Chiral symmetry, $\pi\pi$ scattering and a_μ

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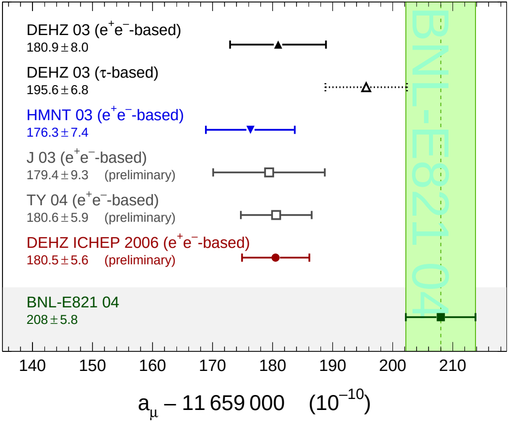
Outline

Status of a_μ



J = Jegerlehner, DEHZ = Davier, Eidelman, Höcker, Zhang

Figure from M. Davier (07)

HMNT = Hagiwara, Martin, Nomura, Teubner, TY = de Tróconiz, Ynduráin

Theory analysis of the two-pion contribution

- About 70% of the hadronic contribution comes from below 1 GeV
- The contributions to $a_\mu^{\rm hvp}$ up to 1 GeV are dominated by the two-pion contribution

$$\sigma(e^+e^- \to \text{hadrons})|_{s\leq 1\text{GeV}} \sim |F_\pi(s)|^2$$

- Analyticity and unitarity relate very strongly the pion form factor and the P-wave $\pi\pi$ phase shift
- The $\pi\pi$ interaction at low energy is known theoretically to high precision

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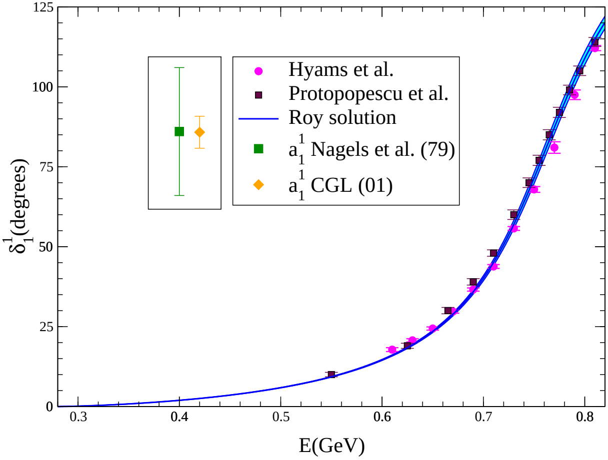
$\Rightarrow$ Use the knowledge on $\delta_{\pi\pi}$ to constrain $F_\pi(s)$

$\pi\pi$ scattering

Using analyticity, unitarity ($\equiv$ Roy eqs.) and chiral symmetry



GC, Gasser and Leutwyler, NPB 01

$\pi\pi$ scattering

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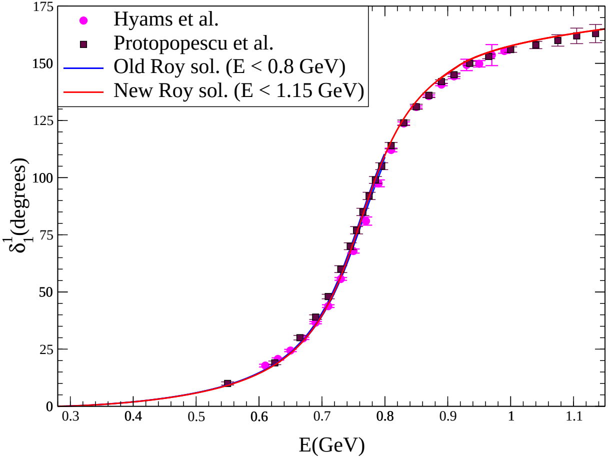

Roy equation for the P wave

$$
\begin{aligned}
t_1^1(s) &= k_1^1(s) + \int_{4M_\pi^2}^{E_0^2} ds' K_{1\,1}^{1\,1}(s,s')\,\mathrm{Im}\, t_1^1(s') \\
&+ \int_{4M_\pi^2}^{E_0^2} ds' K_{1\,0}^{1\,0}(s,s')\,\mathrm{Im}\, t_0^0(s') + \int_{4M_\pi^2}^{E_0^2} ds' K_{1\,0}^{1\,2}(s,s')\,\mathrm{Im}\, t_0^2(s') \\
&+ f_1^1(s) + d_1^1(s)
\end{aligned}
$$

$$
\begin{aligned}
k_1^1(s) &= \frac{s-4M_\pi^2}{36M_\pi^2}(2a_0^0 - 5a_0^2) && E_0 = 1.15 \text{ GeV} \\
f_1^1(s) &= \sum_{I'=0}^{2}\sum_{\ell'=0}^{1}\int_{E_0^2}^{E_1^2} ds' K_{1\ell'}^{1I'}(s,s')\,\mathrm{Im}\, t_{\ell'}^{I'}(s') && E_1 = 1.7 \text{ GeV} \\
d_1^1(s) &= \text{all the rest}
\end{aligned}
$$

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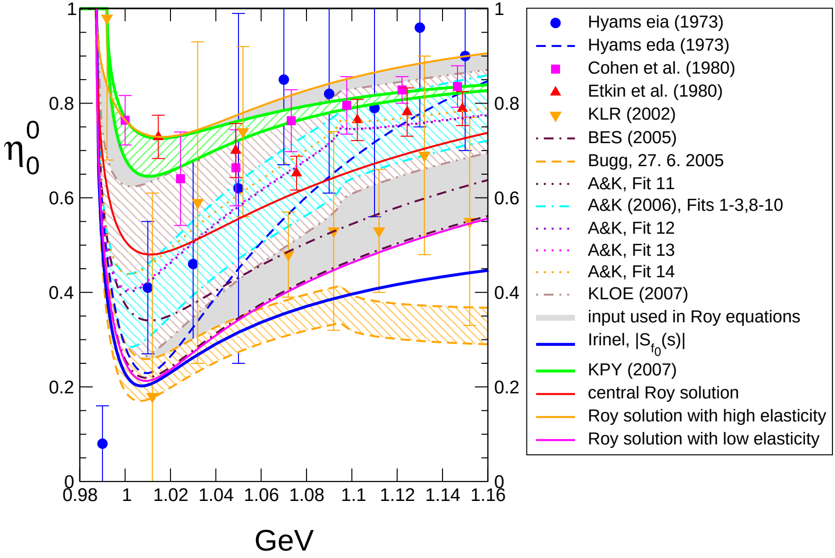
\end{aligned}
$$

$$
t_\ell^I(s) = \frac{1}{2i\sigma(s)}\left\{\eta_\ell^I(s)\,e^{2i\delta_\ell^I(s)} - 1\right\}
$$

η_ℓ^I given $\Leftrightarrow$ Roy eqs. $\equiv$ coupled integral equations for $\delta_\ell^I(s)$
for $4M_\pi^2 < s < E_0^2$

Roy equations: input uncertainties



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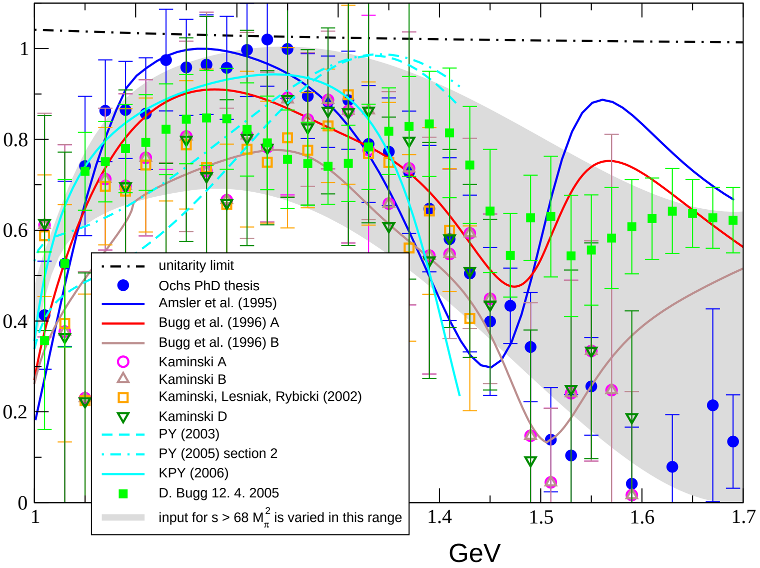

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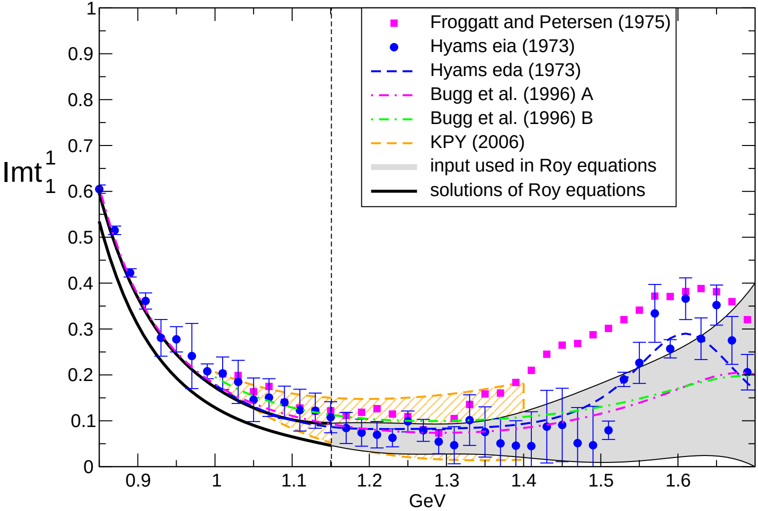

Analyticity and unitarity constraints

- Analyticity $\Rightarrow$ $\delta(s) \Leftrightarrow F_\pi(s)$ (Omnés)

$$F_\pi(s) = \exp\left[\frac{s}{\pi}\int_{4M_\pi^2}^{\infty} ds' \, \frac{\delta(s')}{s'\,(s'-s)}\right] \,, \qquad F_\pi(0) = 1$$

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Representation of $F_\pi(s)$ which automatically satisfies unitarity, analyticity and chiral symmetry

[Heyn and Lang 81, de Trocóniz and Ynduráin 02]

[Caprini, GC, Leutwyler and Smith work in progr.]

Our form factor representation

- Omnés representation (57)

$$F_V(s) = \exp\left[\frac{s}{\pi}\int_{4M_\pi^2}^{\infty} ds' \frac{\delta(s')}{s'(s'-s)}\right] \equiv \Omega(s)$$

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- Split elastic from inelastic contributions

$$\delta = \delta_{\pi\pi} + \delta_{\text{in}} \quad \Rightarrow \quad F_V(s) = \Omega_{\pi\pi}(s)\Omega_{\text{in}}(s)$$

Eidelman-Lukaszuk: unitarity bound on δ_{in}

$$\sin^2\delta_{\text{in}} \leq \frac{1}{2}\left(1-\sqrt{1-r^2}\right) \qquad r = \frac{\sigma^{I=1}_{e^+e^-\to\neq 2\pi}}{\sigma_{e^+e^-\to 2\pi}}$$

$$\Rightarrow \quad \text{Im}\Omega_{\text{in}}(s) \simeq 0 \qquad s \leq (M_\pi + M_\omega)^2$$

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- $\rho-\omega$–mixing must also be explicitly taken into account

$$F_V(s) = \Omega_{\pi\pi}(s)\Omega_{\rm in}(s)G_\omega(s)$$

Free parameters

$$\Omega_{\pi\pi}(s) \quad \Rightarrow \quad \begin{cases} \phi_0 = \delta_{\pi\pi}((0.8\ \mathrm{GeV})^2) \\ \phi_1 = \delta_{\pi\pi}(68 M_\pi^2) \end{cases}$$

$$G_\omega(s) \quad \Rightarrow \quad \begin{cases} \epsilon & \omega - \rho \text{ mixing} \\ M_\omega & \end{cases}$$

$$\Omega_{\mathrm{in}}(s) \quad \Rightarrow \quad \begin{cases} c_1 \\ \vdots \\ c_P \end{cases} \qquad \mathrm{Im} G_2(s) = 0 \quad s \leq s_{\mathrm{in}}$$

$$G_\omega(s) = 1 + \epsilon \frac{s}{s_\omega - s} \qquad \text{where} \qquad s_\omega = (M_\omega - i\Gamma_\omega/2)^2$$

$$\Omega_{\mathrm{in}}(s) = 1 + \sum_{k=1}^{n} c_k (z(s)^k - z(0)^k) \qquad z = \frac{\sqrt{s_{\pi\omega} - s_1} - \sqrt{s_{\pi\omega} - s}}{\sqrt{s_{\pi\omega} - s_1} + \sqrt{s_{\pi\omega} - s}}$$

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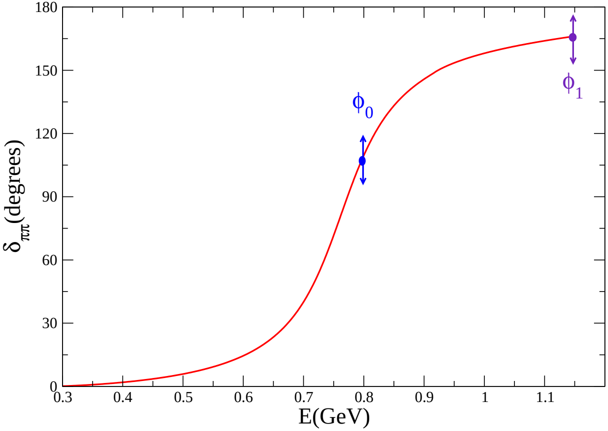
Output:

$$a_{\rho,2M_K} = 10^{10}\left(\frac{\alpha m_\mu}{3\pi}\right)^2 \int_{4M_\pi^2}^{s_\rho,4M_K^2} ds \frac{\hat{K}(s)R(s)}{s^2} \qquad s_\rho = (0.81\mathrm{GeV})^2$$

Free parameters



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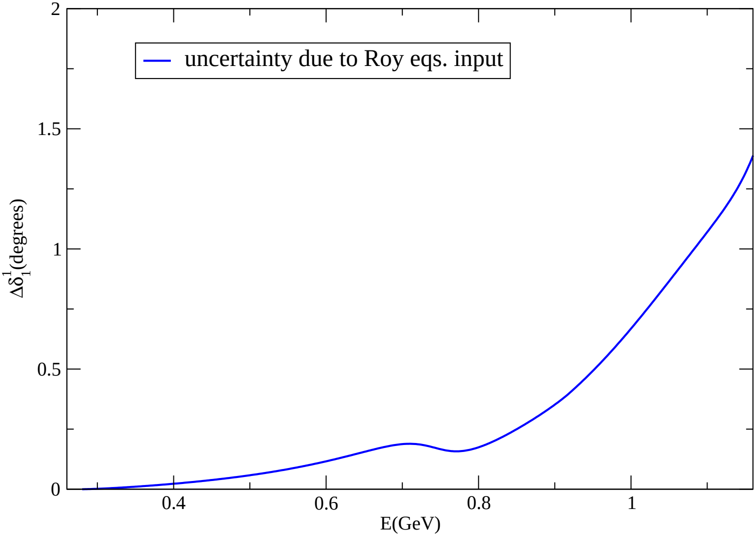
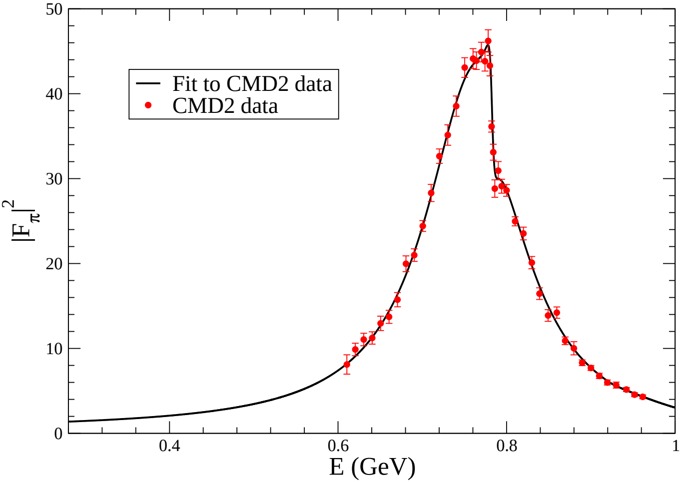


Systematic uncertainty in δ_1^1 due to the Roy eqs. input

Outcome of the fit



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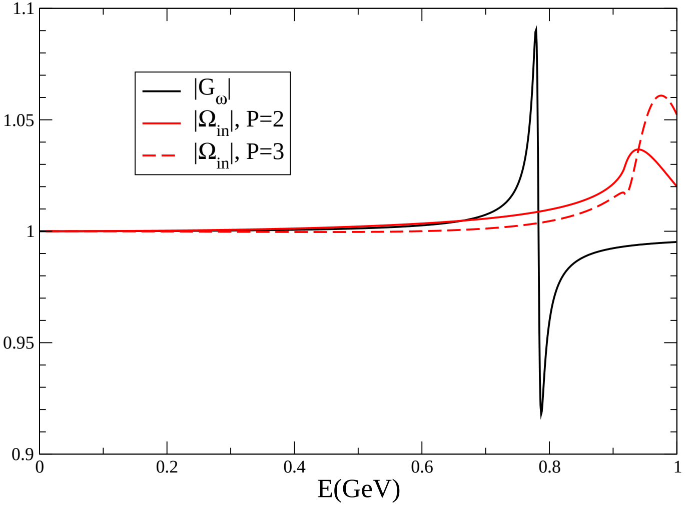

P = number of parameters in Ω_{in}

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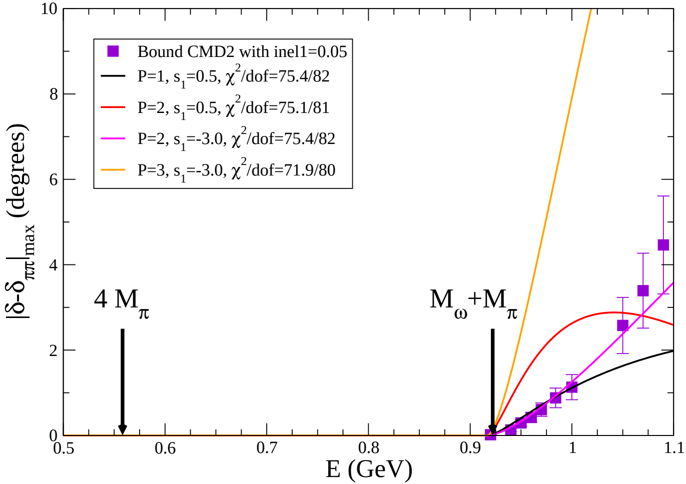

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Fits to different data sets

Data	χ^2/dof	ϕ_0	ϕ_1	$10^{10}a_\rho$	$10^{10}a_{2M_K}$
CMD2(04)	76/82	110.0(7)	167.6(6)	422.2(2.8)	492.4(3.0)
CMD2(06)	82/76	109.0(5)	166.4(4)	421.5(2.3)	491.7(2.4)
CMD2(06-hi)	65/67	108.9(5)	166.2(4)	423.0(2.4)	493.2(2.5)
SND	70/84	108.9(5)	166.9(4)	421.1(2.0)	493.2(2.1)
KLOE	70/98	108.3(4)	168.4(3)	428.2(1.6)	494.9(1.4)

The extrapolation down to threshold does not imply an increase in the relative uncertainty, e.g. for CMD2(04):

$$a_\mu^{\text{hvp}}{}_{(0.6\text{GeV}\leq\sqrt{s}\leq 2M_K)} = (385.2\pm 2.3)\cdot 10^{-10} = 385.2\cdot 10^{-10}(1\pm 0.006)$$

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The 10 data points close to threshold by CMD2(06) cannot be fitted very well – the parametrization is too stiff to adapt to them

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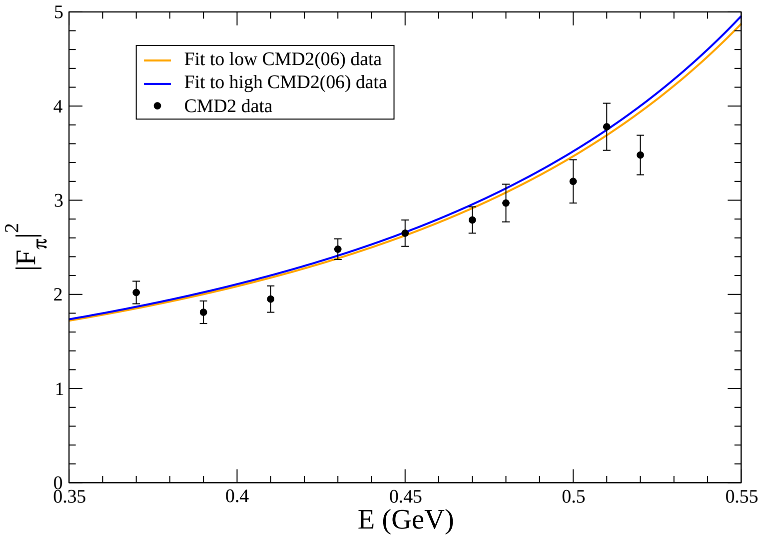

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CMD2 and SND agree well. There is some tension with KLOE as far as the shape is concerned:

$$(\phi_1 - \phi_0)_{\text{without KLOE}} = 57.7 \pm 0.7 \qquad (\phi_1 - \phi_0)_{\text{KLOE}} = 60.1 \pm 0.6$$

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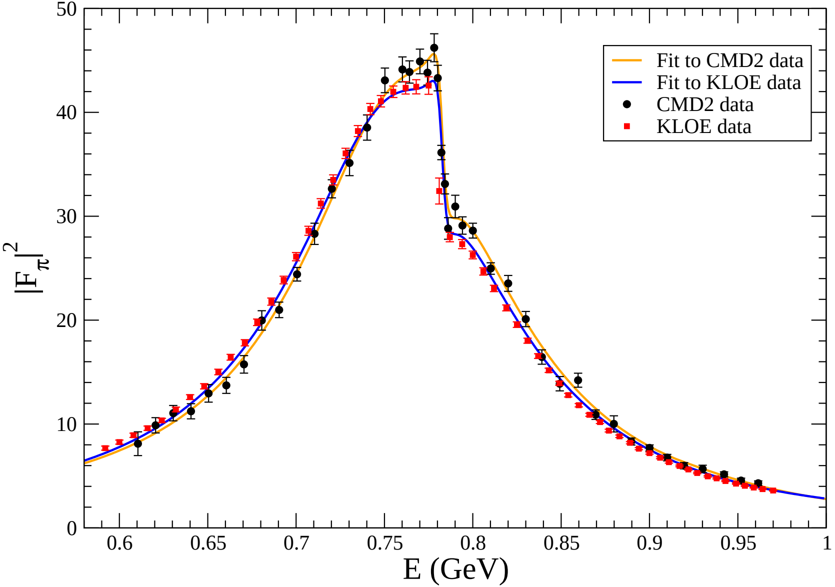
The different determinations of a_{2M_K} can be averaged well:

$$a_{2M_K} = 493.7 \pm 1.0$$

Stat. and syst. errors have been added in squares for all data sets.
Numbers here are preliminary and mainly for illustration purposes.
Final error here too small to be taken at face value – more conservative procedure should be adopted (e.g. adding syst. errors linearly).

Comparison CMD2(04)–KLOE



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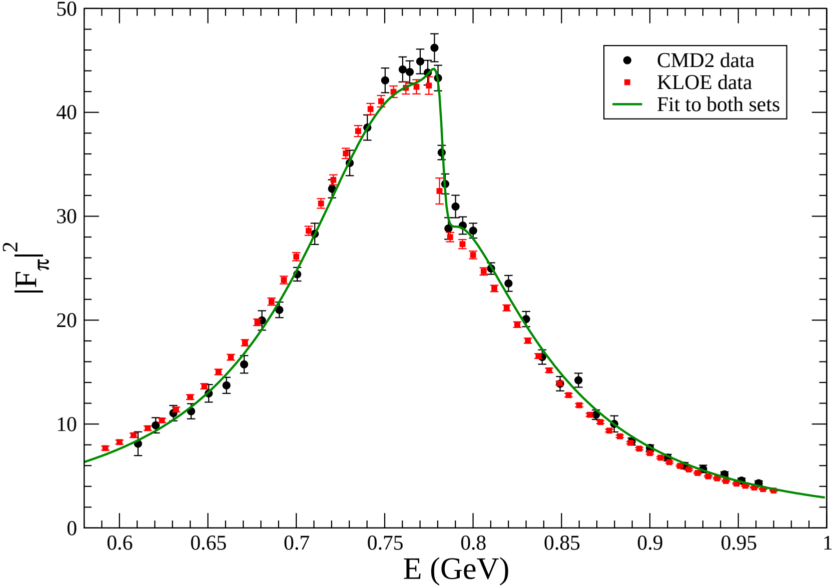

Summary and outlook

- the extension of the Roy equation analysis of $\pi\pi$ scattering has been completed and is ready to be used for the analysis of pion em form factor
- fully exploiting analyticity, unitarity and χ-symmetry is essential in the evaluation of the integral $a_\mu^{\rm hvp}$ by providing a controlled framework in which to analyze the data below 1 GeV and make the extrapolation down to threshold
- I have presented preliminary numbers for $a_\mu^{\rm hvp}$ below 1 GeV obtained by analyzing different data sets
- quite a coherent picture emerges from the different e^+e^- data sets – $a_\mu^{\rm hvp}$ can be determined to better than 1% but there are still some tensions in the data that one would like to understand
- the analysis is in progress and will be completed in the next few months

work in progress, Caprini, GC, Leutwyler and Smith