Constraints on Proton Structure from Precision Atomic-Physics Measurements

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Introduction

- Inconsistencies between atomic and nuclear determinations of proton structure

- The Zemach radius

\[
\langle r \rangle_Z = -\frac{4}{\pi} \int_0^\infty \frac{dQ}{Q^2} \left[ \frac{G_E(Q^2)G_M(Q^2)}{1+\kappa_p} - 1 \right]
\]

ranges from 1.024(16) fm to 1.091(18) fm.

- The rms charge radius, from 0.871(12) fm to 0.897(18) fm.

- Will discuss an atomic estimate of \( \langle r \rangle_Z \) based on the difference between hyperfine splittings (hfs) in hydrogen and muonium.

- The large QED contributions for a pointlike nucleus essentially cancel, when corrected for magnetic moment and reduced mass effects, providing a sum rule for proton form factors and structure functions.
Acknowledgments

• published as S.J. Brodsky, C.E. Carlson, JRH, and D.S Hwang, PRL 94, 022001 (2005); 169902(E) (2005).


• see also comments in J.L Friar and I. Sick, PRL 95, 049101 (2005) and the reply in PRL 95, 049102 (2005).

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Outline

- introduction
- sum-rule derivation
- evaluation
- interpretation
- conclusions
Hyperfine splittings

- Fermi energy: \( E_F^N = \frac{8 \alpha^3}{3 \pi} \frac{\mu_B \mu_N m_e^3}{(1+m_e/m_N)^3}, \)
  with \( N = \mu^+ \) or \( p \), \( \mu_B = \frac{e}{2m_e} \), and \( \mu_N = (1 + \kappa_N) \frac{e}{2m_N} \)

- muonium
  \[ E_{\text{hfs}}(e^- \mu^+) = (1 + \Delta_{\text{QED}} + \Delta_{\mu_R}^{\mu} + \Delta_{\text{hvp}}^{\mu} + \Delta_{\text{weak}}^{\mu}) E_F^{\mu} \]

- hydrogen
  \[ E_{\text{hfs}}(e^- p) = (1 + \Delta_{\text{QED}} + \Delta_{R}^{p} + \Delta_{S}^{p} + \Delta_{\text{hvp}}^{p} + \Delta_{\mu vp}^{p} + \Delta_{\text{weak}}^{p}) E_F^{p} \]

- construct ratio rescaled by \( \mu_N \) and reduced masses
  \[ \Delta_{\text{hfs}} \equiv \frac{E_{\text{hfs}}(e^- p)}{E_{\text{hfs}}(e^- \mu^+)} \frac{\mu_{\mu}}{\mu_p} \frac{(1+m_e/m_p)^3}{(1+m_e/m_u)^3} - 1 \]
  \[ = \frac{1 + \Delta_{\text{QED}} + \Delta_{R}^{p} + \Delta_{S}^{p} + \Delta_{\text{hvp}}^{p} + \Delta_{\mu vp}^{p} + \Delta_{\text{weak}}^{p}}{1 + \Delta_{\text{QED}} + \Delta_{R}^{\mu} + \Delta_{\text{hvp}}^{\mu} + \Delta_{\text{weak}}^{\mu}} - 1 \]
The atomic side

\[ \Delta_S = \Delta_{hfs} + \Delta_R^\mu + \Delta_{hvp}^\mu + \Delta_{weak}^\mu - (\Delta_R^p + \Delta_{hvp}^p + \Delta_{\mu vp}^p + \Delta_{weak}^p) + \Delta_{hfs}(\Delta_{QED} + \Delta_R^\mu + \Delta_{hvp}^\mu + \Delta_{weak}^\mu) \]

- the leading \( \Delta_{QED} \) cancel.
- the remaining \( \Delta_{QED} \) can be replaced by the lowest-order approximation, \( \alpha/2\pi \).
The hadronic side

- \( \Delta_S = \Delta_Z + \Delta_{\text{pol}} \)
- Zemach contribution: \( \Delta_Z = -2\alpha m_e (1 + \delta_Z^{\text{rad}}) \langle r \rangle_Z \), with \( \langle r \rangle_Z = -\frac{4}{\pi} \int_0^\infty \frac{dQ^2}{Q^2} \left[ \frac{G_E(Q^2)G_M(Q^2)}{1+\kappa_p} - 1 \right] \)
  \[ = \int d^3r d^3r' |\vec{r} - \vec{r}'| \rho_E(\vec{r}) \rho_M(\vec{r}') \]
- polarization contribution: \( \Delta_{\text{pol}} = \frac{\alpha m_e}{2\pi m_p(1+\kappa_p)}(\Delta_1 + \Delta_2) \), with
  \[ \Delta_1 = \int_0^\infty \frac{dQ^2}{Q^2} \left\{ \frac{9}{4} F_2^2(Q^2) - 4 m_p \int_{\nu_{\text{th}}}^\infty \frac{d\nu}{\nu^2} \beta_1 \left( \frac{\nu^2}{Q^2} \right) g_1(\nu, Q^2) \right\} , \]
  \[ \Delta_2 = -12 m_p \int_0^\infty \frac{dQ^2}{Q^2} \int_{\nu_{\text{th}}}^\infty \frac{d\nu}{\nu^2} \beta_2 \left( \frac{\nu^2}{Q^2} \right) g_2(\nu, Q^2) , \]
  \( \nu_{\text{th}} = m_{\pi} + (m_{\pi}^2 + Q^2)/2m_p \),
  \[ \beta_1(\theta) = 3\theta - 2\theta^2 - 2(2 - \theta) \sqrt{\theta(\theta + 1)} , \]
  \[ \beta_2(\theta) = 1 + 2\theta - 2 \sqrt{\theta(\theta + 1)} \]
Radiative correction to Zemach radius

\[ \delta_{Z}^{\text{rad}} \] estimated by Bodwin & Yennie, PRD 37, 498 (1988).

Karshenboim, PLA 225, 97 (1997) calculated analytically for dipole form factors:

\[ \delta_{Z}^{\text{rad}} = (\alpha/3\pi) \left[ 2 \ln(\Lambda^2/m_e^2) - 4111/420 \right]. \]

with \( \Lambda^2 = 0.71 \text{ GeV}^2 \), this yields \( \delta_{Z}^{\text{rad}} = 0.0153 \).
Recoil corrections in muonium

- muonium: $\Delta^\mu_R = \Delta^\mu_{\text{rel}} + \Delta^\mu_{\text{rad}}$.

- relativistic recoil [Bodwin & Yennie, PRD 37, 498 (1988)]:

$$
\Delta^\mu_{\text{rel}} = \frac{1}{1+\kappa_\mu} \left[ \frac{-3\alpha}{\pi} \frac{m_em_\mu}{m_\mu^2 - m_e^2} \ln \frac{m_\mu}{m_e} + \alpha^2 \frac{m_e}{m_\mu} \left( 2 \ln \frac{1}{2\alpha} - 6 \ln 2 + \frac{65}{18} \right) \right]
$$


$$
\Delta^\mu_{\text{rad}} =
\frac{1}{1+\kappa_\mu} \left[ \frac{\alpha^2}{\pi^2} \frac{m_e}{m_\mu} \left( -2 \ln^2 \frac{m_\mu}{m_e} + \frac{13}{12} \ln \frac{m_\mu}{m_e} + \frac{21}{2} \zeta(3) + \zeta(2) + \frac{35}{9} \right) 
+ \alpha^3 \frac{m_e}{\pi^3 m_\mu} \left( -\frac{4}{3} \ln^3 \frac{m_\mu}{m_e} + \frac{4}{3} \ln^2 \frac{m_\mu}{m_e} - \left[ 4\pi^2 \ln 2 + \frac{29}{12} \right] \ln \frac{m_\mu}{m_e} + 47.7213 \right) 
+ \alpha^2 \left( \frac{m_e}{m_\mu} \right)^2 \left( -6 \ln 2 - \frac{13}{6} \right) \right]
$$
Recoil corrections in hydrogen

- Bodwin & Yennie, PRD 37, 498 (1988): \( \Delta^p_R = -1.55 \text{ ppm} \).
- finite-size corrections \( \rightarrow +5.68(1) \text{ ppm} \).
- radiative recoil corrections [Karshenboim, PLA 225, 97 (1997)] \( \rightarrow 5.77(1) \text{ ppm} \).
- Volotka et al, EPJD 33, 23 (2005):
  - re-evaluation of finite-size corrections \( \rightarrow 5.86 \text{ ppm} \).
  - forced \( G_M \) to reproduce their \( \langle r \rangle_Z \) \( \rightarrow 6.01 \text{ ppm} \).
  - chose \( \Delta^p_R = 5.86(15) \text{ ppm} \).
Atomic inputs

  \[ E_{\text{hfs}}(e^- p) = 1 420.405 751 766 7(9) \text{ MHz}. \]

- W. Liu et al., PRL 82, 711 (1999):
  \[ E_{\text{hfs}}(e^- \mu^+) = 4 463.302 765(53) \text{ MHz}. \]

- S. Eidelman et al., PLB 592, 1 (2004):
  \[ m_p = 938.272 029(80) \text{ MeV}, \ m_\mu = 105.658 369(9) \text{ MeV}, \]
  \[ m_e = 0.510 998 918(44) \text{ MeV}, \ \alpha^{-1} = 137.035 999 11(46). \]

- G.W. Bennett et al., PRL 92, 161802 (2004):
  \[ \kappa_\mu = 0.001 165 920 8(6). \]

  \[ m_\mu/m_e = 206.768 2838(54), \ m_p/m_e = 1836.152 672 61(85). \]

- P.J. Mohr, private communication:
  \[ \mu_\mu/\mu_p = 3.183 345 20(20), \text{ free of muonium hfs.} \]
Cross-check with QED

- muonium:
  \[ \Delta_{\text{QED}} = \frac{E_{\text{hfs}}(e^-\mu^+)}{E_{\mu}^F} - 1 - (\Delta_{R}^\mu + \Delta_{hvp}^\mu + \Delta_{\text{weak}}^\mu) \]
  \[ = 1136.12(13) \text{ ppm.} \]

- hydrogen:
  \[ \Delta_{\text{QED}} = \frac{E_{\text{hfs}}(e^-p)}{E_{p}^F} - 1 - (\Delta_{R}^p + \Delta_{S} + \Delta_{hvp}^p + \Delta_{\mu vp}^p + \Delta_{\text{weak}}^p) \]

- mix:
  \[ \Delta_{\text{QED}} = \frac{E_{\text{hfs}}(e^-p)}{E_{p}^F} - 1 - (\Delta_{R}^\mu + \Delta_{hvp}^\mu + \Delta_{\text{weak}}^\mu + \Delta_{\text{hfs}}) \]
  \[ - \Delta_{\text{hfs}}\left(\frac{\alpha}{2\pi} + \Delta_{R}^\mu + \Delta_{hvp}^\mu + \Delta_{\text{weak}}^\mu\right) \]
  \[ = 1136.09(14) \text{ ppm.} \]

- consistent with Dupays et al., PRA 68, 052503 (2003) and Volotka et al., EPJD 33, 23 (2005).
Evaluation of atomic side

\[ \Delta_{\text{hfs}} = 145.51(4) \text{ ppm.} \]

\[ \Delta_{R}^{\mu} = -178.34 \text{ ppm.} \]

more inputs [Volotka et al., EPJD 33, 23 (2005)]:

\[ \Delta_{hvp}^{\mu} = 0.05 \text{ ppm, } \Delta_{\text{weak}}^{\mu} = -0.01 \text{ ppm, } \Delta_{hvp}^{p} = 0.01 \text{ ppm, } \Delta_{\muvp}^{p} = 0.07 \text{ ppm, } \Delta_{\text{weak}}^{p} = 0.06 \text{ ppm.} \]

\[ \Delta_{S} = -38.62(16) \text{ ppm.} \]

\[ \rightarrow \text{constraint on } G_{E}, G_{M}, g_{1}, \text{ and } g_{2} \text{ that is better than 1%}. \]
Interpretation of hadronic side

- if use estimate of \( \Delta_{\text{pol}} = 1.4(6) \text{ ppm} \) by Faustov and Martynenko [EPJC 24, 281 (2002)], then \( \Delta_Z = -40.0(6) \text{ ppm} \) and \( \langle r \rangle_Z = 1.043(16) \text{ fm} \).

- Griffioen et al. from preliminary CLAS \( g_1 \) data, SLAC E155x \( g_2 \) data, and MAID parameterization in resonance region: \( \Delta_{\text{pol}} = 0.72(37) \text{ ppm} \) → \( \Delta_Z = -39.3(4) \text{ ppm} \) and \( \langle r \rangle_Z = 1.024(16) \text{ fm} \).

- if use estimate of \( \langle r \rangle_Z = 1.086(12) \text{ fm} \) by Friar and Sick from fits to form factor data [PLB 579, 285 (2004)], then \( \Delta_{\text{pol}} = 3.05(49) \text{ ppm} \).

- new estimate of \( \langle r \rangle_Z = 1.091(18) \text{ fm} \) by Blunden and Sick [nucl-th/0508037], where attempt to incorporate two-photon effects beyond Coulomb distortion, only makes the situation worse.
**$\langle r \rangle_z$ from form-factor models**

- **dipole** → 1.025 fm
- fit to standard Rosenbluth separation  
  \[ G_E(Q^2), G_M(Q^2)/(1 + \kappa_p) = 1/(1 + p_2Q^2 + p_4Q^4 + \cdots) \]  
  → 1.081 fm
- fit constrained by polarization transfer data  
  [Arrington, Table II] → 1.050 fm
Electric charge radius

\[ r_{E,\text{rms}} = \sqrt{-6 \frac{d}{dQ^2} G_E(Q^2)}|_{Q^2=0}. \]

obtain estimates from

- a standard empirical fit: 0.862(12) fm
  [G.G. Simon et al., NPA 333, 381 (1980)]

- Lamb-shift measurements: 0.871(12) fm
  [K. Pachucki, PRA 63, 042503 (2001);
   K. Pachucki and U.D. Jentschura, PRL 91, 113005 (2003); updated by M. Eides, private communication]

- a continued-fraction fit for \( G_E \): 0.895(18) fm
  [I. Sick, PLB 576, 62 (2003)]

- the 2002 CODATA value: 0.8750(68) fm
  [P.J. Mohr and B.N. Taylor, RMP 77, 1 (2005)]
Plot of $\langle r \rangle_Z$ vs $r_{E,\text{rms}}$
Conclusions

atomic physics provides a very precise constraint on proton structure, to better than 1%.

the subtraction method removes uncertainties associated with pure QED contributions to hfs.

the method could also be applied to Lamb shifts, to extract $r_{E,rms}$ with less uncertainty.

the interpretation of individual structure contributions requires more data and analysis, particularly for $g_1$, $g_2$, and $\Delta_{\text{pol}}$.

two-photon contributions to electron-proton scattering.