★ NuFact '05

Large Non-perturbative Effects of Small $\Delta m^2_{21}/\Delta m^2_{31}$ and $\sin\theta_{13}$

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Main Results

1. Non-perturbative effects of small $\Delta m^2_{21}/\Delta m^2_{31}$ and $\sin\theta_{13}$ become more than 50% at the MSW resonance point.
2. We propose a method to include these effects in approximate formulas for oscillation probabilities.

Two Small Parameters

- Atmospheric Neutrino

$$\theta_{23} \simeq \frac{\pi}{4}, \qquad |\Delta m^2_{31}| \simeq 2.0 \times 10^{-3} \text{eV}^2$$

- Solar Neutrino

$$\theta_{12} \simeq \frac{\pi}{6}, \qquad \Delta m^2_{21} \simeq 8.3 \times 10^{-5} \text{eV}^2$$

- CHOOZ

$$\theta_{13} \lesssim 0.23$$

★ Two small parameters

1. $\alpha = \Delta m^2_{21}/\Delta m^2_{31} \simeq 0.03$
2. $s_{13} = \sin\theta_{13} \lesssim 0.23$

★ Double suppression of CP violating effects

$$P(\nu_e \to \nu_\mu) = A\cos\delta + B\sin\delta + C$$

$$\lim_{\alpha\to 0} A = 0 \qquad \lim_{s_{13}\to 0} A = 0$$

$$\lim_{\alpha\to 0} B = 0 \qquad \lim_{s_{13}\to 0} B = 0$$

$$A = O(\alpha s_{13}) \qquad B = O(\alpha s_{13})$$

Remark

It is important to measure CP violation we take into account corrections due to small parameters.

Second Order Perturbation

★ Second order formula for $P(\nu_e \to \nu_\mu)$

$$A \simeq 2\alpha s_{13} \sin 2\theta_{12} \sin 2\theta_{23} \cos\Delta \frac{\sin \bar{a}}{\bar{a}} \frac{\sin(\bar{a}-1)\Delta}{\bar{a}-1}$$

$$B \simeq 2\alpha s_{13} \sin 2\theta_{12} \sin 2\theta_{23} \sin\Delta \frac{\sin \bar{a}}{\bar{a}} \frac{\sin(\bar{a}-1)\Delta}{\bar{a}-1}$$

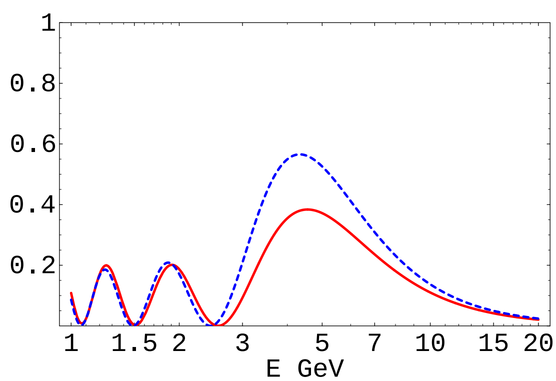
$$C \simeq \alpha^2 \sin^2 2\theta_{12} s_{23}^2 \frac{\sin^2 \bar{a}\Delta}{\bar{a}^2} + 4 s_{13}^2 c_{23}^2 \frac{\sin^2(\bar{a}-1)\Delta}{(\bar{a}-1)^2}$$

$$\Delta = \frac{\Delta m_{31}^2 L}{4E}, \qquad \bar{a} = \frac{2Ea}{\Delta m_{31}^2}$$

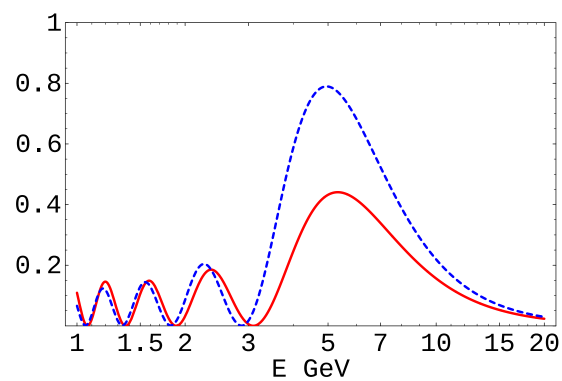
★ Perturbation vs exact calculation

Reference values: $\delta = 90^\circ$, $\theta_{13} = 0.23$

$L = 4500$ km



$L = 6000$ km



dotted line=perturbation, solid line=exact calculation

Remark

Perturbative calculation is significantly different from numerical calculation MSW resonance region.

Comparison in Two Generations

★ Hamiltonian in two generations

$$H = O\,\mathrm{diag}(0, \Delta) O^T + \mathrm{diag}(a, 0)$$

★ Second order perturbation due to small mixing

$$P = \frac{4\Delta^2 \sin^2\theta}{(\Delta - a)^2} \sin^2 \frac{(\Delta - a)L}{2} = O(\sin^2\theta)$$

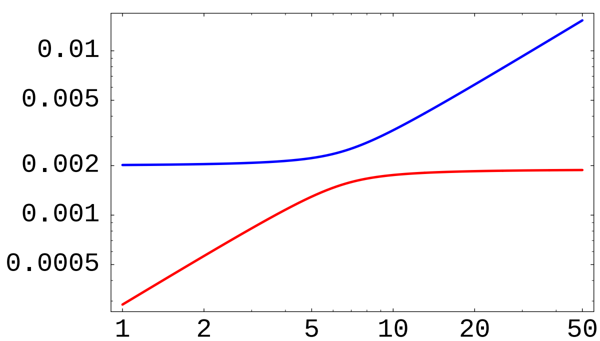
★ Exact calculation

$$P = \frac{\Delta^2}{\tilde{\Delta}^2} \sin^2 \frac{\tilde{\Delta} L}{2} = O(\sin^2\theta) + O(\sin^4\theta) + \cdots$$

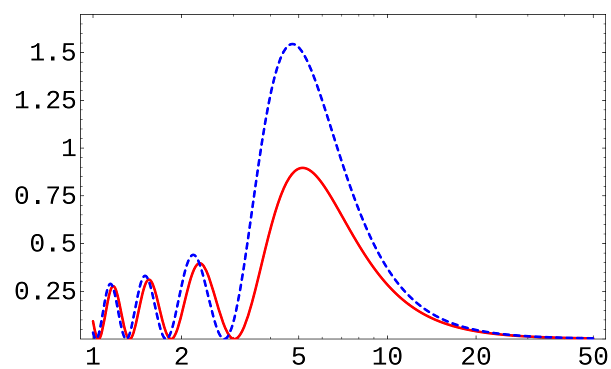
★ Perturbation vs exact calculation

$$\Delta m^2 = 2.0 \times 10^{-3} \mathrm{eV}^2, \ \sin\theta = 0.23$$

Level Crossing



Probability



dotted line=perturbation, solid line=exact

Two Energy Regions

★ Standard Parametrization

$$U = O_{23}\Gamma_\delta O_{13}\Gamma_\delta^\dagger O_{12}$$

★ General relation

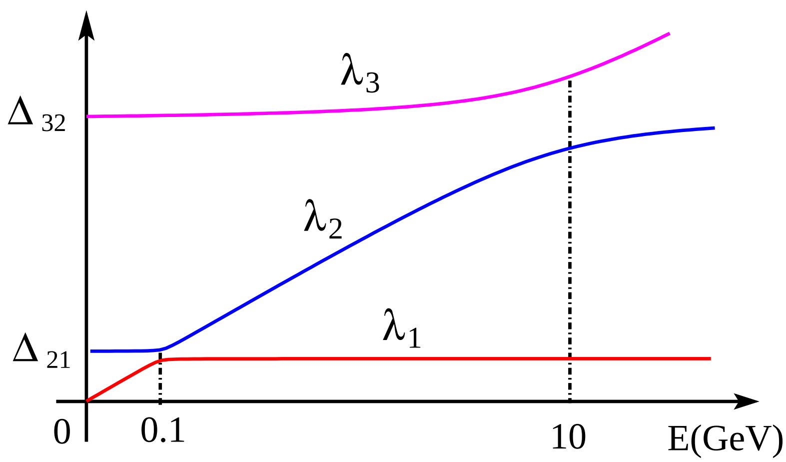
$$H(t) = [O_{23}\Gamma_\delta] H'(t) [O_{23}\Gamma_\delta]^\dagger$$

$$\begin{aligned} H' &= O_{13}O_{12}\mathrm{diag}(0, \Delta_{21}, \Delta_{31}) O_{12}^T O_{13}^T \\ &+ \mathrm{diag}(a(t), 0, 0) \end{aligned}$$

★ Two energy regions

$(\theta_{12}, \Delta m_{21}^2)$: Low Energy Phenomenon

$(\theta_{13}, \Delta m_{31}^2)$: High Energy Phenomenon



Remark

Parameters α and s_{13} control low and high energy MSW resonance, respectively.

New Proposal

★ Double expansion

$$S' = \begin{array}{|cccccc|} O(1) & + & O(\alpha) & + & O(\alpha^2) & \cdots \\ O(s_{13}) & + & O(\alpha s_{13}) & + & O(\alpha^2 s_{13}) & \cdots \\ O(s_{13}^2) & + & O(\alpha s_{13}^2) & + & O(\alpha^2 s_{13}^2) & \cdots \\ O(s_{13}^3) & + & O(\alpha s_{13}^3) & + & O(\alpha^2 s_{13}^3) & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{array}$$

★ Second order perturbation

$$S' = \begin{array}{ccccc} O(1) & + & O(\alpha) & + & O(\alpha^2) \\ O(s_{13}) & + & O(\alpha s_{13}) & & \\ O(s_{13}^2) & & & & \end{array}$$

★ Our method

$$S' = \begin{array}{cccccccc} O(1) & + & O(\alpha) & + & O(\alpha^2) & + & O(\alpha^3) & \cdots \\ O(s_{13}) & & & & & & & \\ O(s_{13}^2) & & & & & & & \\ O(s_{13}^3) & & & & & & & \\ O(s_{13}^4) & & & & & & & \\ \cdots & & & & & & & \end{array}$$

Remark

Approximate formula in our method contains the higher order terms of small α and s_{13}.

Two Generation Approximations

★ Double expansion

$$S' = \boxed{\begin{array}{cccccc} O(1) & + & O(\alpha) & + & O(\alpha^2) & \cdots \\ O(s_{13}) & + & O(\alpha s_{13}) & + & O(\alpha^2 s_{13}) & \cdots \\ O(s_{13}^2) & + & O(\alpha s_{13}^2) & + & O(\alpha^2 s_{13}^2) & \cdots \\ O(s_{13}^3) & + & O(\alpha s_{13}^3) & + & O(\alpha^2 s_{13}^3) & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{array}}$$

$$= \lim_{s_{13}\to 0} S' + \lim_{\alpha\to 0} S' - \lim_{s_{13},\alpha\to 0} S'$$

$$+ \; O(\alpha s_{13}) + O(\alpha^2 s_{13}) + O(\alpha s_{13}^2) + \cdots$$

★ Our method

$$S' \sim \lim_{s_{13}\to 0} S' + \lim_{\alpha\to 0} S' - \lim_{s_{13},\alpha\to 0} S'$$

Where

$$\lim_{s_{13}\to 0} S' = \mathrm{T}\exp\left(-i\int \lim_{s_{13}\to 0} H' dt\right)$$

$$\lim_{\alpha\to 0} S' = \mathrm{T}\exp\left(-i\int \lim_{\alpha\to 0} H' dt\right)$$

$$\lim_{s_{13},\alpha\to 0} S' = \mathrm{T}\exp\left(-i\int \lim_{s_{13},\alpha\to 0} H' dt\right)$$

Remark

We need to calculate Hamiltonian in the vanishing limit of small parameters.

Hamiltonian in Two Generations

★ Hamiltonian

$$\begin{aligned} H' &= O_{13} O_{12} \mathrm{diag}(0, \Delta_{21}, \Delta_{31}) O_{12}^T O_{13}^T \\ &+ \mathrm{diag}(a(t), 0, 0) \end{aligned}$$

1. Vanishing limit of s_{13}

$$\begin{aligned} H^\ell &= \lim_{s_{13}\to 0} H' \\ &= \begin{pmatrix} \Delta_{21} s_{12}^2 + a(t) & \Delta_{21} s_{12} c_{12} & 0 \\ \Delta_{21} s_{12} c_{12} & \Delta_{21} c_{12}^2 & 0 \\ 0 & 0 & \Delta_{31} \end{pmatrix} \end{aligned}$$

2. Vanishing limit of α

$$\begin{aligned} H^h &= \lim_{\alpha\to 0} H' \\ &= \begin{pmatrix} \Delta_{31} s_{13}^2 + a(t) & 0 & \Delta_{31} s_{13} c_{13} \\ 0 & 0 & 0 \\ \Delta_{31} s_{13} c_{13} & 0 & \Delta_{31} c_{13}^2 \end{pmatrix} \end{aligned}$$

3. Vanishing limit of s_{13} and α

$$\begin{aligned} H^d &= \lim_{s_{13},\alpha\to 0} H' \\ &= \begin{pmatrix} \Delta_{31} s_{13}^2 + a(t) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \Delta_{31} c_{13}^2 \end{pmatrix} \end{aligned}$$

Main Results

★ Zeros of Amplitudes

$$S^\ell = \begin{pmatrix} S^\ell_{ee} & S^\ell_{e\mu} & 0 \\ S^\ell_{\mu e} & S^\ell_{\mu\mu} & 0 \\ 0 & 0 & S^\ell_{\tau\tau} \end{pmatrix} \qquad S^h = \begin{pmatrix} S^h_{ee} & 0 & S^h_{e\tau} \\ 0 & S^h_{\mu\mu} & 0 \\ S^h_{\tau e} & 0 & S^h_{\tau\tau} \end{pmatrix}$$

$$S^d = \begin{pmatrix} S^d_{ee} & 0 & 0 \\ 0 & S^d_{\mu\mu} & 0 \\ 0 & 0 & S^d_{\tau\tau} \end{pmatrix} \qquad (S^h_{\mu\mu} = S^d_{\mu\mu},\ S^\ell_{\tau\tau} = S^d_{\tau\tau})$$

★ Our method

$$S' \sim S^\ell + S^h - S^d$$

★ Main Results

$$\begin{aligned} S'_{\mu e} &\sim S^\ell_{\mu e} \\ S'_{\tau e} &\sim S^h_{\tau e} \\ S'_{\tau\mu} &\sim 0 \\ S'_{ee} &\sim S^\ell_{ee} + S^h_{ee} - S^d_{ee} \\ S'_{\mu\mu} &\sim S^\ell_{\mu\mu} \\ S'_{\tau\tau} &\sim S^h_{\tau\tau} \end{aligned}$$

Remark

We obtain very simple results except for S'_{ee}.

Large Non-perturbative Effects

★ Simple CP dependence

$$S(t) = [O_{23}\Gamma_\delta] S'(t) [O_{23}\Gamma_\delta]^\dagger$$
$$P(\nu_e \to \nu_\mu) = |S(t)_{\mu e}|^2 = A\cos\delta + B\sin\delta + C$$

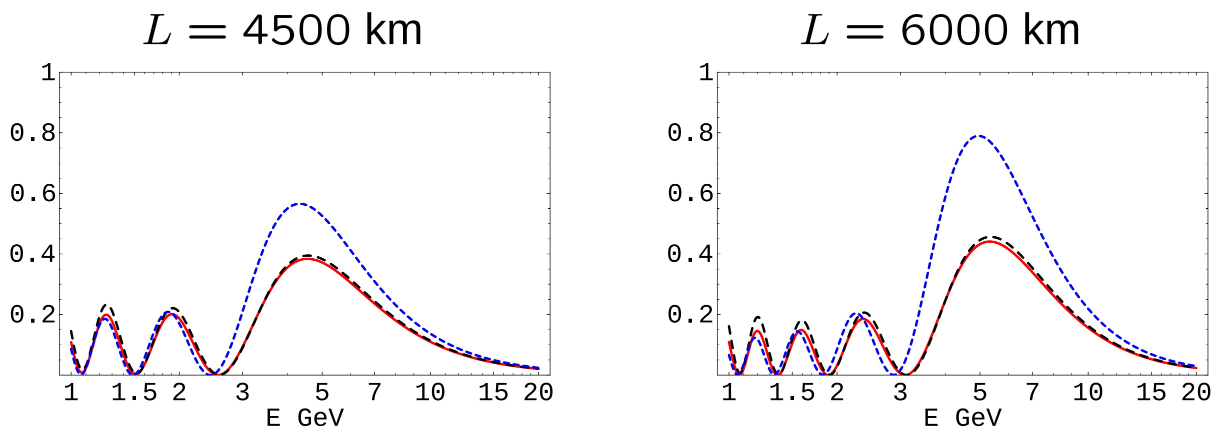
★ Our formula for $P(\nu_e \to \nu_\mu)$

$$A \simeq \sin 2\theta_{12}^\ell \sin 2\theta_{23} \sin 2\theta_{13}^h \sin\frac{\Delta_{21}^\ell L}{2} \sin\frac{\Delta_{31}^h L}{2} \cos\frac{\Delta_{32} L}{2}$$
$$B \simeq \sin 2\theta_{12}^\ell \sin 2\theta_{23} \sin 2\theta_{13}^h \sin\frac{\Delta_{21}^\ell L}{2} \sin\frac{\Delta_{31}^h L}{2} \sin\frac{\Delta_{32} L}{2}$$
$$C \simeq c_{23}^2 \sin^2 2\theta_{12}^\ell \sin^2\frac{\Delta_{21}^\ell L}{2} + s_{23}^2 \sin^2 2\theta_{13}^h \sin^2\frac{\Delta_{31}^h L}{2}$$

$$\sin 2\theta_{13}^h = O(s_{13}) + O(s_{13}^2) + O(s_{13}^3) + \cdots$$
$$\sin\frac{\Delta_{21}^\ell L}{2} = O(1) + O(\alpha) + O(\alpha^2) + O(\alpha^3) + \cdots$$

★ Our method vs perturbation

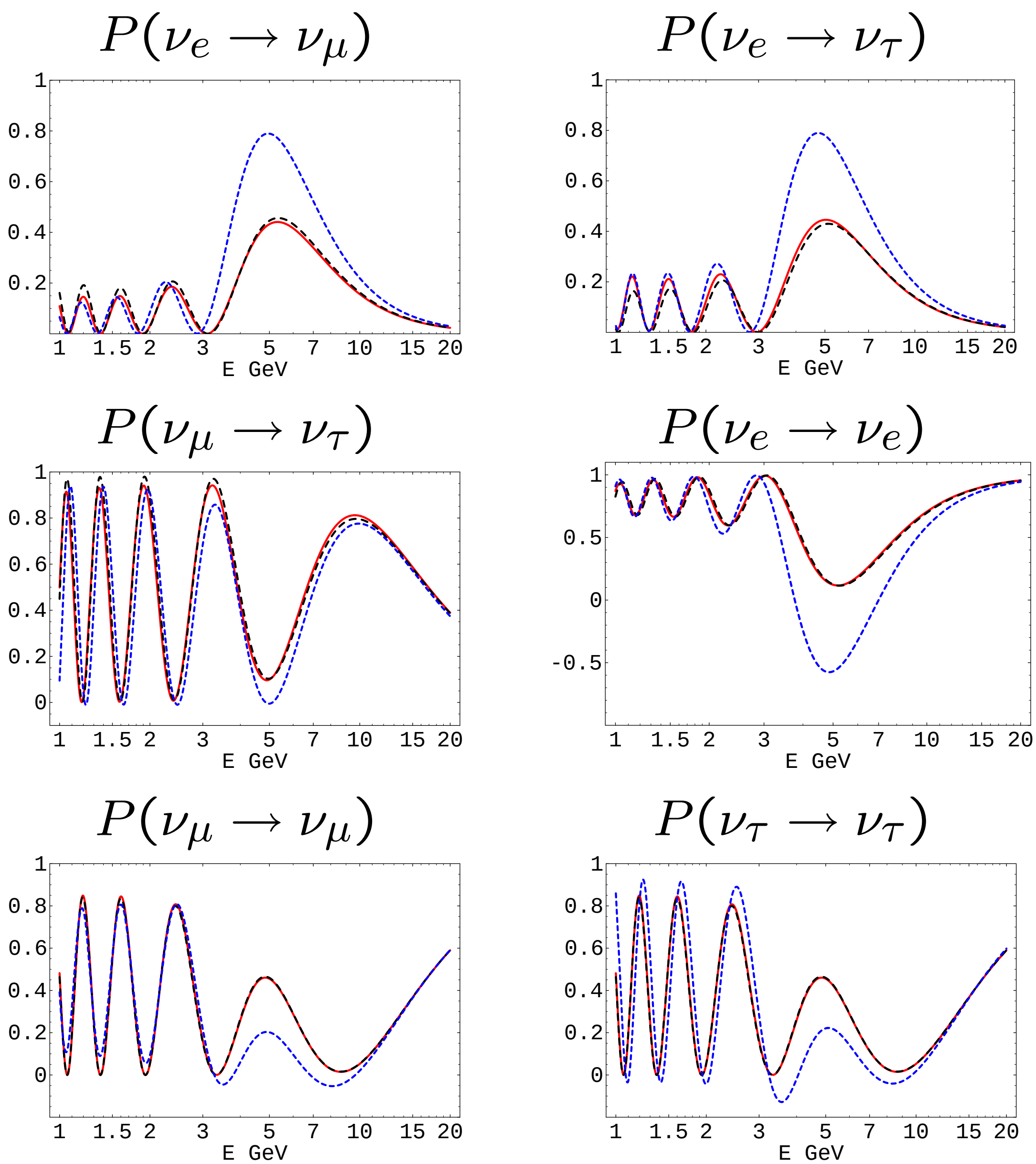
Reference values: $\delta = 90^\circ,\ \theta_{13} = 0.23$



dotted line=perturbation, line=ours, dashed line=numerical

Comparison in All Channels

$$L = 6000\ \text{km}, \quad \delta = 90^\circ, \quad \theta_{13} = 0.23$$



dotted line=perturbation, line=ours, dashed line=numerical

Remark

Our formulas are a good agreement with numerical calculation.

Summary

We have studied non-perturbative effects due to small parameters $\Delta m^2_{21}/\Delta m^2_{31}$ and $\sin\theta_{13}$.

★ Second order perturbation

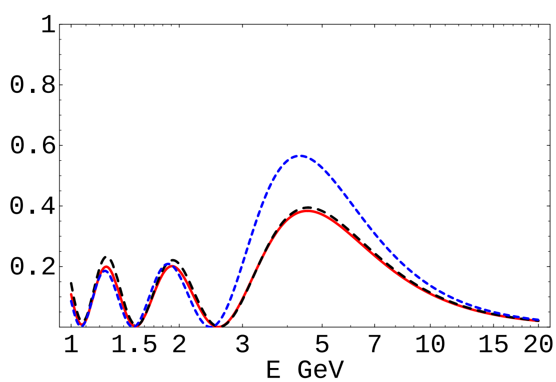
$$S' = \begin{array}{|lll} O(1) & + \; O(\alpha) & + \; O(\alpha^2) \\ O(s_{13}) & + \; O(\alpha s_{13}) & \\ O(s_{13}^2) & & \end{array}$$

★ Our method

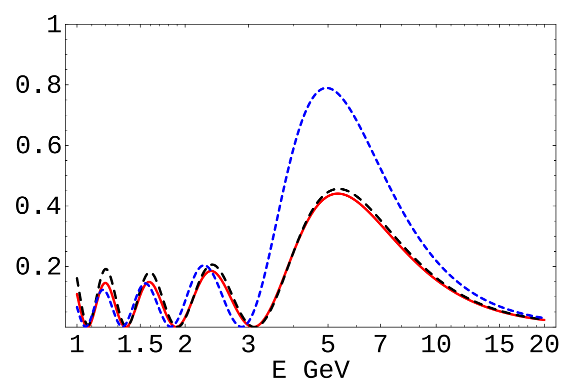
$$S' = \begin{array}{|lllll} O(1) & + \; O(\alpha) & + \; O(\alpha^2) & + \; O(\alpha^3) & \cdots \\ O(s_{13}) & & & & \\ O(s_{13}^2) & & & & \\ O(s_{13}^3) & & & & \\ O(s_{13}^4) & & & & \\ \cdots & & & & \end{array}$$

★ Our method vs perturbation

$L = 4500$ km



$L = 6000$ km



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