Status of QED prediction of the electron g-2

Ettore Remiddi

Dipartimento di Fisica, Università di Bologna

INFN, Sezione di Bologna

ettore.remiddi@bo.infn.it

Frascati, 7 April 2008

plan of the talk

.

- current experimental value of the anomaly
- current status of the theoretical calculation
- recent results (mostly by S. Laporta)

Anomalous magnetic moment of the electron

if $\vec{\mu}$ is the `magnetic moment` of the electron and $\vec{s}$ its `spin`

$$\vec{\mu} = g_e \frac{e\hbar}{2mc}\vec{s}$$

where g_e is the `gyromagnetic ratio`; the Dirac equation gives $g_e = 2$ for a free electron – but $g_e \neq 2$ (`Breit`) for a *bound* electron (g_e-*bound*), so in general

$$g_e = 2(1 + a_e)$$

where $a_e = \frac{1}{2}(g-2)$ is the *anomaly*.

in `QED`, $g_e \neq 2$, or $a_e \neq 0$ even for the *free* electron.

Experimental values of the *free* electron a_e (current main problem: cavity shift)

$10^{12}\ a_e$(`exp`)	=	1 159 652 188.4 (4.3)[3.7 ppb]	Dehmelt 1987, UW
	=	1 159 652 180.85(.76)[0.66 ppb]	Gabrielse 2006, Harvard
	=	1 159 652 180.73(.28)[0.24 ppb]	Gabrielse 2008, Harvard

PRL **100,** 120801 (2008) PHYSICAL REVIEW LETTERS week ending 28 MARCH 2008

New Measurement of the Electron Magnetic Moment and the Fine Structure Constant

D. Hanneke, S. Fogwell, and G. Gabrielse*

Department of Physics, Harvard University, Cambridge, Massachusetts 02138, USA

(Received 4 January 2008; published 26 March 2008)

A measurement using a one-electron quantum cyclotron gives the electron magnetic moment in Bohr magnetons, $g/2 = 1.001\,159\,652\,180\,73\,(28)$ [0.28 ppt], with an uncertainty 2.7 and 15 times smaller than for previous measurements in 2006 and 1987. The electron is used as a magnetometer to allow line shape statistics to accumulate, and its spontaneous emission rate determines the correction for its interaction with a cylindrical trap cavity. The new measurement and QED theory determine the fine structure constant, with $\alpha^{-1} = 137.035\,999\,084\,(51)$ [0.37 ppb], and an uncertainty 20 times smaller than for any independent determination of α.

DOI: 10.1103/PhysRevLett.100.120801 PACS numbers: 06.20.Jr, 12.20.Fv, 13.40.Em, 14.60.Cd

The electron magnetic moment $\boldsymbol{\mu}$ is one of the few measurable properties of one of the simplest of elementary particles—revealing its interaction with the fluctuating QED vacuum, and probing for size or composite structure not yet detected. What can be accurately measured is $g/2$, the magnitude of $\boldsymbol{\mu}$ scaled by the Bohr magneton, $\mu_B = e\hbar/(2m)$. For an eigenstate of spin $\mathbf{S}$,

$$\boldsymbol{\mu} = -\frac{g}{2}\mu_B\frac{\mathbf{S}}{\hbar/2}, \tag{1}$$

order QED calculation (underway [7]) than by the measurement uncertainty in $g/2$. The accuracy of the new g sets the stage for an improved CPT test with leptons. It also will allow an improved test of QED, and will be part of the discovery of low-mass dark-matter particles or the elimination of this possibility [8], when a better independent measurement of α becomes available.

Figure 3 represents the lowest cyclotron and spin energy levels for an electron weakly confined in a vertical magnetic field $B\hat{\mathbf{z}}$ and an electrostatic quadrupole potential.

the theoretical expression for a_e is

$$\begin{aligned} a_e &= a_e(QED) \\ &\quad + \Delta a_e(\mu) + \Delta a_e(\tau) + \Delta a_e(\text{hadronic}) + \Delta a_e(\text{weak}) \\ &\quad + \Delta a_e(\texttt{NP}) \end{aligned}$$

small terms:

$$\Delta a_e(\mu) = 2.71 \times 10^{-12}$$

$$\Delta a_e(\tau) = 0.01 \times 10^{-12}$$

$$\Delta a_e(\text{hadronic}) = 1.671(19) \times 10^{-12} \simeq \left(\frac{m_e}{m_\mu}\right)^2 \Delta a_\mu(\text{hadronic})$$

$$\Delta a_e(\text{weak}) = 0.030(01) \times 10^{-12}$$

The constraint on NP depends on the experimental error – but the dependence on unknown new particles is quadratic in the ratio of the masses (for large masses)

$$\Delta a_e(\mathtt{NP}) \simeq \left(\frac{m_e}{m_\mu}\right)^2 \Delta a_\mu(\mathtt{NP})$$

experimental error on a_e: $\Delta a_e(\mathtt{exp}) = 0.28 \times 10^{-12}$

experimental values of the μ anomaly

$$10^{12}\ a_\mu(\mathtt{exp}) = \mathtt{1\ 165\ 920\ 800(600)[517\ ppb]} \qquad \mathtt{E821\ Brookhaven, 2004}$$

$$\frac{\Delta a_\mu(\mathtt{exp})}{\Delta a_e(\mathtt{exp})} = 2100 \qquad\qquad \left(\frac{m_\mu}{m_e}\right)^2 = 43000$$

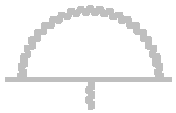
the μ anomaly is still more constraining on NP by a factor 20.

(strictly) QED contribution

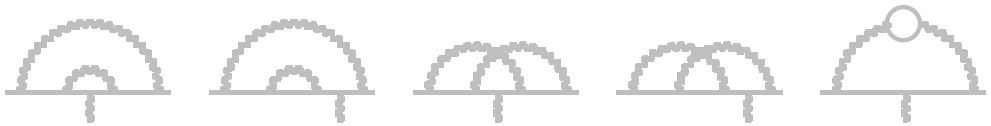
$a_e(\texttt{QED})$ is the QED contribution with electron loops only
and depends only on the fine structure constant α

$$a_e(QED) = C_1\left(\frac{\alpha}{\pi}\right) + C_2\left(\frac{\alpha}{\pi}\right)^2 + C_3\left(\frac{\alpha}{\pi}\right)^3 + C_4\left(\frac{\alpha}{\pi}\right)^4 + C_5\left(\frac{\alpha}{\pi}\right)^5 + \ldots$$

$C_1 = \frac{1}{2}$ (Schwinger 1948) *1 diagram*



$C_2 = \frac{197}{144} + \frac{1}{12}\pi^2 - \frac{1}{2}\pi^2 \ln 2 + \frac{3}{4}\zeta(3)$



$= -0.328\ 478\ 965\ 579\ \ldots,$ (Petermann, Sommerfield 1957) *7 diagrams*

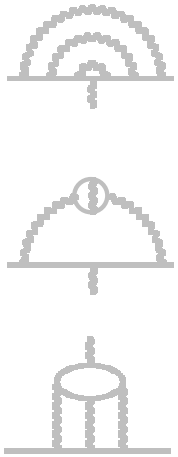
(strictly) QED contribution

3 loops:

$$C_3 = \frac{83}{72}\pi^2\zeta(3) - \frac{215}{24}\zeta(5) + \frac{100}{3}\left[\left(a_4 + \frac{1}{24}\ln^4 2\right) - \frac{1}{24}\pi^2\ln^2 2\right]$$

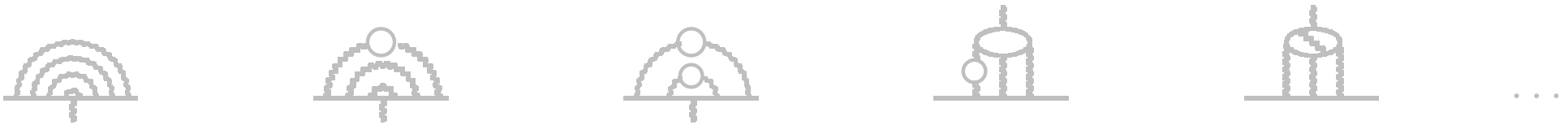
$$- \frac{239}{2160}\pi^4 + \frac{139}{18}\zeta(3) - \frac{298}{9}\pi^2\ln 2 + \frac{17101}{810}\pi^2 + \frac{28259}{5184}$$

$$= 1.181\ 241\ 456\ \ldots, \quad \text{(S.Laporta, E.R. 1996)} \quad \textit{72 diagrams}$$



$$\zeta(p) = \sum_{n=0}^{\infty} \frac{1}{n^p}, \qquad a_4 = \sum_{n=0}^{\infty} \frac{1}{2^n n^4},$$

$$C_3\left(\frac{\alpha}{\pi}\right)^3 = 14\ 804.20 \times 10^{-12}\ , \qquad \Delta a_e(\mathtt{exp}) = 0.28 \times 10^{-12}$$



891 diagrams

The 4-loop coefficient is known only numerically.

Only a few diagrams are known in analytical form.

Numerical values are obtained by using MonteCarlo integration of huge 11-dimensional integrands (... and fixing minor bugs underway)

$C_4 = -1.5098(384)$ (Kinoshita, 1999)

$C_4 = -1.7283(35)$ (Kinoshita, 2003) shift of -0.24 due to the discovery of one error

$C_4 = -1.9144(35)$ (Kinoshita, 2007) shift of -0.22 due to the discovery of another error

$$C_4\left(\frac{\alpha}{\pi}\right)^4 = (55.73 \pm 0.10) \times 10^{-12}\ , \qquad \Delta a_e(\mathrm{exp}) = 0.28 \times 10^{-12}$$

the numerical evaluation of the 5 loop contribution is underway (Kinoshita)
guess: $C_5 = 0.0 \pm 4$

$$C_5\left(\frac{\alpha}{\pi}\right)^5 = (0 \pm 0.27) \times 10^{-12} \ , \qquad \Delta a_e(\mathrm{exp}) = 0.28 \times 10^{-12}$$

same order of magnitude as the current experimental error !

QED=QED(α)

$$\text{QED} + \alpha \rightarrow a_e$$

$$\text{QED} + a_e \rightarrow \alpha$$

$$\text{QED} + a_e + \alpha \rightarrow \text{real check}$$

best independent experimental values of α from

$$\alpha^2 = \frac{2R_\infty}{c} \left(\frac{m_u}{m_e}\right) \left(\frac{M_X}{m_u}\right) \left(\frac{h}{M_X}\right)$$

where m_u is the unit of mass, M_X the mass of the atom X and (h/M_X) is measured by recoil spectroscopy
best results

$$\alpha^{-1}(Cs) = 137.036\ 000\ 00(110) \qquad (8.0\ ppb) \qquad \texttt{NIST, 2006}$$

$$\alpha^{-1}(Rb) = 137.035\ 998\ 78\ \ (91) \qquad (6.7\ ppb) \qquad \texttt{ENS, 2006}$$

much larger than the (0.24 ppb) error in a_e(exp);
(relatively) weak check of QED

by combining available theory and the Harvard 2008 value of a_e

$$\begin{aligned} \alpha^{-1}(a_e) &= 137.035\,999\,084\ (33)(39) \\ &= 137.035\,999\,084\ (51) \qquad [0.37\ \text{ppb}] \end{aligned}$$

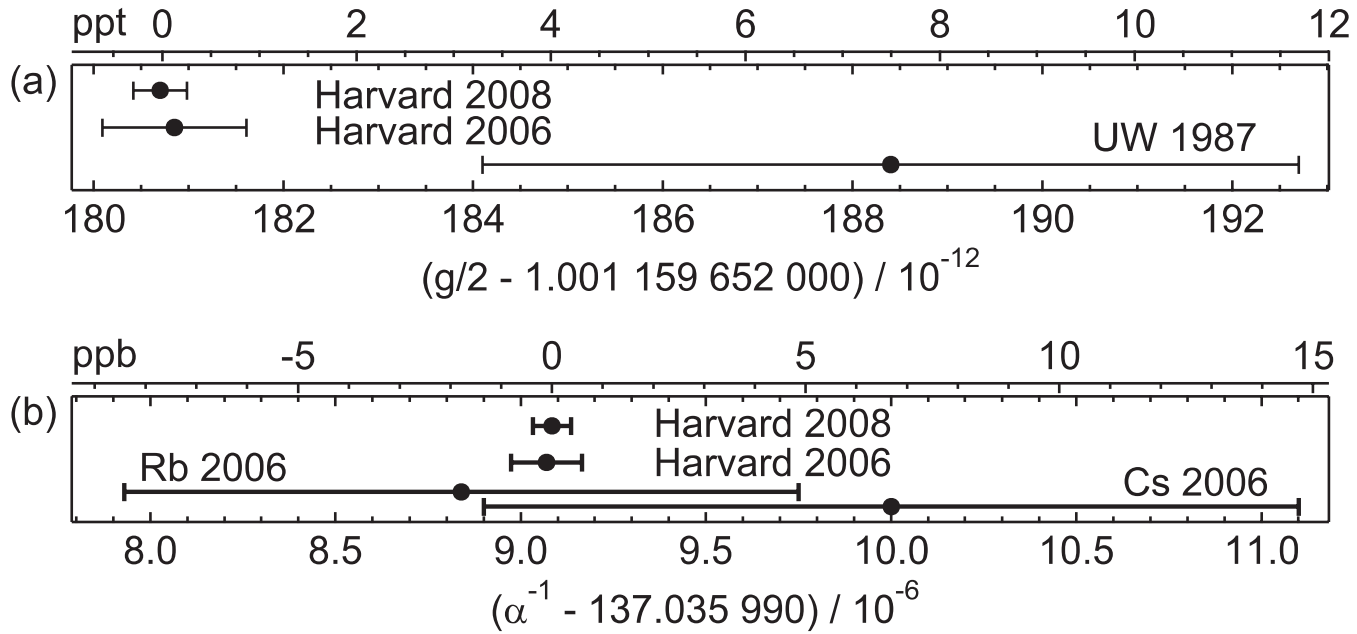


FIG. 1. Most accurate measurements of the electron $g/2$ (a), and most accurate determinations of α (b).

Improving the 4 loop calculation

We (mostly S. Laporta) are trying to improve C_4

- to provide a check ;
- to reduce the error.

The by now standard approach (implemented by S.L. in his computer programs) is:

- Extraction of the contribution to g-2 from each diagram as a sum of scalar loop integrals.
- Reduction of each contribution to the combination of a (small) number of `Master Integrals`.
- high-precision numerical calculation of the `Master Integrals`.
- Cross-checks of procedures, intermediate and final results

the first step is almost trivial (with an algebraic program like, say, `SCHOONSCHIP` or `FORM`)

Reduction to Master Integrals

The scalar loop integrals are reduced to `Master Integrals` by exploiting the `integration by parts identities` (Chetyrkin, Tkachov 1981) in the D-continuous regularization scheme.

Not surprisingly, the `Laporta` algorithm is used for solving the identities:

• an essential point of the algorithm is the *ordering* of the scalar integrals;

• a *sufficiently* large system of identities is built, containing all the relevant scalar integrals and more identities than integrals;

• as a rule, a large linear system of identities is obtained, up to $10^6 \ldots 10^7$ identities for graph.

• the system is then solved, one equation at the time, with the `Gauss` substitution rule, by expressing "more difficult" integrals (within the *ordering rule*) in terms of "simpler" ones;

As a result all the integrals are reduced to a combination of irreducible master integrals M_j

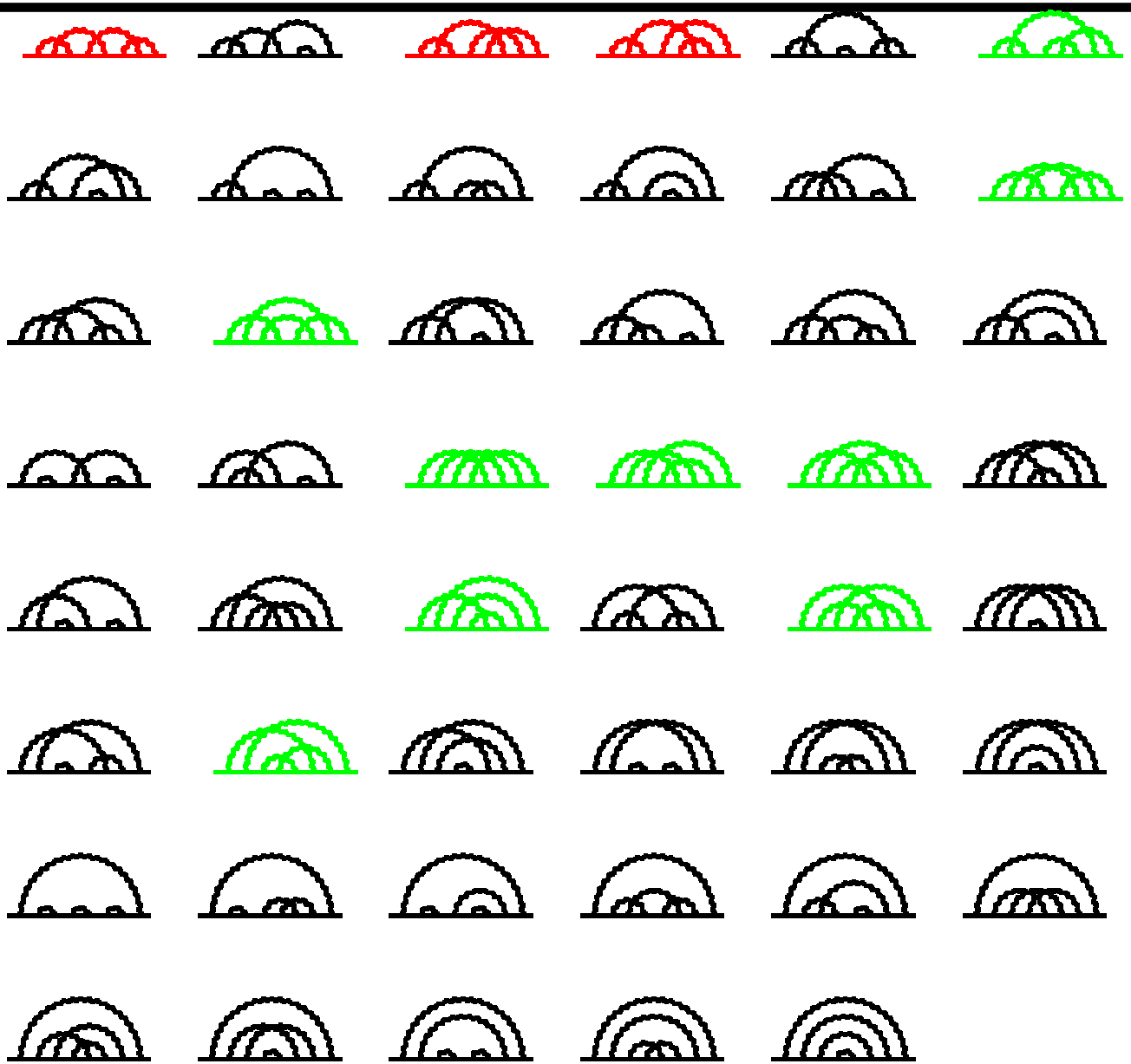
$$a_e(\texttt{diagram}) = \sum_{j=1}^{N} \frac{p_j(D)}{q_j(D)} M_j(D)$$

where $r_j(D), s_j(D)$ are polynomials in the dimension D with integer coefficients.

• at 3-loop, $N = 18$, *i.e.* the final result from all the graphs is expressed in terms of 18 *Master Integrals* only;
• at 4-loop, $N \sim 300$ is expected.

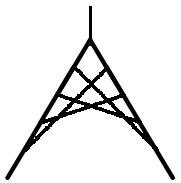
some graphs without electron loops were already processed

the 47 4-loop electron self-masses without electron loops



green,red = reduced to `Master Integrals` (work in progress...)

Contribution of a 4-loop quadruple-cross diagram



$$= \{23299 \text{ integrals}\} = \sum_{j=1}^{140} \frac{p_j(D)}{q_j(D)} M_j(D)$$

140 Master Integrals

M_1 M.I. with 11 denominators (still to be evaluated)

$p_1(D)$ =polynomial of degree 11

$q_1(D) = 5(D-1)(D-2)(D-3)(5D-16)(5D-18)(5D-22)$

M_{140} M.I. with 4 denominators (factorizes into 4 1-loop tadpoles)

$p_{140}(D)$ =polynomial of degree 58

$$q_{140}(D) = 5184000 D(D+4)(D+2)(D-1)^2(D-3)^5(D-4)^4$$
$$(D-5)^3(D-6)(D-8)(2D-5)^2(2D-7)^4(2D-9)^2(2D-11)$$
$$(2D-13)(3D-8)^2(3D-10)^3(3D-11)^2(3D-13)(4D-11)$$
$$(5D-12)(5D-13)(5D-14)(5D-16)(5D-17)(5D-18)$$
$$(5D-19)(5D-21)(5D-22)(7D-16)(10D^2-59D+86)$$

M_{140} is trivial!

$$= M_{140} = \frac{1}{\epsilon^4(1-\epsilon)^4}$$

with $\epsilon = \frac{4-D}{2}$.

The evaluation of the other `MI's` is *not* trivial.
A few analytic results are available, but the analytic evaluation of *all* the `MI's` does not seem to be at present a realistic possibility.

But a High precision numerical algorithm has been developped by S. Laporta a few yeas ago (2000).

High-precision numerical calculation of MI's

Given a master integral $M(D)$

- choose a denominator in $M(D)$
- consider the set of the integrals $M(n, D)$ where that denominator is raised to the power n
- construct a (system of) difference equation in n for the $M(n, D)$
- solve the system by expansion in factorial series or by Laplace transformation.

Advantages:

- No numerical integration
- The calculation of multi-loop integrals (in *any* number of loops) is reduced to sums of series in one variable.
- Arbitrary precision obtainable (such as 100-1000 digits).

as an example, consider M_{139}, which is *not* trivial; a semi-analytic approach gives

$$= M_{139} = -\frac{5}{2\epsilon^4} - \frac{45}{4\epsilon^3} - \frac{4255}{144\epsilon^2} - \frac{106147}{1728\epsilon}$$

$$+ \frac{\pi\sqrt{3}}{240}\,(297C - 1477E) - \frac{2320981}{20736} + O(\epsilon)$$

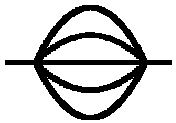
$$C = \frac{4\pi^2}{27}\int_0^1 \frac{dx}{\sqrt{1-x}}\;{}_2F_1^2\left(\begin{matrix}\frac{1}{3} & \frac{2}{3}\\ & 1\end{matrix};x\right)$$

$$= 7.396\;099\;534\;768\;919\;553\;449\;114\;417\;961\;526\;519\;642\ldots$$

$$E = \frac{4\pi^2}{27}\int_0^1 \frac{dx}{\sqrt{1-x}}\;{}_2F_1^2\left(\begin{matrix}\frac{1}{3} & -\frac{1}{3}\\ & 1\end{matrix};x\right)$$

$$= 2.376\;887\;326\;184\;666\;003\;152\;855\;958\;761\;330\;926\;023\ldots$$

The Laporta High-precision numerical algorithm gives



$$= M_{139} = -\frac{5}{2\epsilon^4} - \frac{45}{4\epsilon^3} - \frac{4255}{144\epsilon^2} - \frac{106147}{1728\epsilon} - 141.722156186647686949967 91$$

$$- 521.146545686002504417754 66\epsilon - 3347.993365078288611786534 1\epsilon^2$$

$$- 17951.377477480994493109762 2\epsilon^3 - 101753.816533117318213956038 6\epsilon^4 + \ldots$$