} + \frac{1}{b_0 g^3} - \frac{b_1}{b_0^2 g} \right] \right\}$$

$$M \equiv \bar{m}(\mu) \left( 2b_0 \bar{g}^2 \right)^{-d_0/(2b_0)} \exp \left\{ -\int_0^{\bar{g}} \! dg \left[ \frac{\tau(g)}{\beta(g)} - \frac{d_0}{b_0 g} \right] \right\}$$

RG invariant quark mass
Scale dependence of QCD parameters

- Renormalization group (RG) equations, $\bar{g} \equiv \bar{g}(\mu)$

$$\mu \frac{\partial \bar{g}}{\partial \mu} = \beta(\bar{g}) \quad \bar{g} \to 0 \quad -\bar{g}^3 \{ b_0 + b_1 \bar{g}^2 + b_2 \bar{g}^4 + \ldots \}$$

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- Simple relations between different renormalization schemes:

$$S \to S' \quad \alpha \to \alpha' = \alpha + c \alpha^2 + O(\alpha^3)$$

$$\Rightarrow M \text{ is scale & scheme independent}$$
Scale dependence of QCD parameters

- **Renormalization group (RG) equations**, $\bar{g} \equiv \bar{g}(\mu)$
  \[
  \mu \frac{\partial \bar{g}}{\partial \mu} = \beta(\bar{g}) \quad \bar{g} \rightarrow 0 \quad -\bar{g}^3 \left\{ b_0 + b_1 \bar{g}^2 + b_2 \bar{g}^4 + \ldots \right\}
  \]
  \[
  \mu \frac{\partial \bar{m}}{\partial \mu} = \tau(\bar{g}) \bar{m} \quad \tau(\bar{g}) \rightarrow 0 \quad -\bar{g}^2 \left\{ d_0 + d_1 \bar{g}^2 + \ldots \right\}
  \]

- **Solution leads to exact equations in a mass-independent scheme**
  \[
  \Lambda \equiv \mu \left( b_0 \bar{g}^2 \right)^{-b_1/(2b_0^2)} e^{-1/(2b_0 \bar{g}^2)} \exp \left\{ - \int_0^{\bar{g}} dg \left[ \frac{1}{\beta(g)} + \frac{1}{b_0 \bar{g}^3} - \frac{b_1}{b_0^2 \bar{g}} \right] \right\}
  \]
  \[
  M \equiv \bar{m} \left( 2b_0 \bar{g}^2 \right)^{-d_0/(2b_0)} \exp \left\{ - \int_0^{\bar{g}} dg \left[ \frac{\tau(g)}{\beta(g)} - \frac{d_0}{b_0 \bar{g}} \right] \right\} \quad \text{RG invariant quark mass}
  \]

- **Simple relations between different renormalization schemes:**
  \[
  \frac{\Lambda_S'}{\Lambda_S} = e^{\frac{c}{4\pi b_0}}, \quad \frac{\bar{m}_S'(\mu)}{\bar{m}_S(\mu)} = 1 + O(\alpha(\mu)) \quad \mu \rightarrow \infty \quad 1 \Rightarrow \quad M_S' = M_S \equiv M
  \]

  \Rightarrow Choose convenient scheme (i.e. physical coupling) to compute $M$
The basic equation for the RGI quark mass

Definition of the running mass through the PCAC relation:

\[ \partial_\mu A_{\mu}^{su} = (m_u + m_s) p^{su} \]

\[ \left\{ \begin{array}{l}
A_{\mu}^{su} = \bar{u} \gamma_\mu \gamma_5 s : \text{axial vector current} \\
p^{su} = \bar{u} \gamma_5 s : \text{pseudoscalar density}
\end{array} \right. \]

Upon renormalization in the lattice regularized theory (\( g_0 \leftrightarrow a \)):

\[ \overline{m}_u(\mu) + \overline{m}_s(\mu) = \frac{Z_A(g_0)}{Z_P(g_0, \mu)} \times \frac{\langle 0 | \partial_\mu A_{\mu}^{su} | K^- (p = 0) \rangle}{\langle 0 | p^{su} | K^- (p = 0) \rangle} \]

\[ m_u + m_s = \frac{F_K m_K^2}{G_K} : \text{bare PCAC} \]
The basic equation for the RGI quark mass

Definition of the running mass through the PCAC relation:

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\[ \begin{align*}
A_{\mu}^{su} &= \bar{u} \gamma_{\mu} \gamma_5 s : \text{ axial vector current} \\
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Upon renormalization in the lattice regularized theory \((g_0 \leftrightarrow \alpha)\):

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\overline{m}_u(\mu) + \overline{m}_s(\mu) = \frac{Z_A(g_0)}{Z_P(g_0, \mu)} \times \frac{\langle 0 | \partial_{\mu} A^{su}_\mu | K^-(p = 0) \rangle}{\langle 0 | p^{su} | K^-(p = 0) \rangle} \\
\overbrace{m_u + m_s = \frac{F_K m_K^2}{G_K}}^{\text{bare PCAC}}
\]

- Scale & scheme dependence via \(Z_P\), which is poorly convergent in PT \[\text{[ALPHA Collaboration 1999 (N_f = 0) & 2005 (N_f = 2)]}\]

- \(Z_A\) fixed by requiring a chiral Ward identity from Euclidean current algebra

- Analogously for the charm sector:

\[
\overline{m}_s(\mu) + \overline{m}_c(\mu) = \frac{Z_A(g_0)}{Z_P(g_0, \mu)} \times \frac{\langle 0 | \partial_{\mu} A^{cs}_\mu | D_s^+(p = 0) \rangle}{\langle 0 | p^{cs} | D_s^+(p = 0) \rangle} \\
\overbrace{m_s + m_c = \frac{F_K m_K^2}{G_K}}^{\text{bare PCAC}}
\]

with same \(Z_P\) in a mass/flavour independent renormalization scheme
Now split up the problem according to the generic strategy:

\[
M_s = \frac{M}{\overline{m}(\mu)} \overline{m}_s(\mu) = \frac{M}{\overline{m}(\mu)} \frac{Z_A(g_0)}{Z_P(g_0, \mu)} \underbrace{m_s(g_0)}_{\text{bare PCAC}} \\
\equiv Z_M(g_0) \times m_s(g_0)
\]

\[
Z_M(g_0) = \frac{M}{\overline{m}(\mu)} \frac{Z_A(g_0)}{Z_P(g_0, \mu)} = \frac{M}{\overline{m}(\mu_{\text{pert}})} \overline{m}(\mu_{\text{pert}}) \overline{m}(\mu_{\text{had}}) \frac{Z_A(g_0)}{Z_P(g_0, \mu_{\text{had}})} \\
\text{PT} \quad \text{NP} \quad \text{“easy”}
\]
Basic equation for the RGI quark mass

\[ Z_M(g_0) = \frac{Z_A(g_0)}{Z_P(g_0, \mu_{\text{had}})} \frac{\bar{m}(\mu_{\text{pert}})}{\bar{m}(\mu_{\text{had}})} \frac{M}{\bar{m}(\mu_{\text{pert}})} \]

\[ M_i = Z_M(g_0) m_i(g_0) \quad (\text{here: } i = s, c) \]
NP calculation of $\bar{m}(\mu_{\text{pert}})/\bar{m}(\mu_{\text{had}})$ in the intermediate SF scheme

Schrödinger Functional = QCD partition function in a cylinder (Dirichlet BCs in $x_0$)

⇒ NP running from (finite-volume) correlation functions & a recursive technique
Physics inputs & Lattice QCD computation

- Set bare strange mass in the Lagrangian s.th. \( m_K / F_K = \text{experiment} \)
  \[ \Rightarrow \] amounts to slightly extrapolate to physical quark mass: \( m_{\text{PS}} \rightarrow m_K \)

- MeV-scale by fixing the static quark potential to phenomenology: \( r_0 = 0.5 \text{ fm} \)

- Evaluate ratio \( R \equiv F_K / G_K \) of K-meson matrix elements in numerical lattice simulations and combine this with \( Z_M \):
  \[
  M_s + M_u = Z_M \frac{F_K}{G_K} m_K^2
  \]
  \[
  \text{[NP } Z_M: \text{ALPHA Collaboration 1999 & 2005 (} N_f = 0, 2 \text{)]}
  \]
  (ChPT “cheat”: We actually compute \( M_s + M_\ell, M_\ell = \frac{1}{2}(M_u + M_d), \frac{M_s}{M_\ell} = 24.4 \))

\[ N_f = 0 \text{ result: } \]
\[ \overline{m}_s^{\text{MS}}(2 \text{ GeV}) = 97(4) \text{ MeV (} \lesssim 15\% \text{ quenching error)} \]
The charm quark’s mass

Calculation in same spirit as for strange

- **Physics input:** bare charm mass in $\mathcal{L}_{QCD}$ s.th. $m_{D_s}/F_K = \text{experiment}$
- **Additional complication in the charm sector:**
  - $O(\alpha m_{q,c})$ cutoff effects become relevant

\[
M_c = Z_M \left[ 1 + (b_A - b_P) \alpha m_{q,c} \right] m_c = Z_M \frac{Z_m}{Z_A} m_{q,c} (1 + b_m \alpha m_{q,c})
\]

$\rightarrow$ remove $O(\alpha m_{q,c})$ NP’ly

[ \text{LPHA} \text{ Collaboration 2000 (}N_f = 0\text{) & in progress (}N_f = 2\text{)} ]


The charm quark’s mass

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$$

$\rightarrow$ remove $O(\alpha m_{q,c})$ NP’ly

[**LPHA** Collaboration 2000 ($N_f = 0$) & in progress ($N_f = 2$)]

$N_f = 0$, $r_0 = 0.5 \text{ fm}$: $M_c = 1654(45) \text{ MeV}$

$\Rightarrow \bar{m_c}^{\overline{MS}}(M_c) = 1301(34) \text{ MeV}$

- **Similar analysis for** $N_f > 0$ in progress
The charm quark’s mass

Calculation in same spirit as for strange

- **Physics input**: bare charm mass in $\mathcal{L}_{\text{QCD}}$ s.th. $m_{D_s}/F_K = \text{experiment}$
- **Additional complication in the charm sector**:
  - $O(\alpha m_{q,c})$ cutoff effects become relevant

\[
M_c = Z_M \left[ 1 + (b_A - b_P) \alpha m_{q,c} \right] m_c = Z_M \frac{Z_m Z_P}{Z_A} m_{q,c} \left( 1 + b_m \alpha m_{q,c} \right)
\]

$\rightarrow$ remove $O(\alpha m_{q,c})$ NP’ly

[Alfa Collaboration 2000 ($N_f = 0$) & in progress ($N_f = 2$)]

- $N_f = 0$, $r_0 = 0.5 \, \text{fm}$: $M_c = 1654(45) \, \text{MeV}$
  \[\Rightarrow \overline{m}_{\text{MS}}(\overline{m}_c) = 1301(34) \, \text{MeV}\]
- Similar analysis for $N_f > 0$ in progress

- Charm *just* doable, but $\alpha M_b \simeq 4 \alpha M_c$!
  $\rightarrow$ for b-quarks, continuum limit $\alpha \rightarrow 0$
  can’t be controlled in this way
  $\Rightarrow$ need for an effective theory

\[
\alpha \simeq 0.03 \, \text{fm added}
\]
(byproduct of work with A. Jüttner)
The bottom quark’s mass

- Non-perturbative Heavy Quark Effective Theory
- Application: Computation of $M_b$
- Status in two-flavour QCD
Lattice HQET — Why?

- **Light quarks: too light**
  - Widely spread objects
  - Finite-volume errors due to light pions

- **b-quark: too heavy**
  - Extremely localized object
  - B-mesons with a propagating b-quark on the lattice require very small \( a < \frac{1}{m_b} \), beyond today's computing resources

⇒ Recourse to an *effective* theory for the b-quark:

**Heavy Quark Effective Theory**

\[ \bar{\psi}_b \left\{ \gamma_\mu D_\mu + m_b \right\} \psi_b \rightarrow \]
Lattice HQET — Why?

- Light quarks: too light
  - Widely spread objects
  - Finite-volume errors due to light pions
- b-quark: too heavy
  - Extremely localized object
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⇒ Recourse to an effective theory for the b-quark:

**Heavy Quark Effective Theory**

$[\text{Eichten, 1988; Eichten \\ & Hill, 1990}]

\mathcal{L}_{HQET}(x) = \bar{\psi}_h(x) \left[ D_0 + m_b - \frac{\omega_{\text{kin}}}{2m_b} D^2 - \frac{\omega_{\text{spin}}}{2m_b} \sigma \cdot B \right] \psi_h(x) + \ldots$
Non-perturbative formulation of HQET

Beyond the static approximation

\[ \bar{\psi}_h \left( -\frac{1}{2} \sigma \cdot B \right) \psi_h \equiv \mathcal{O}_{\text{spin}} \rightarrow \{ \text{chromomagnetic interaction with the gluon field} \} \]

\[ \bar{\psi}_h \left( -\frac{1}{2} D^2 \right) \psi_h \equiv \mathcal{O}_{\text{kin}} \rightarrow \{ \text{kinetic energy from heavy quark's residual motion} \} \]

\[ \bar{\psi}_h \left[ \nabla_0^* + \delta m \right] \psi_h \rightarrow \text{Eichten-Hill action} \]

\[ \delta m, \omega_i(g_0, m) \] must be determined such that HQET matches QCD

Analogously: Composite fields in the effective theory, e.g.

\[ A_0^{\text{HQET}}(x) = Z_A^{\text{HQET}} \frac{\bar{\psi}_1(x) \gamma_0 \gamma_5 \psi_h(x)}{1 + \mathcal{O}(g_0^2)} + c_A^{\text{HQET}} \frac{\bar{\psi}_1(x) \gamma_j \gamma_5 \hat{D}_j \psi_h(x)}{1} + \ldots \]

\[ \propto 1/m \]
Non-perturbative formulation of HQET

Beyond the static approximation

\[ \overline{\psi}_h \left[ \nabla_0^* + \delta m \right] \psi_h \rightarrow \text{Eichten-Hill action} \]
\[ \overline{\psi}_h \left( -\frac{1}{2} D^2 \right) \psi_h \equiv \mathcal{O}_{\text{kin}} \rightarrow \{ \text{kinetic energy from heavy quark's residual motion} \} \]
\[ \overline{\psi}_h \left( -\frac{1}{2} \sigma \cdot B \right) \psi_h \equiv \mathcal{O}_{\text{spin}} \rightarrow \{ \text{chromomagnetic interaction with the gluon field} \} \]

- **EVs = Functional integral representation at the quantum level:**

\[
\langle O \rangle = \frac{1}{\mathcal{Z}} \int \mathcal{D}[\varphi] \ O[\varphi] \ e^{-\left(S_\text{rel} + S_{\text{HQET}}\right)} \quad \mathcal{Z} = \int \mathcal{D}[\varphi] \ e^{-\left(S_\text{rel} + S_{\text{HQET}}\right)}
\]

Now the integrand is expanded in a power series in \(1/m\)

\[
\exp\{-S_{\text{HQET}}\} = \\
\exp \left\{ -a^4 \sum_x \mathcal{L}_{\text{stat}}(x) \right\} \times \left\{ 1 - a^4 \sum_x \mathcal{L}^{(1)}(x) + \frac{1}{2} \left[ a^4 \sum_x \mathcal{L}^{(1)}(x) \right]^2 - a^4 \sum_x \mathcal{L}^{(2)}(x) + \ldots \right\}
\]
Beyond the static approximation

\[ \overline{\psi}_h \left[ \nabla_0^* + \delta m \right] \psi_h \quad \rightarrow \quad \text{Eichten-Hill action} \]

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\[ \overline{\psi}_h \left( - \frac{1}{2} \sigma \cdot B \right) \psi_h \equiv \mathcal{O}_{\text{spin}} \quad \rightarrow \quad \{ \text{chromomagnetic interaction with the gluon field} \} \]

\[ \Rightarrow \quad \langle O \rangle = \frac{1}{\mathcal{Z}} \int \mathcal{D}[\varphi] \ e^{-S_{\text{rel}} - a^4 \sum_x \mathcal{L}_{\text{stat}}(x)} \ O \left\{ 1 - a^4 \sum_x \mathcal{L}^{(1)}(x) + \ldots \right\} \]

Important (but not automatic) implications of this definition of HQET

1/\( m \) – terms appear only as *insertions* of local operators

⇒ Power counting: *Renormalizability* at any given order in 1/\( m \)

⇔ Existence of the *continuum limit with universality*

Effective theory = *Continuum asymptotic expansion in 1/\( m \)
Mass renormalization pattern in HQET

Already at the level of $\mathcal{L}_{\text{stat}}(x) = \overline{\psi}_h(x) \left[ \nabla^*_0 + \delta m \right] \psi_h(x)$:

Linear divergence $\delta m \propto a^{-1}$ originates from mixing of $\overline{\psi}_h D_0 \psi_h$ with $\overline{\psi}_h \psi_h$.

$$m_{\overline{\text{MS}}} = Z_{\text{pole}} \times m_{\text{pole}} \quad m_{\text{pole}} = m_B - E_{\text{stat}} - \delta m$$

\[\begin{bmatrix}
m_B : \text{(exp.) B-meson mass} \\
E_{\text{stat}} : \text{static binding energy}
\end{bmatrix}\]

$$\delta m = \frac{c(g_0)}{a} \sim e^{1/(2b_0 g_0^2)} \times \left\{ c_1 g_0^2 + c_2 g_0^4 + \ldots \right\}$$
Mass renormalization pattern in HQET

Already at the level of \( \mathcal{L}_{\text{stat}}(x) = \overline{\psi}_h(x) \left[ \nabla_0^* + \delta m \right] \psi_h(x) \):

Linear divergence \( \delta m \propto a^{-1} \) originates from mixing of \( \overline{\psi}_h D_0 \psi_h \) with \( \overline{\psi}_h \psi_h \):

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m_b^{\overline{\text{MS}}} = Z^{\overline{\text{MS}}} \times m_{\text{pole}} \quad m_{\text{pole}} = m_B - E_{\text{stat}} - \delta m
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\]

- In PT:
  uncertainty = truncation error \( \sim e^{1/(2b_0 g_0^2)} c_{n+1} g_0^{2n+2} \xrightarrow{g_0 \to 0} \infty ! \)
  \( \Rightarrow \) Non-perturbative \( c(g_0) \) needed
  \( \Rightarrow \) NP renormalization (resp. matching to QCD) of HQET required for the continuum limit to exist

- Power-law divergences even worse at the level of \( 1/m \) – corrections: \( a^{-1} \to a^{-2} \)
NP matching in *finite* volume

Need to treat the b as particle with finite mass (while $\alpha m_b \ll 1$ to keep $\alpha$–effects small) and $L m_b \gg 1$ to apply HQET

⇒ **Trick:** start with QCD in a *small* volume, $V = L^4 \approx (0.4 \, \text{fm})^4$

**QCD**

\[
\frac{1}{m_b} \gg \alpha
\]

**HQET**

Matching conditions

\[
\Phi_{QCD}^k = \Phi_{HQET}^k
\]

for observables $\Phi_k$

(renormalized quantities, computable for $\alpha \to 0$)

- HQET parameters fixed by relating them to QCD observables in small $V$
- Rather than simulating a propagating ‘real’ relativistic b-quark, one aims at determining the *NP heavy quark mass dependence* of $\Phi_{QCD}^k$ in finite $V$
Connecting small and large volumes

- Matching volume: $L = L_1 \simeq 0.4 \text{ fm}$, very small lattice spacings
- Gap to large volumes and practicable lattice spacings, where physical quantities (e.g. $m_B$, $F_{B_s}$) may be extracted, bridged by a . . .

Finite-size scaling step


HQET in large volume:

\[ \Phi_{k}^{\text{HQET}}(L_1, M_b) \]

\[ \Phi_{k}^{QCD}(L_1, M_b) \]

\[ \Phi_{k}^{\text{HQET}}(L_2, M_b) \]

Matching

\[ \sigma_k \]

0.4 fm
Finite-size scaling step


- Fully non-perturbative, continuum limit can be taken everywhere
- Use the \textit{QCD Schrödinger Functional}, \( L \rightarrow 2L \) via \textit{Step Scaling Functions}

\[
\Phi_k^{\text{HQET}}(2L) = \sigma_k \left( \{ \Phi_j^{\text{HQET}}(L), j = 1, \ldots, N \} \right)
\]

\( 2L = 2L_1 \simeq 0.8 \text{ fm} \)

\[\rightarrow\] Large \( V \) (\( L \simeq 2 \text{ fm} \)) at same resolution, where a B-meson fits comfortably
Computation of $M_b$

Non-trivial matching problem:

\[ \{m_{\text{bare}} + \delta m, \omega_{\text{kin}}, \omega_{\text{spin}}\} \text{ from QCD s.th. } m_{\text{pole}} + \delta m \leftrightarrow M_b \]

\[ \Rightarrow N = 3 \text{ matching conditions:} \]

\[ \Phi_{k}^{QCD}(L, M) = \Phi_{k}^{HQET}(L, M) \quad k = 1, 2, 3 \]

[ $M$: RGI heavy quark mass ]
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Basic equation in leading order of HQET (static approximation)

\[ m_B = E_{\text{stat}} - E_{\text{stat}}^\text{sub} + E_{\text{stat}}^\text{sub} \quad \text{with} \quad E_{\text{stat}}^\text{sub} = E(L_1, M_b) \]  [matching to QCD]

\[ = \underbrace{E_{\text{stat}} - E_{\text{stat}}(L_2)}_{\alpha \to 0 \text{ in HQET}} + \underbrace{E_{\text{stat}}(L_2) - E_{\text{stat}}(L_1)}_{\alpha \to 0 \text{ in HQET} (\sigma_m)} + \underbrace{E(L_1, M_b)}_{\alpha \to 0 \Phi_2/L_1 \text{ in QCD}} \]

- Divergent static quark’s self-energy \( \delta m \) cancels in differences!
- \( \Phi_2(L_1, M) \) carries entire (relativistic) heavy quark mass dependence
Basic equation in leading order of HQET (static approximation)

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\[ = \mathcal{E}_{\text{stat}} - \mathcal{E}_{\text{stat}}(L_2) + \mathcal{E}_{\text{stat}}(L_2) - \mathcal{E}_{\text{stat}}(L_1) + \mathcal{E}(L_1, M_b) \quad (\star) \]

\[ \alpha \to 0 \text{ \ in HQET} \quad \alpha \to 0 \text{ \ in HQET } (\sigma_m) \quad a \to 0 \text{ \ in HQET } \equiv \Phi_2/L_1 \text{ \ in QCD} \]

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Quenched result

Della Morte, Garron, Sommer & Papinutto
JHEP01(2007)007

\( L_1 \simeq 0.4 \text{ fm}, \quad L_2 = 2L_1 \)

- Use \( r_0 m_B^{(\text{exp})} \), \( r_0 = 0.5 \text{ fm} \) & solve \( (\star) \)

\[ \Rightarrow M_b^{\text{stat}} = (6771 \pm 99) \text{ MeV} \]

- NP renormalization & Continuum limit
- Error dominated by that on \( Z_M \) (\( \simeq 1\% \)) in \( L_1 M = Z_M Z (1 + b_m a m_q) \times L_1 m_q \)
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\[ m_B = E_{\text{stat}} - E_{\text{sub stat}} + E_{\text{sub stat}} \]

with

\[ E_{\text{sub stat}} = E(L_1, M_b) \] [matching to QCD]

\[ a \to 0 \text{ in HQET} \]

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Inclusion of $1/m$–terms

$m_B$ at next-to-leading order of HQET

\[ m_B = m_{\text{bare}} + E_{\text{stat}} + \omega_{\text{kin}} E_{\text{kin}} + \omega_{\text{spin}} E_{\text{spin}} \]

- $E_{\text{kin}}, E_{\text{spin}}$ associated with $\bar{\psi}_h(-\frac{1}{2}D^2)\psi_h$ and $\bar{\psi}_h(-\frac{1}{2}\sigma \cdot B)\psi_h$ in $\mathcal{L}^{(1)}$
  → Three observables $\Phi_1, \Phi_2, \Phi_3$ required in the matching step
- Considering the *spin-averaged* B-meson instead, $\omega_{\text{spin}}$ cancels:
  \[ m^{(\text{av})}_B = \frac{1}{4} m_B + \frac{3}{4} m^*_B = m_{\text{bare}} + E_{\text{stat}} + \omega_{\text{kin}} E_{\text{kin}} \]
  → Only *two* observables $\Phi_1, \Phi_2$ necessary
Inclusion of $1/m$–terms

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### Strategy

<table>
<thead>
<tr>
<th>Experiment</th>
<th>Lattice with $am_b \ll 1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$m_B = 5.4 \text{ GeV}$</td>
<td>$\Phi_1(L_1, M), \Phi_2(L_1, M)$</td>
</tr>
<tr>
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</tr>
<tr>
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<td>$\sigma_m(u_1)$</td>
</tr>
<tr>
<td>$u_1 = \bar{g}^2(L_1)$</td>
<td>$\sigma_{\text{kin}}^1(u_1), \sigma_{\text{kin}}^2(u_1)$</td>
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</table>
Matching formula = Static part + $1/m$–correction:

$$L_2m_B^{(av)} = L_2m_B^{stat} \left\{ \begin{array}{c}
L_2 \left[ E_{stat} - \Gamma_{stat}^1(L_2) \right] + \sigma_m(u_1) + 2\Phi_2(L_1, M_b) \\
+ L_2m_B^{(1)} \end{array} \right\} = \sigma_2^{kin}(u_1)\Phi_1(L_1, M_b) + L_2 \left[ E_{kin} - \Gamma_{kin}^1(L_2) \right] \omega_{kin}$$

Most difficult piece encountered in the calculation
Large-volume HQET matrix element $\left[ E_{kin} - \Gamma_{kin}^1(L_2) \right]$ in the $1/m_b$–contribution

Result in the quenched approximation

$$\overline{m}_b^{MS}(\overline{m}_b) = 4.374(64) \text{ GeV} - 0.027(22) \text{ GeV} + O(\Lambda^3/m_b^2)$$

Further ingredients:
- Finite- and large-volume B-meson energies extracted from static-light SF correlators
- HYP-smeared static actions [Hasenfratz & Knechtli, 2001; ALPHA Collaboration 2004/05]
- Consistency with results from different matching observables/conditions checked
Matching to QCD in finite $V$ with a relativistic $b$-quark

- $N_f = 2$ degenerate massless sea quarks
- Evaluation of QCD heavy-light valence quark correlation functions
  ($m_{\text{val}} = m_{\text{sea}} = 0$ in the SF)
- NP quark mass renormalization in $V = L_1^4 \approx (0.5 \text{ fm})^4$ finished:
  \[ z \equiv L_1 M = Z_M Z (1 + b_m a m_q) \times L_1 m_q \]
  known in terms of bare parameters
  \[ \rightarrow \text{enables to calculate the heavy quark mass dependence of observables} \]
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  → enables to calculate the heavy quark mass dependence of observables

Example for a CL extrapolation: spin-averaged B-energy \( \equiv L_1 \times \left[ \frac{1}{4} \Gamma_{\text{PS}} + \frac{3}{4} \Gamma_{\text{V}} \right] \)
Challenges & Outlook

Challenges:

- $(M_s), M_c$ for $N_f = 2$:
  - smaller lattice spacings and smaller quark masses are under way by various groups/collaborations
  - appear feasible owing to algorithmic/technical improvements: domain decomposition, low-mode deflation & all-to-all quark propagators

[Lüscher, ’03 – ’05; Del Debbio et al., ’07; Lüscher, ’07; Foley et al., ’05]
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    - \([\text{Lüscher, '03 – '05; Del Debbio et al., '07; Lüscher, '07; Foley et al., '05}]
  - with current results (i.e. status reported @ Lattice 2007), it is still quite a bit early for a continuum limit
    - (e.g. B. Blossier for ETMC: \(\overline{m}_c^{\overline{MS}}(\overline{m}_c) = 1478(75)\) MeV)
Challenges & Outlook

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► \(M_b\) for \(N_f = 2\) through non-perturbative HQET is in progress:
  ◦ error for \(M_b\) is dominated by \(Z_M(g_0)\) → reduce by a factor 2 ?!
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- Light up & down quarks, large volumes and chiral extrapolation

Why should we face the challenges?

- LQCD: “No” assumptions to determine fundamental SM parameters
- Superb experimental input available: $m_p, m_K, m_D, m_B, \ldots$
Bottomline conclusions:

- Non-perturbative (NP) tools for renormalization are fully developed
- Quenched computations are under good control, particularly thanks to
  - NP $O(\alpha)$ improvement
  - NP renormalization
- LQCD with dynamical quarks, $N_f \geq 2$:
  Significant progress & even more expected in the next 2–3 years