Generalised parton distributions from full lattice QCD

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long term goal:
structure of the nucleon in terms of its constituents
(quarks and gluons)

information on the internal structure of the nucleon
from, e.g., lepton-nucleon scattering experiments

electromagnetic form factors (FFs)

↘

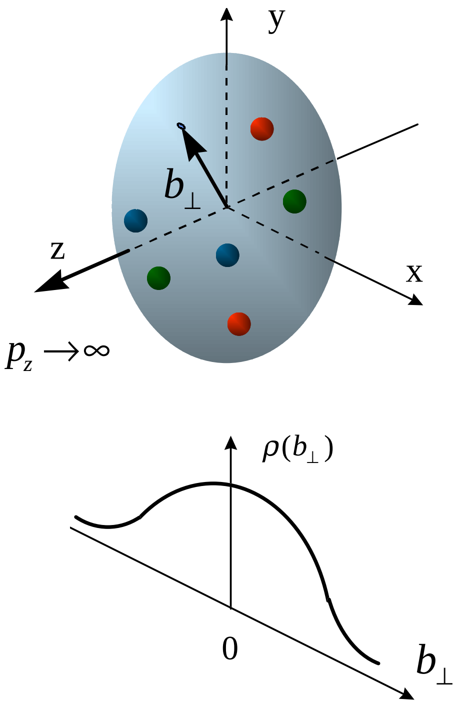
generalised parton distributions (GPDs)

↗ ↑

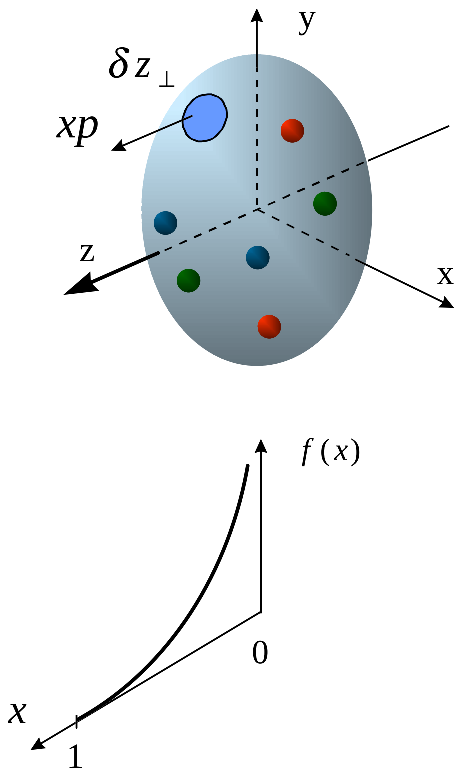
structure functions (parton densities)   lattice calculations required!

further results from other groups: LHPC, SESAM, . . .
no detailed comparison attempted

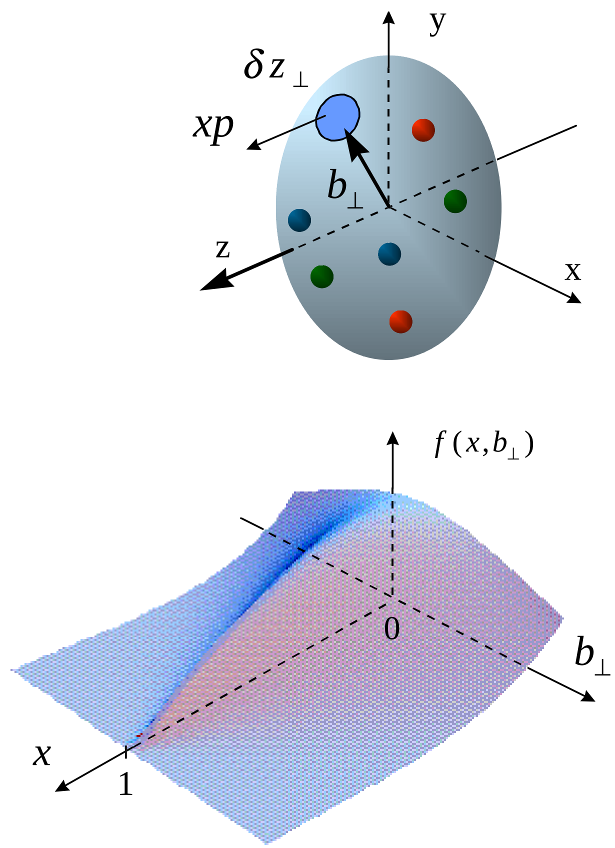
form factor



parton density



GPD at $\xi = 0$

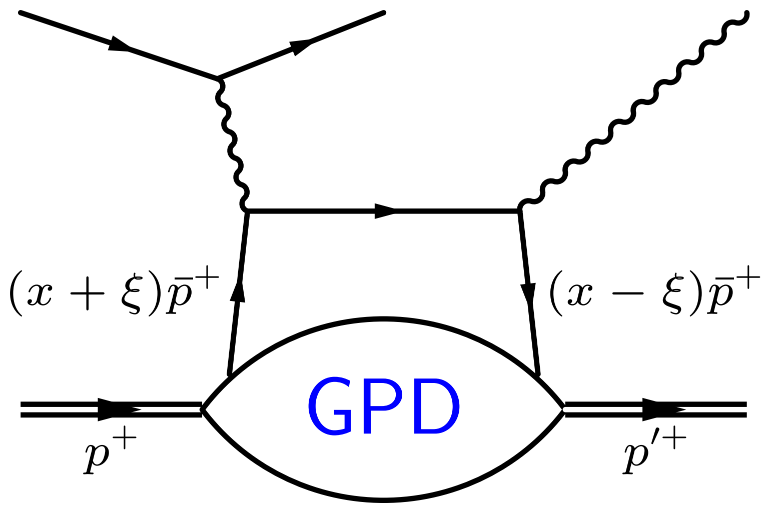


pictures by Dieter Müller

probabilistic interpretation in impact parameter space (M. Burkardt)

expect: $f(x, \vec{b}_\perp) \stackrel{x \to 1}{\longrightarrow} \delta(\vec{b}_\perp)$

Formal definition of GPDs



Wilson line
↓

$$\int \frac{\mathrm{d}\lambda}{2\pi} \mathrm{e}^{\mathrm{i}\lambda x} \langle p' | \bar{q}(-\tfrac{1}{2}\lambda n) \not{n}\, \mathcal{U} q(\tfrac{1}{2}\lambda n) | p \rangle$$
$$= H_q(x,\xi,t)\bar{u}(p')\not{n} u(p) + E_q(x,\xi,t)\bar{u}(p')\frac{\mathrm{i}\sigma^{\mu\nu} n_\mu \Delta_\nu}{2M_N} u(p)$$

dependence on renormalisation scale suppressed

$\bar{p} = \frac{1}{2}(p'+p)$, $\Delta = p' - p$, n: light-like vector with $\bar{p}\cdot n = 1$

GPDs depend on x, $\xi = -n\cdot\Delta/2$, $t = \Delta^2$

normal parton distributions: $\Delta = 0$ (forward matrix elements) $\Rightarrow \xi = 0$

for a lattice calculation: matrix elements of local operators

$$\langle p'|\mathcal{O}^q_{\mu_1\cdots\mu_n}|p\rangle = \bar{u}(p')\gamma_{(\mu_1}u(p)\sum_{i=0}^{\left[\frac{n-1}{2}\right]} A^q_{n,2i}(t)\Delta_{\mu_2}\cdots\Delta_{\mu_{2i+1}}\overline{p}_{\mu_{2i+2}}\cdots\overline{p}_{\mu_n)}$$
$$-\frac{\bar{u}(p')\mathrm{i}\Delta^\alpha\sigma_{\alpha(\mu_1}u(p)}{2M_N}\sum_{i=0}^{\left[\frac{n-1}{2}\right]} B^q_{n,2i}(t)\Delta_{\mu_2}\cdots\Delta_{\mu_{2i+1}}\overline{p}_{\mu_{2i+2}}\cdots\overline{p}_{\mu_n)}$$
$$+ C^q_n(t)\mathrm{Mod}(n+1,2)\frac{1}{M_N}\bar{u}(p')u(p)\Delta_{(\mu_1}\cdots\Delta_{\mu_n)}$$

$(\cdots)$: symmetrisation and subtraction of trace terms

twist-2 operators: $\mathcal{O}^q_{\mu_1\cdots\mu_n} = \left(\frac{\mathrm{i}}{2}\right)^{n-1}\bar{q}\gamma_{(\mu_1}\overset{\leftrightarrow}{D}_{\mu_2}\cdots\overset{\leftrightarrow}{D}_{\mu_n)}q$ with $\overset{\leftrightarrow}{D} = \vec{D} - \overset{\leftarrow}{D}$

A, B, C: generalised form factors (GFFs) $\leftrightarrow$ moments of GPDs

$n=1$: $\langle p'|\bar{q}\gamma_\mu q|p\rangle = \bar{u}(p')\gamma_\mu u(p)A^q_{1,0}(t) - \frac{\bar{u}(p')\mathrm{i}\Delta^\alpha\sigma_{\alpha\mu}u(p)}{2M_N}B^q_{1,0}(t)$

hence: $A^q_{1,0}(t) = F^q_1(t)$, $B^q_{1,0}(t) = F^q_2(t)$ contribution of flavour q

particularly convenient for a lattice calculation: flavour non-singlet (isovector) operators $(u-d)$

Calculating nucleon matrix elements on the lattice

interpolating field for the proton (neutron by interchanging u and d)

$$B_\alpha(t,\vec{p}) = \sum_{x,\, x_4=t} \mathrm{e}^{-\mathrm{i}\vec{p}\cdot\vec{x}} \epsilon_{ijk} u^i_\alpha(x) u^j_\beta(x) (C^{-1}\gamma_5)_{\beta\gamma} d^k_\gamma(x)$$
$$\bar{B}_\alpha(t,\vec{p}) = \sum_{x,\, x_4=t} \mathrm{e}^{\mathrm{i}\vec{p}\cdot\vec{x}} \epsilon_{ijk} \bar{d}^i_\beta(x) (\gamma_5 C)_{\beta\gamma} \bar{u}^j_\gamma(x) \bar{u}^k_\alpha(x)$$

$+$ smearing

$$C\gamma_\mu^{\mathrm{T}} C^{-1} = -\gamma_\mu$$

On a lattice with time extent T we have for the nucleon 2-point function:

$$\langle B(t)\bar{B}(0)\rangle \overset{T\to\infty}{=} \langle 0|B\mathrm{e}^{-Ht}\bar{B}|0\rangle \overset{t\to\infty}{=} \langle 0|B|N\rangle \mathrm{e}^{-E_N t}\langle N|\bar{B}|0\rangle + \cdots$$

momentum 0: $E_N = M_N$

3-point function with $t > \tau > 0$:

$$\begin{aligned} \langle B(t)\mathcal{O}(\tau)\bar{B}(0)\rangle &\overset{T\to\infty}{=} \langle 0|B\mathrm{e}^{-H(t-\tau)}\mathcal{O}\mathrm{e}^{-H\tau}\bar{B}|0\rangle \\ &= \langle 0|B|N\rangle \mathrm{e}^{-E_N(t-\tau)}\langle N|\mathcal{O}|N\rangle \mathrm{e}^{-E_N\tau}\langle N|\bar{B}|0\rangle + \cdots \\ &= \langle 0|B|N\rangle \mathrm{e}^{-E_N t}\langle N|\bar{B}|0\rangle \langle N|\mathcal{O}|N\rangle + \cdots \end{aligned}$$

ratio (for forward matrix elements) $R \equiv \dfrac{\langle B(t)\mathcal{O}(\tau)\bar{B}(0)\rangle}{\langle B(t)\bar{B}(0)\rangle} = \langle N|\mathcal{O}|N\rangle + \cdots$

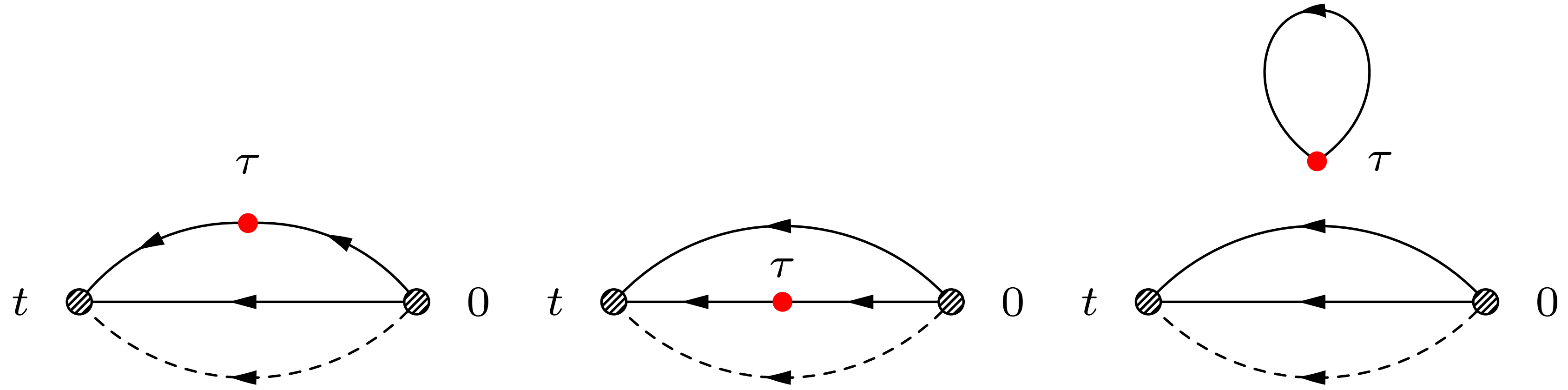
(with an appropriate "projection" on the spinor indices)

should be independent of τ, t for $0 \ll \tau \ll t$ i.e. if excited states can be neglected.

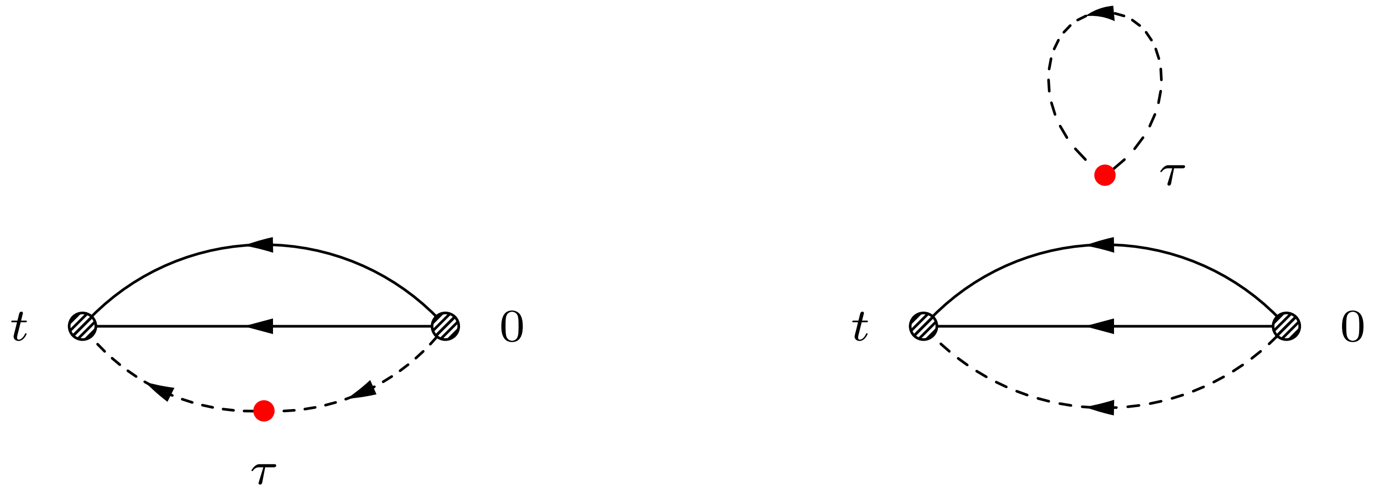
proton 3-point function of a local operator $\bar{\psi} \cdots \psi$ schematically:

$$\langle \psi(t)\psi(t)\psi(t)\bar{\psi}(\tau)\psi(\tau)\bar{\psi}(0)\bar{\psi}(0)\bar{\psi}(0)\rangle$$

$\bar{u} \cdots u$



$\bar{d} \cdots d$



(quark-line) disconnected contribution drops out in non-singlet quantities (for exact isospin invariance)

The simulations

- non-perturbatively $O(a)$-improved Wilson (clover) fermions
- $N_f = 2$ dynamical configurations
- 3 sink momenta $\vec{p}\,'$: $(0,\ 0,\ 0)$, $(1,\ 0,\ 0)$, $(0,\ 1,\ 0)$ in units of $2\pi/L$
- 16 non-zero momentum transfers
- three polarisations: $\Gamma^{\rm unpol} = \frac{1}{2}(1+\gamma_4)$, $\Gamma_1 = \frac{1}{2}(1+\gamma_4)\mathrm{i}\gamma_5\gamma_1$, $\Gamma_2 = \frac{1}{2}(1+\gamma_4)\mathrm{i}\gamma_5\gamma_2$
- renormalisation: non-perturbatively
- scale set by $r_0 = 0.467\,$fm
  r_0 is a length scale derived from the heavy-quark potential $V(R)$:

$$R^2 \frac{\mathrm{d}V(R)}{\mathrm{d}R}\Big|_{R=r_0} = 1.65$$

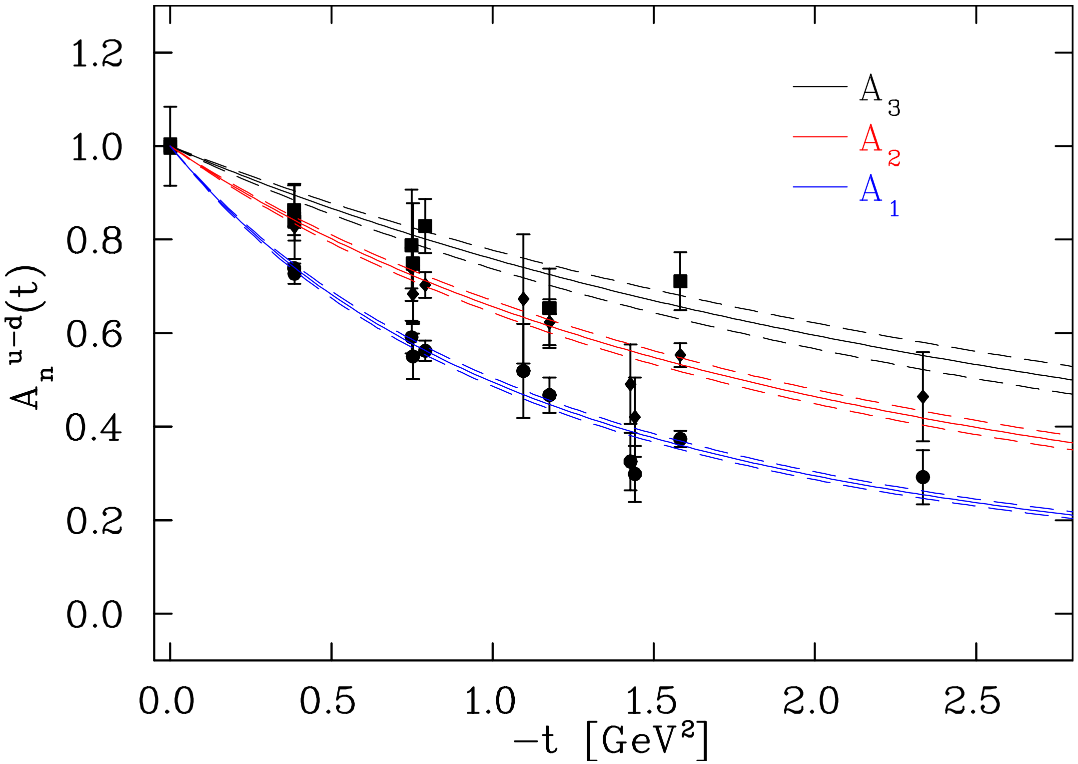
β	κ	Size	m_π[GeV]
5.20	0.13420	$16^3 \times 32$	1.007(17)
5.20	0.13500	$16^3 \times 32$	0.833(8)
5.20	0.13550	$16^3 \times 32$	0.619(7)
5.25	0.13460	$16^3 \times 32$	0.987(11)
5.25	0.13520	$16^3 \times 32$	0.830(9)
5.25	0.13575	$24^3 \times 48$	0.597(4)
5.29	0.13400	$16^3 \times 32$	1.173(20)
5.29	0.13500	$16^3 \times 32$	0.929(13)
5.29	0.13550	$24^3 \times 48$	0.769(9)
5.29	0.13590	$24^3 \times 48$	0.596(10)
5.40	0.13500	$24^3 \times 48$	1.037(11)
5.40	0.13560	$24^3 \times 48$	0.842(7)
5.40	0.13610	$24^3 \times 48$	0.626(6)

lattice spacing $a = 0.07 - 0.11$ fm
length of the spatial box $L = 1.4 - 2.0$ fm

analysis still preliminary!

Results for generalised form factors

GFFs $A_{10}^{u-d} = F_1^{u-d}(t)$, A_{20}^{u-d}, A_{30}^{u-d} (non-singlet), normalised to unity at $t = 0$ (dipole fits)



$\beta = 5.4$, $\kappa = 0.1350$
$24^3 \times 48$ lattice, $m_\pi \approx 970\text{MeV}$

dipole fit:

$$A_{n0}(t) = \frac{A_{n0}(0)}{(1 - t/M_n^2)^2}$$

form factor $A_{n0}(t)$ flattens as n grows

$$\int_{-1}^{1} \mathrm{d}x\, x^{n-1} H_q(x, \xi = 0, t) = A_{n0}^q(t)$$

H_q (as a function of t) becomes wider as x grows

$$q(x, \vec{b}_\perp) = \int \frac{\mathrm{d}^2\Delta_\perp}{(2\pi)^2}\, \mathrm{e}^{\mathrm{i}\vec{b}_\perp \cdot \vec{\Delta}_\perp} H_q(x, 0, -\vec{\Delta}_\perp^2)$$

M. Burkardt

q (as a function of $\vec{b}_\perp$) becomes narrower as x grows (as expected)

Q^2 dependence of F_1 and F_2

common expectation: $F_1 \propto \dfrac{1}{Q^4}\,,\ F_2 \propto \dfrac{1}{Q^6} \quad \Rightarrow \quad Q^2\dfrac{F_2}{F_1} \propto \text{const.}$ $\quad Q^2 = -q^2 = -t$

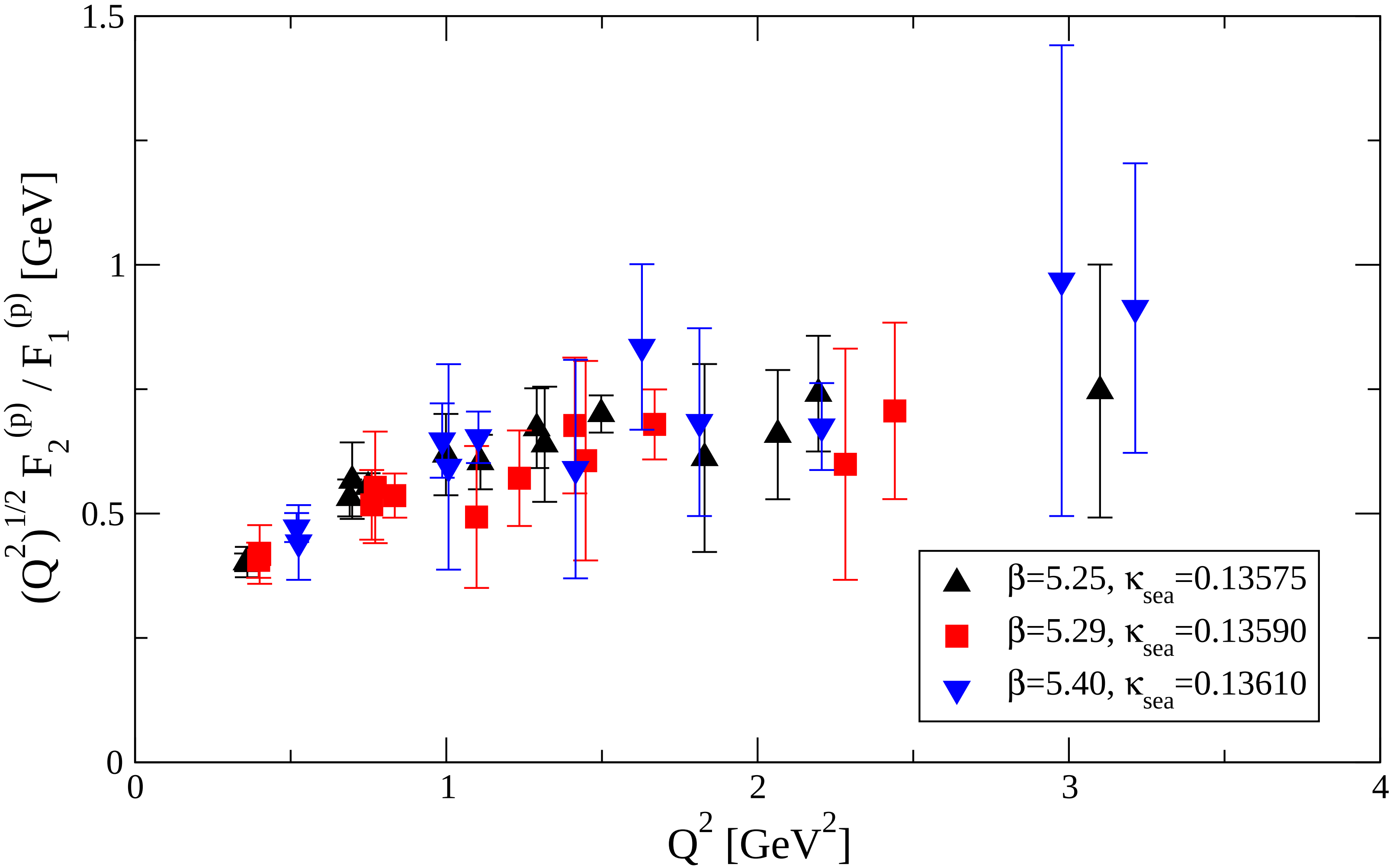
experiment: $\sqrt{Q^2}\dfrac{F_2}{F_1} \propto \text{const.}$ for the proton

perturbative QCD + . . . *: $\dfrac{Q^2}{\ln^2(Q^2/\Lambda^2)}\dfrac{F_2}{F_1} \propto \text{const.}$

* Belitsky, Ji, Yuan, Phys. Rev. Lett. 91 (2003) 092003

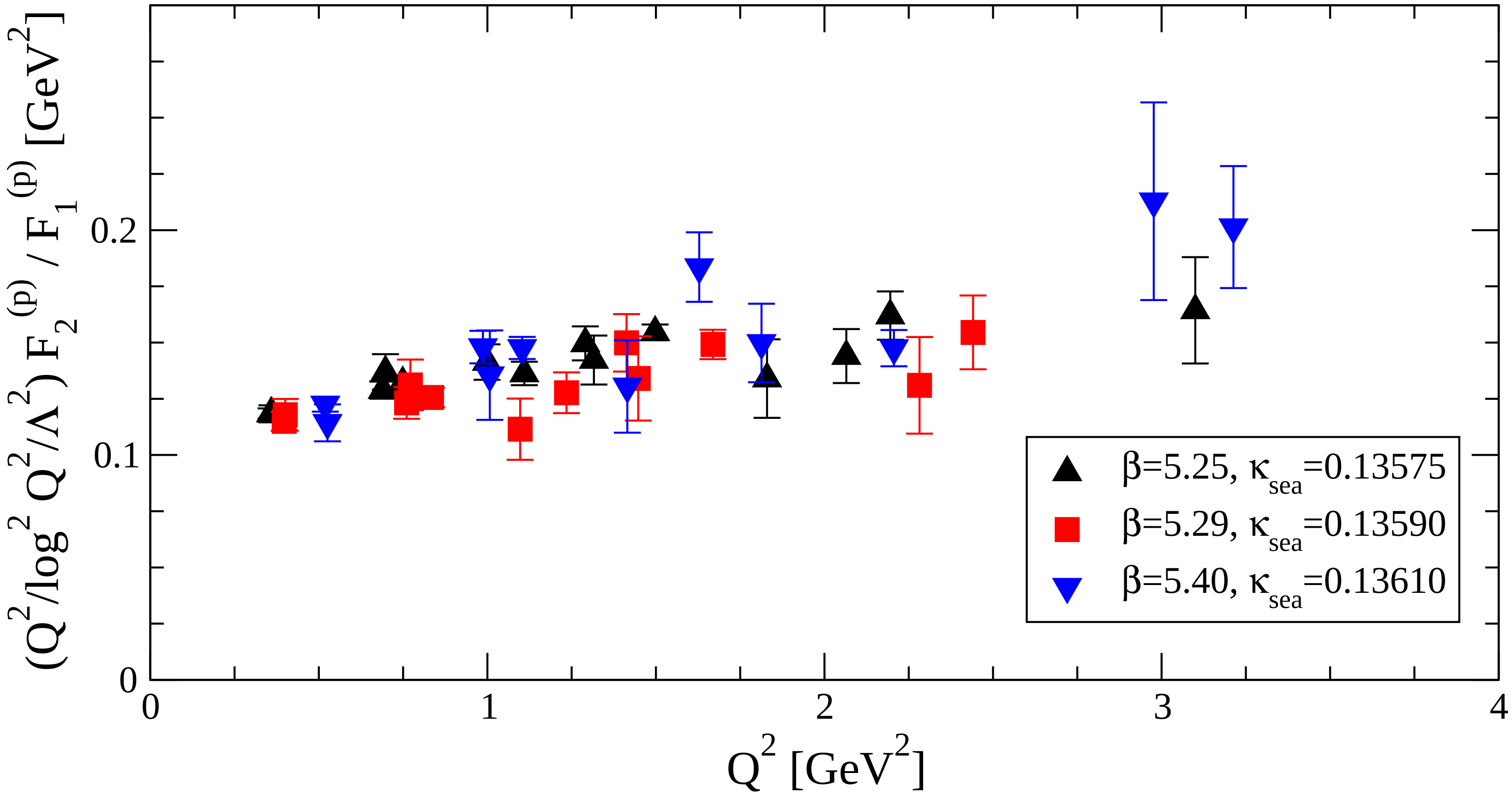
$m_\pi \approx 600\,\text{MeV}$

disconnected contributions neglected!



$1/\sqrt{Q^2}$ behaviour suggested by experimental data

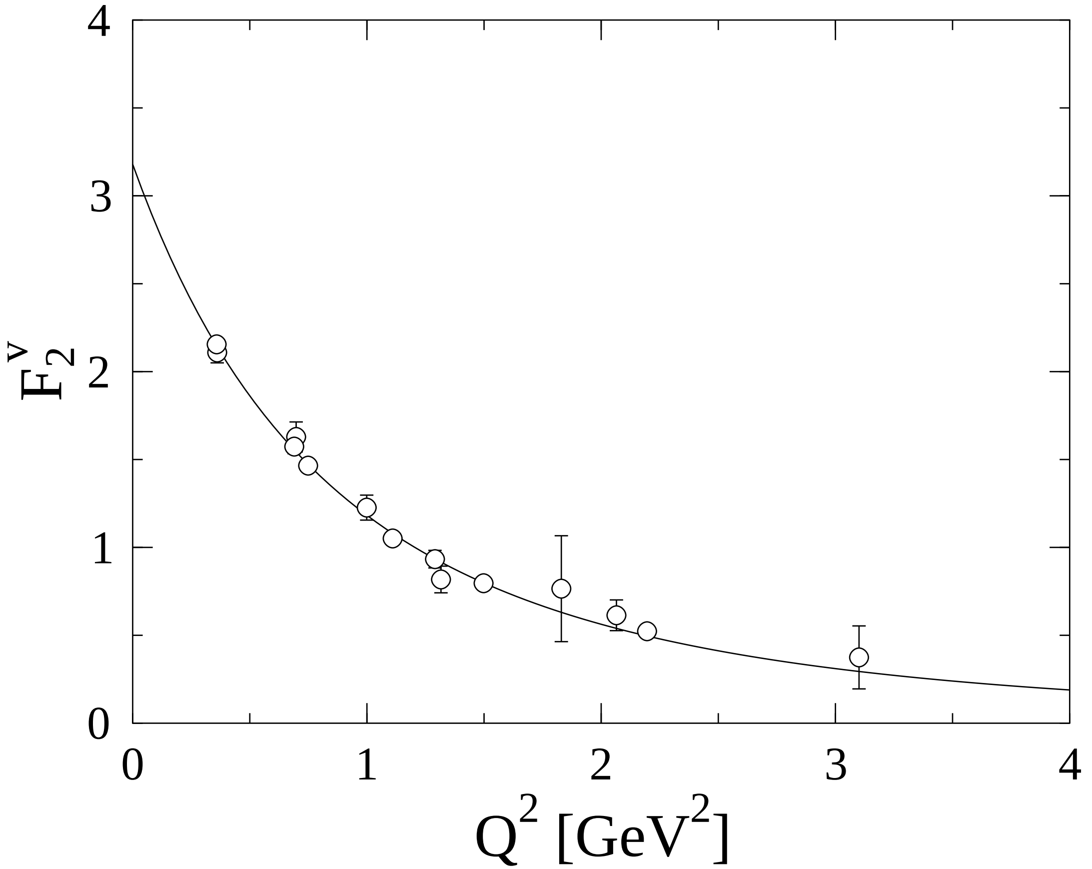
$m_\pi \approx 600\,\mathrm{MeV}$ disconnected contributions neglected!



Belitsky et al.: $\ln^2(Q^2/\Lambda^2)/Q^2$ behaviour asymptotically $\Lambda = 200\,\mathrm{MeV}$

dipole ansatz for F_1: $F_1(Q^2) = \dfrac{A_1}{(1 + Q^2/M_1^2)^2}$

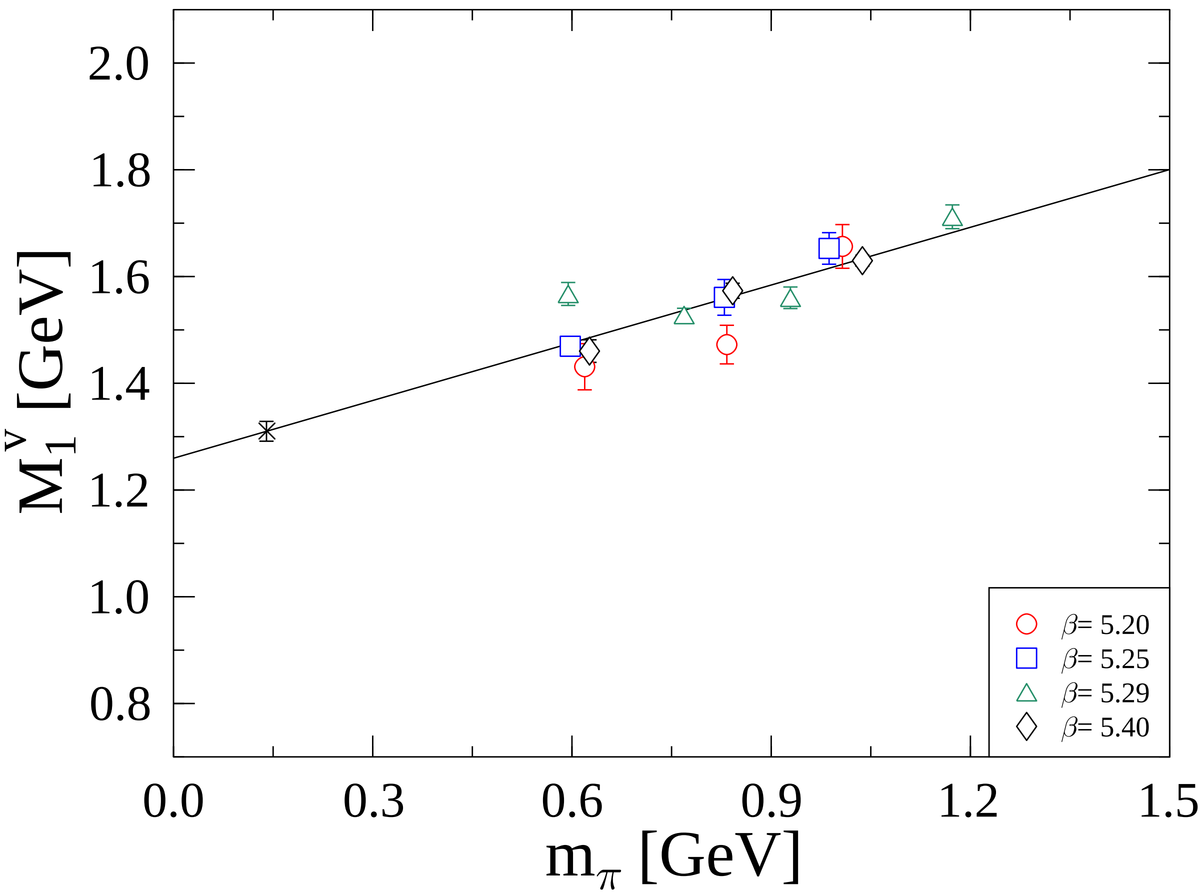
tripole ansatz for F_2: $F_2(Q^2) = \dfrac{A_2}{(1 + Q^2/M_2^2)^3}$



$\beta = 5.25$, $\kappa = 0.13575$
$V = 24^3 \times 48$, $m_\pi \approx 600$ MeV

isovector channel

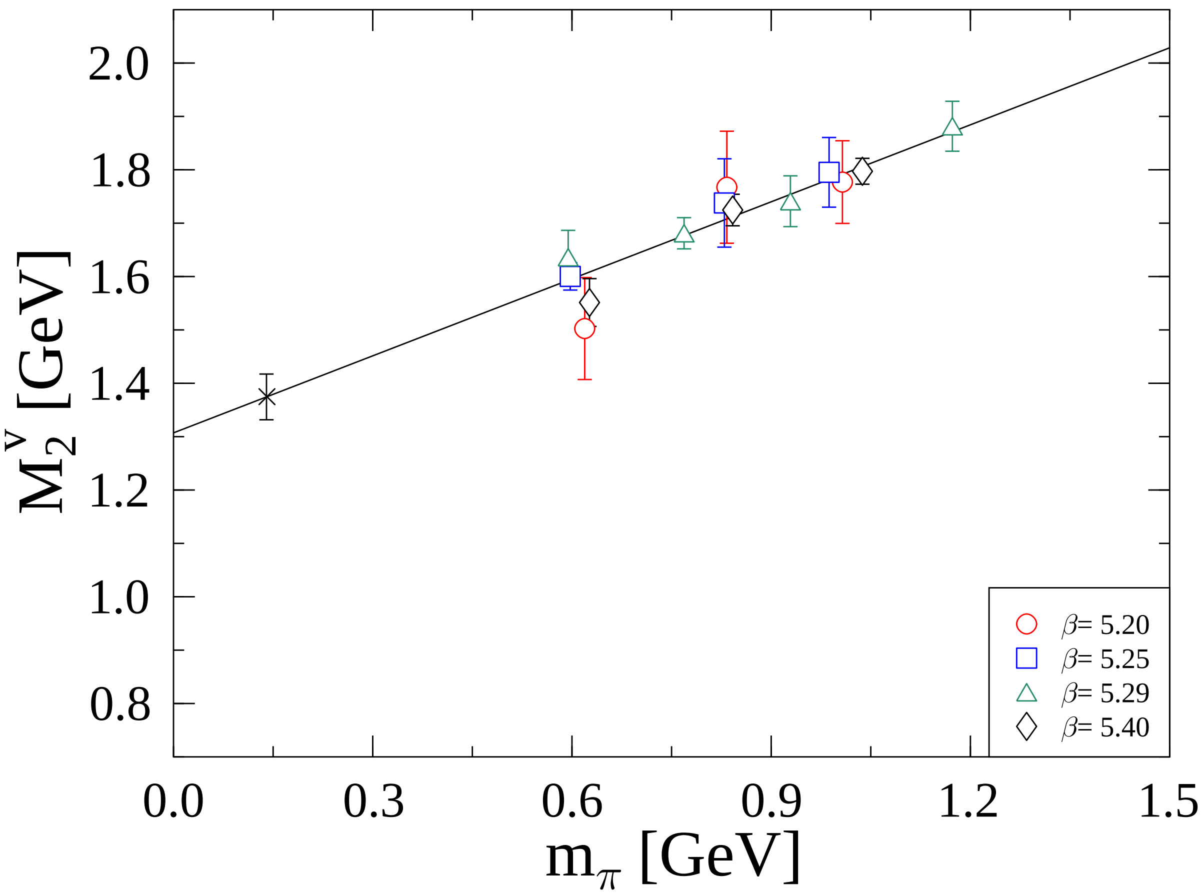
F_1 dipole masses:



discretisation errors apparently small
chiral extrapolation linear in m_π

isovector channel

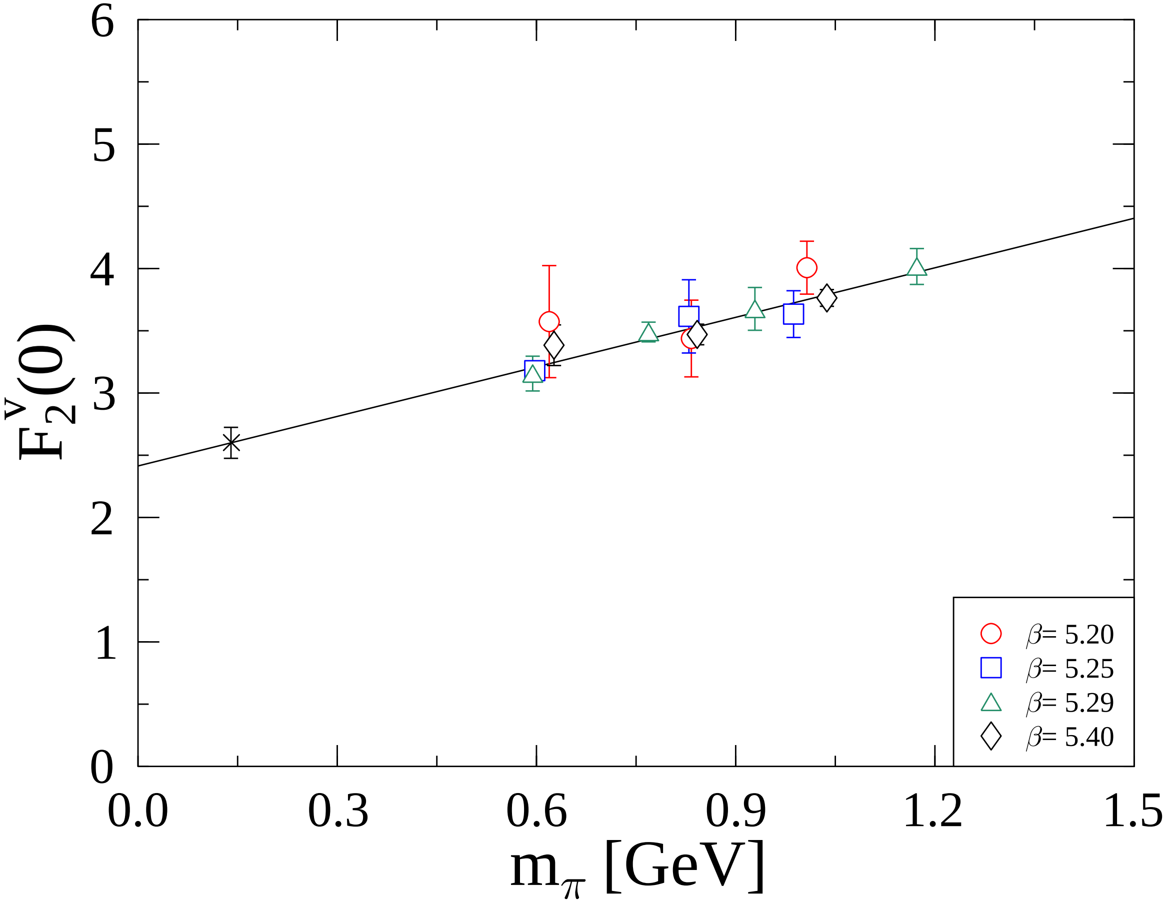
F_2 tripole masses:



chiral extrapolation linear in m_π

isovector channel

$F_2(0)$ from tripole ansatz:



chiral extrapolation linear in m_π

isovector channel

Chiral effective field theory

wanted: guidance for chiral extrapolation $\longrightarrow$ contact with chiral effective field theory (ChEFT)?

ChEFT has problems with large quark masses (pion masses)
large values of Q^2 ($Q^2 \gtrsim 0.4$ GeV^2)
$\uparrow$
experimental data

direct comparison with ChEFT expressions for the form factors questionable
$\longrightarrow$ compare data for radii (dipole masses) and the magnetic moment

isovector channel: no disconnected contributions!

more specifically: small scale expansion SSE (ChEFT with explicit nucleon and Δ degrees of freedom)
V. Bernard, H.W. Fearing, T.R. Hemmert, U.-G. Meißner, Nucl. Phys. A635 (1998) 121
T.R. Hemmert, W. Weise, Eur. Phys. J. A15 (2002) 487

radii r_1, r_2 defined by $$F_i(Q^2) = F_i(0)\left[1 - \frac{1}{6}\, r_i^2 Q^2 + \mathcal{O}(Q^4)\right]$$

related to the (di, tri)pole masses M_1, M_2 $$r_1^2 = \frac{12}{M_1^2}\,,\; r_2^2 = \frac{18}{M_2^2}$$

SSE for the (square of the) isovector Dirac radius:

$$(r_1^v)^2 = -\frac{1}{(4\pi F_\pi)^2}\left\{1 + 7g_A^2 + \left(10g_A^2 + 2\right)\log\left[\frac{m_\pi}{\lambda}\right]\right\} - \frac{12B_{10}^{(r)}(\lambda)}{(4\pi F_\pi)^2}$$

$$+ \frac{c_A^2}{54\pi^2 F_\pi^2}\left\{26 + 30\log\left[\frac{m_\pi}{\lambda}\right] + 30\frac{\Delta}{\sqrt{\Delta^2 - m_\pi^2}}\log R\right\}$$

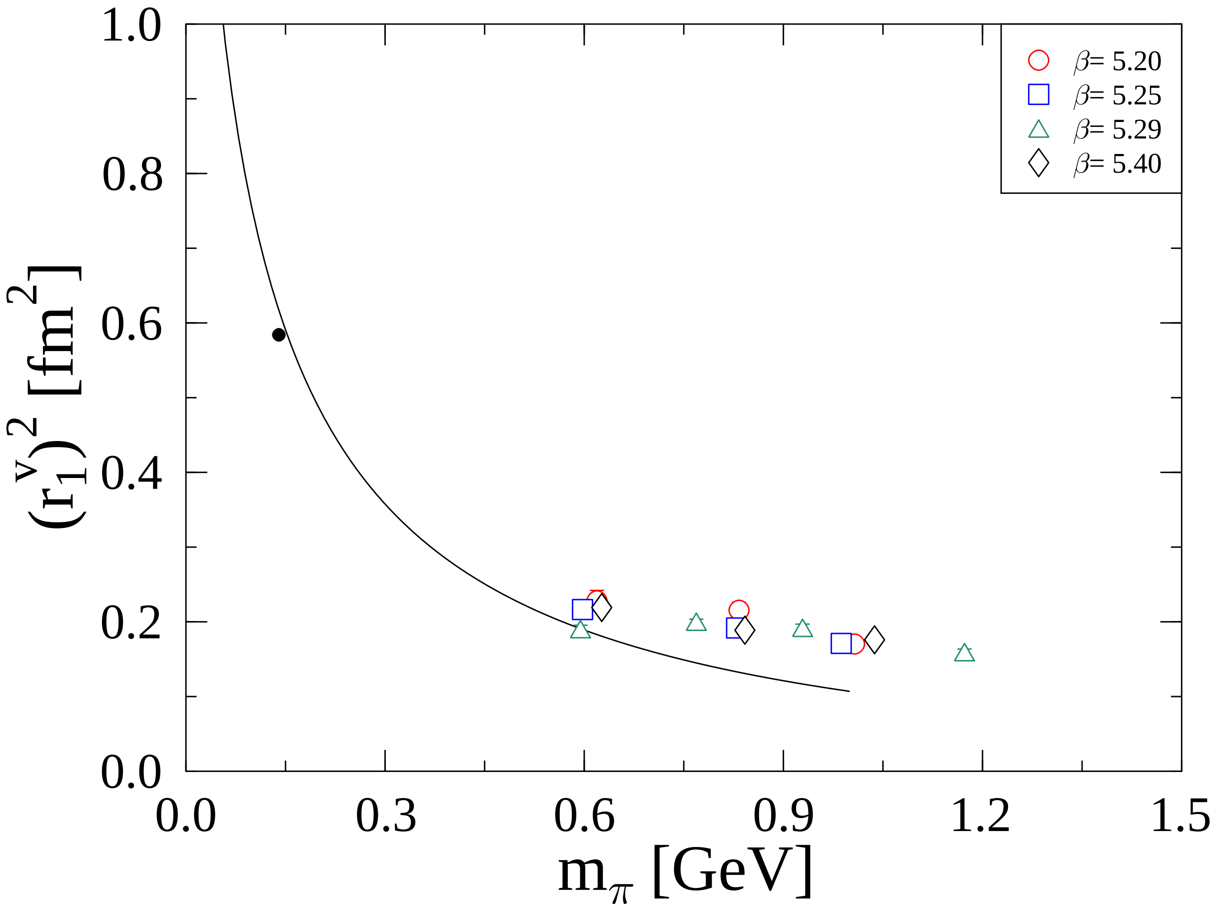
$$R = \frac{\Delta}{m_\pi} + \sqrt{\frac{\Delta^2}{m_\pi^2} - 1}$$

Parameter		Empirical value	used in plot
g_A	axial coupling of the nucleon	1.267	1.2
c_A	leading axial $N\Delta$ coupling	1.1 - 1.5	1.4
F_π	pion decay constant	0.0924 GeV	0.0924 GeV
Δ	Δ-nucleon mass splitting	0.2711 GeV	0.2711 GeV

$B_{10}^{(r)}(\lambda)$: counterterm at renormalisation scale λ

$B_{10}^{(r)}(600\,\text{MeV}) = 0.6$ reproduces approximately the phenomenological value of r_1^v
$\longrightarrow$ negative core contribution to $(r_1^v)^2$

$\mathcal{O}(\epsilon^3)$ SSE formula not accurate enough?



1.0
0.8
0.6
0.4
0.2
0.0
0.0
0.3
0.6
0.9
1.2
1.5
$(r_1^v)^2$ [fm^2]
m_π [GeV]
β= 5.20
β= 5.25
β= 5.29
β= 5.40

no fit!

SSE for the (square of the) isovector Pauli radius:

$$\left(r_2^v\right)^2 = \frac{g_A^2 M_N}{8F_\pi^2 \kappa_v(m_\pi)\pi m_\pi} + \frac{c_A^2 M_N}{9F_\pi^2 \kappa_v(m_\pi)\pi^2\sqrt{\Delta^2 - m_\pi^2}} \log R$$

$\kappa_v(m_\pi^{\rm phys}) = \kappa_v(140\,\text{MeV}) = 3.71$ $\qquad$ M_N = nucleon mass $= 0.9389\,\text{GeV}$

SSE: κ_v shows a rather strong m_π dependence
T.R. Hemmert, W. Weise, Eur. Phys. J. A15 (2002) 487

$$\kappa_v = \kappa_v^0 - \frac{g_A^2\, m_\pi M_N}{4\pi F_\pi^2} + \frac{2c_A^2 \Delta M_N}{9\pi^2 F_\pi^2}\left\{\sqrt{1 - \frac{m_\pi^2}{\Delta^2}}\log R + \log\left[\frac{m_\pi}{2\Delta}\right]\right\}$$
$$- 8E_1^{(r)}(\lambda) M_N m_\pi^2 + \frac{4c_A c_V g_A M_N m_\pi^2}{9\pi^2 F_\pi^2}\log\left[\frac{2\Delta}{\lambda}\right] + \frac{4c_A c_V g_A M_N m_\pi^3}{27\pi F_\pi^2 \Delta}$$
$$- \frac{8c_A c_V g_A \Delta^2 M_N}{27\pi^2 F_\pi^2}\left\{\left(1 - \frac{m_\pi^2}{\Delta^2}\right)^{3/2}\log R + \left(1 - \frac{3m_\pi^2}{2\Delta^2}\right)\log\left[\frac{m_\pi}{2\Delta}\right]\right\}$$

additional parameters:		
	κ_v^0	isovector anomalous magnetic moment in the chiral limit
	c_V	leading isovector $N\Delta$ coupling
	$E_1^{(r)}(\lambda)$	counterterm

How to compare with the lattice data?

Note: κ_v is the anomalous magnetic moment in units of the nuclear magneton

dimensionful anomalous magnetic moment: $\kappa_v \dfrac{e}{2M_N}$ $\swarrow$ nucleon mass for the quark mass considered

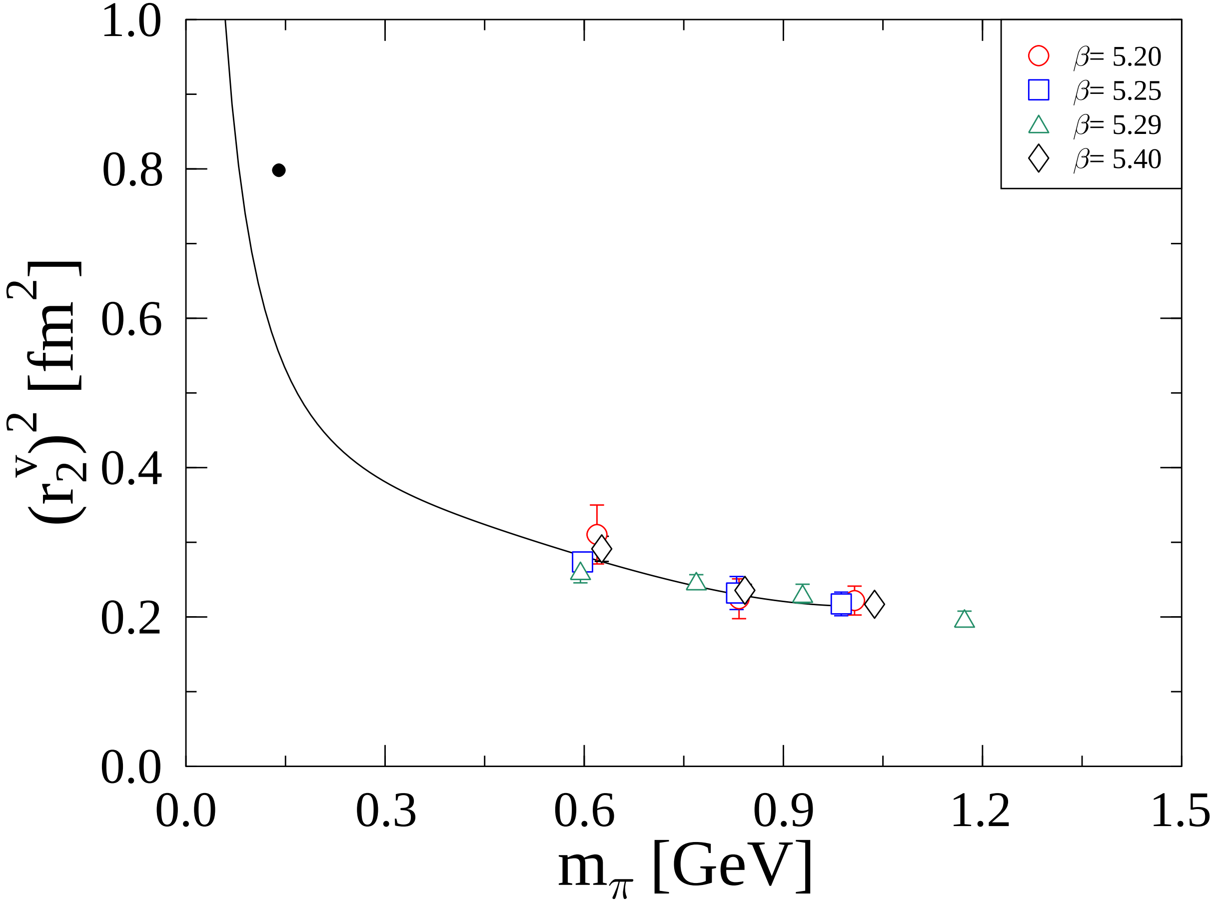
SSE relation refers to the nuclear magneton calculated with $M_N^{\rm phys}$

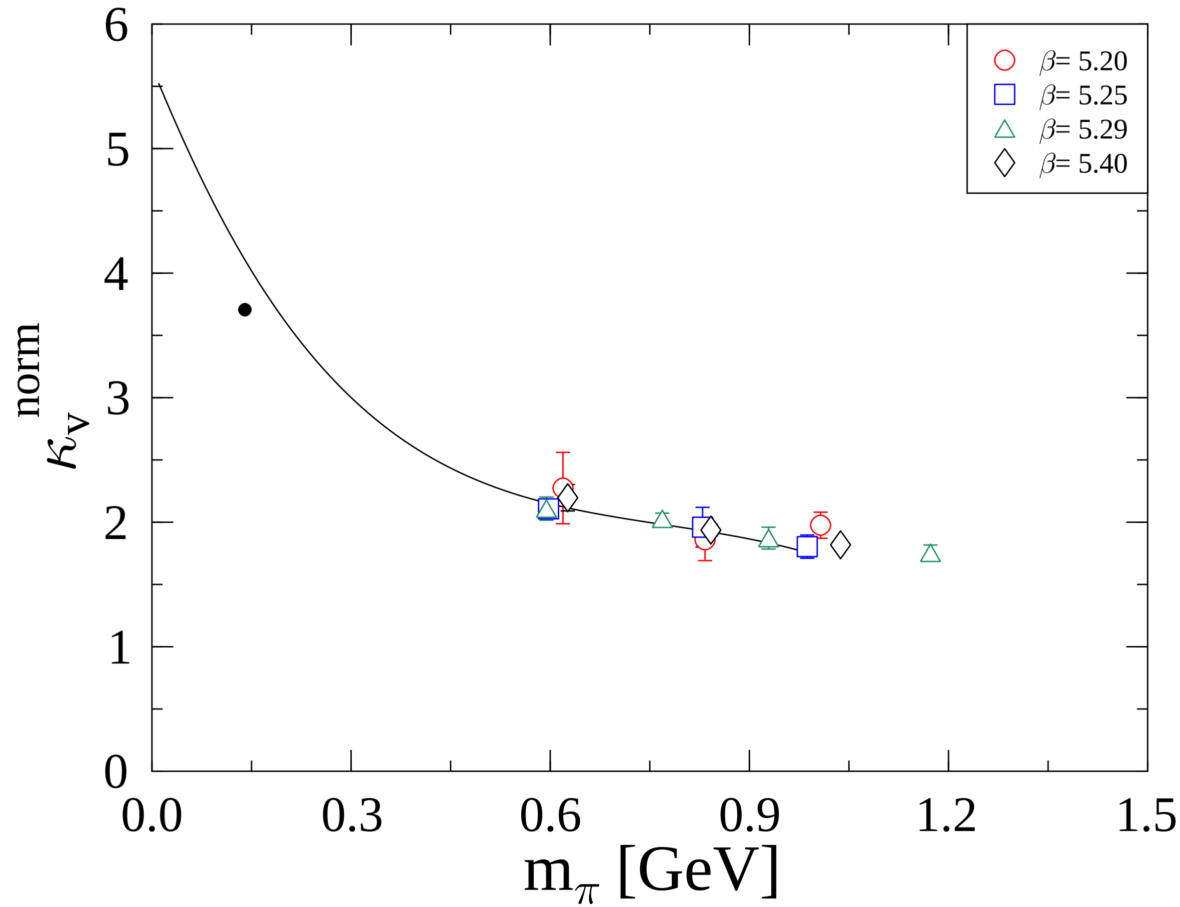
$\rightarrow$ compare "normalised" $\kappa_v^{\rm norm}$ with SSE: $\kappa_v^{\rm norm} \dfrac{e}{2M_N^{\rm phys}} = \kappa_v \dfrac{e}{2M_N} \quad \Rightarrow \quad \kappa_v^{\rm norm} = \kappa_v \dfrac{M_N^{\rm phys}}{M_N}$

combined fit of $(r_2^v)^2(m_\pi)$ and $\kappa_v(m_\pi)$ of all data with $m_\pi < 1$ GeV

g_A	1.2^*	fixed	
c_A	1.4	fixed	
F_π	0.0924 GeV	fixed	
M_N	0.9389 GeV	fixed	
Δ	0.2711 GeV	fixed	
κ_v^0		fitted	5.7(3)
c_V		fitted	$-1.7(2)$ GeV^{-1}
$E_1^{(r)}(0.6$ GeV$)$		fitted	$-4.7(4)$ GeV^{-3}

* T.R. Hemmert, M. Procura, W. Weise, Phys. Rev. D68 (2003) 075009


1.0
0.8
0.6
0.4
0.2
0.0
$(r_2^v)^2$ [fm^2]
0.0
0.3
0.6
0.9
1.2
1.5
m_π [GeV]
β= 5.20
β= 5.25
β= 5.29
β= 5.40



β= 5.20
β= 5.25
β= 5.29
β= 5.40
κ_v^{norm}
m_π [GeV]
0.0
0.3
0.6
0.9
1.2
1.5
0
1
2
3
4
5
6

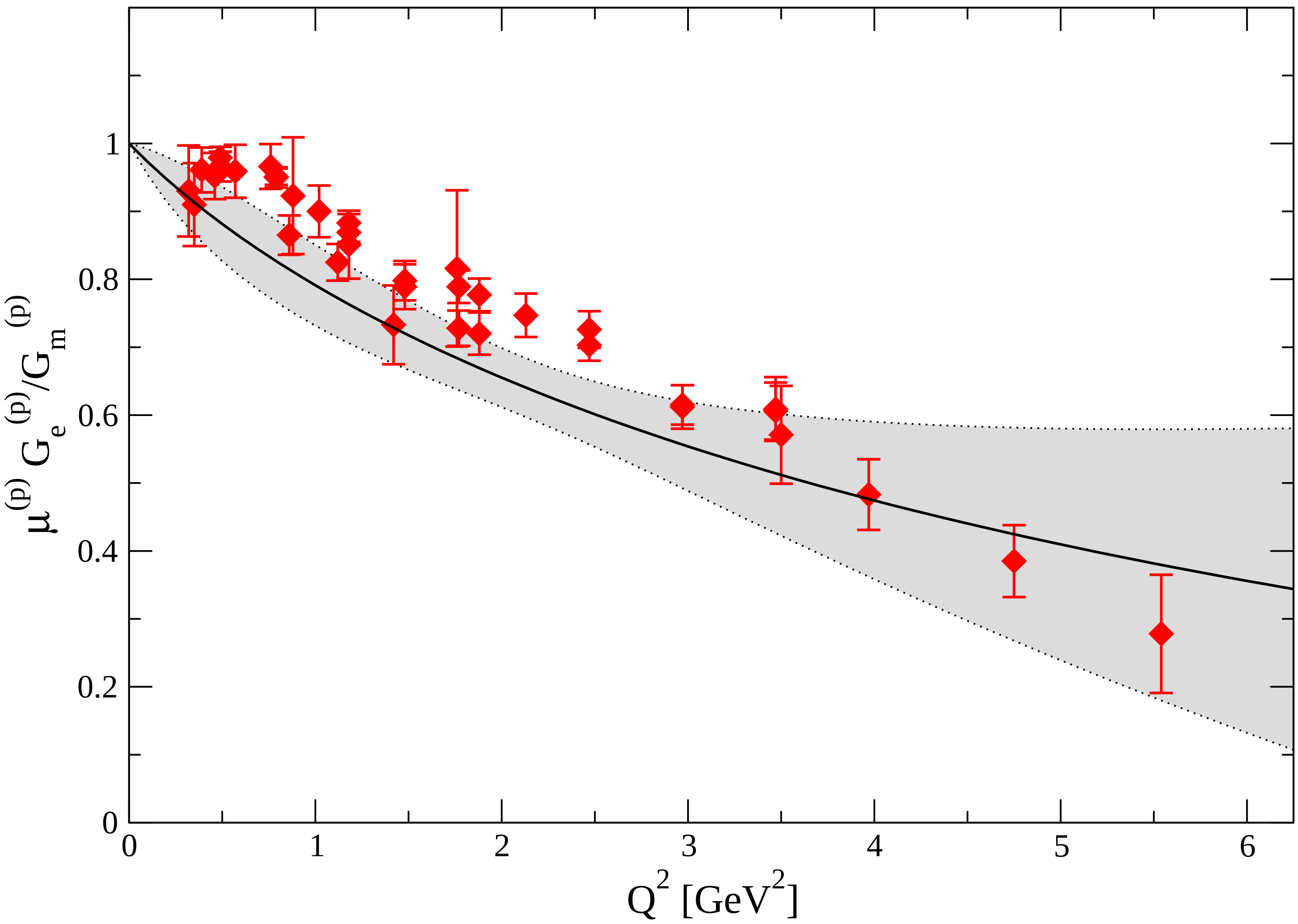
G_e/G_m for the proton

- consider the flavour combination $\frac{2}{3}u - \frac{1}{3}d$ (electromagnetic current in the proton)
  disconnected contributions neglected
- dipole (tripole) fits of F_1 (F_2)
- naive chiral extrapolation of M_1, $F_2(0)$ and M_2
- reconstruct F_1, F_2 in the chiral limit from these extrapolated values
- compute G_e and G_m from

$$G_e(q^2) = F_1(q^2) + \frac{q^2}{(2M_N)^2} F_2(q^2)$$

$$G_m(q^2) = F_1(q^2) + F_2(q^2)$$

compared with JLAB data



Summary and outlook

- GPDs have attracted a lot of interest recently.
- QCDSF-UKQCD compute matrix elements which allow to evaluate moments of GPDs
- The data reveal the expected "shrinking" of the nucleon for $x \to 1$.
- Special case of GPDs: (electromagnetic) form factors
- Dipole (tripole) fits describe the Q^2 dependence of the form factors.
- Naive chiral extrapolation surprisingly successful
- Comparison with ChEFT difficult at present
  - combined fit of $\kappa_v(m_\pi)$ and the Pauli radius $r_2^v(m_\pi)$
    compatible both with the lattice data and with phenomenology
  - Dirac radius $r_1^v(m_\pi)$ problematic

for the future:

- Simulations on larger lattices
  for smaller momenta in the (generalised) form factors
- Simulations for smaller quark masses (overlap fermions?)
  to make the chiral extrapolation more reliable
- More trustworthy fits with expressions from ChEFT