Introduction to Chiral Perturbation Theory – III
One loop: Unitarity and Analyticity

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Outline of Lecture 3

- Dispersion relation for the scalar form factor of the pion
- Chiral counting on the dispersion relation
- Going beyond one loop CHPT with dispersion relations
- Contribution of resonances to the dispersive integrals
- Summary and conclusions of lectures 2 and 3

Dispersion relation for $\Gamma(t)$

For $t \geq 4M_\pi^2$ $\mathrm{Im}\,\Gamma(t) \neq 0$. $\Gamma(t)$ is analytic everywhere else in the complex t plane, and obeys the following dispersion relation:

$$\bar{\Gamma}(t) = 1 + bt + \frac{t^2}{\pi}\int_{4M_\pi^2}^{\infty} \frac{dt'}{t'^2}\frac{\mathrm{Im}\,\bar{\Gamma}(t')}{t'-t} \quad .$$

where $\bar{\Gamma}(t) = \Gamma(t)/\Gamma(0)$.

Unitarity implies $[\sigma(t) = \sqrt{1-4M_\pi^2/t}]$

$$\begin{aligned} \mathrm{Im}\,\bar{\Gamma}(t) &= \sigma(t)\bar{\Gamma}(t){t_0^0}^*(t) = \bar{\Gamma}(t)e^{-i\delta_0^0}\sin\delta_0^0 \\ &= |\bar{\Gamma}(t)|\sin\delta_0^0 \end{aligned}$$

where t_0^0 is the S–wave, isospin zero amplitude of $\pi\pi$ scattering.

Strictly speaking, the above unitarity relations are valid only for $t \leq 16M_\pi^2$. To a good approximation, however, they hold up to the $K\bar{K}$ threshold.

Chiral counting on the dispersion relation

$$\begin{aligned}\bar{\Gamma}(t) &\sim 1+O(p^2)\\ b &\sim O(1)\left(1+O(M_\pi^2)\right)\\ \delta_0^0 &\sim O(p^2)\left(1+O(p^2)\right)\end{aligned}$$

There are two $O(p^2)$ correction to $\bar{\Gamma}$:

1. $O(1)$ contribution to b;
2. the dispersive integral containing the $O(p^2)$ phase δ_0^0.

Notice that the latter is fixed by unitarity and analyticity.

Are these respected by the one loop calculation?

Dispersion relation and one–loop CHPT

The full one–loop expression of $\bar{\Gamma}(t)$ reads as follows:

$$\bar{\Gamma}(t) = 1 + \frac{t}{16\pi^2 F_\pi^2}(\bar{l}_4 - 1) + \frac{2t - M_\pi^2}{2F_\pi^2}\bar{J}(t)$$

where

$$\bar{J}(t) = \frac{1}{16\pi^2}\left[\sigma(t)\ln\frac{\sigma(t)-1}{\sigma(t)+1} + 2\right]$$

To prove that unitarity and analyticity are respected at this order is sufficient to add:

$$\begin{aligned} \delta_0^0(t) &= \sigma(t)\frac{2t - M_\pi^2}{32\pi F_\pi^2} + O(p^4) \\ \bar{J}(t) &= \frac{t}{16\pi^2}\int_{4M_\pi^2}^{\infty}\frac{dt'}{t'}\frac{\sigma(t')}{t'-t} \end{aligned}$$

Hints

1. Subtract $\bar{J}(t)$ once more:

$$\bar{J}(t) = \frac{t}{96\pi^2} + \frac{t^2}{16\pi^2}\int_{4M_\pi^2}^{\infty} \frac{dt'}{t'^2}\frac{\sigma(t')}{t'-t}$$

2. Trick to pull out a linear term from the dispersive integral:

$$\int_{4M_\pi^2}^{\infty} \frac{dt'}{t'^2}\frac{t'\sigma(t')}{t'-t} = t\int_{4M_\pi^2}^{\infty} \frac{dt'}{t'^2}\frac{\sigma(t')}{t'-t} + \int_{4M_\pi^2}^{\infty} \frac{dt'}{t'^2}\sigma(t')$$

Exact solution of the dispersion relation

Mathematical problem:

1. $F(t)$ is an analytic function of t in the whole complex plane, with the exception of a cut for $4M_\pi^2 \leq t < \infty$;
2. approaching the real axis from above $e^{-i\delta(t)}F(t)$ is real on the real axis, where $\delta(t)$ is a known function.

Omnès ('58) found an exact solution to this mathematical problem:

$$F(t) = P(t)\Omega(t) = P(t)\exp\left\{\frac{t}{\pi}\int_{4M_\pi^2}^{\infty}\frac{dt'}{t'}\frac{\delta(t')}{t'-t}\right\} \quad ,$$

where $P(t)$ is a generic polynomial. $P(t)$ can only be constrained by the behaviour of the function $F(t)$ at infinity.
$\Omega(t)$ is called the Omnès function.

Combining CHPT and the Omnès representation

At order p^4 we have:

$$\Gamma^{(4)}_{\mathsf{CHPT}}(t) = 1 + b^{(2)} t + \Delta^{(2)}(t) \quad , \qquad \Delta^{(2)}(t) = \frac{t}{\pi} \int_{4M_\pi^2}^{\infty} \frac{dt'}{t'} \frac{\delta^{(2)}(t')}{t' - t}$$

It is easy to imagine how to improve the above representation. A few proposals:

$$\Downarrow$$

$$\Gamma^{\alpha}(t) = (1 + b^{(2)} t) e^{\Delta^{(2)}(t)}$$

$$\Downarrow$$

$$\Gamma^{\beta}(t) = (1 + b^{(2)} t) e^{\Delta^{(4)}(t)}$$

$$\Downarrow$$

$$\Gamma^{\gamma}(t) = (1 + b^{(2)} t) e^{\Delta_{\mathsf{phys}}(t)}$$

Which one is best?

Subtraction point

The Omnès representation could be written more generally as follows:

$$\begin{aligned} F(t) &= P'(t)\Omega(t,t_0) \\ &= P'(t)\exp\left\{\frac{t-t0}{\pi}\int_{4M_\pi^2}^{\infty} dt' \frac{\delta(t')}{(t'-t_0)(t'-t)}\right\} \quad , \end{aligned}$$

[Exercise: show that the ratio $P(t)\Omega(t)/P'(t)\Omega(t,t_0)$ is a rational function.]

This opens up a new degree of freedom in improving the chiral representation:

$$\Downarrow$$

$$\Gamma^{\delta}(t) = (1+b^{(2)}t)\exp\left\{\Delta^{(4)}(t,t_0) - \Delta^{(4)}(0,t_0)\right\}$$

Which t_0 is best?

Exercises

1. Show that:

$$\Delta(t,t_0) - \Delta(0,t_0) = \Delta(t)$$

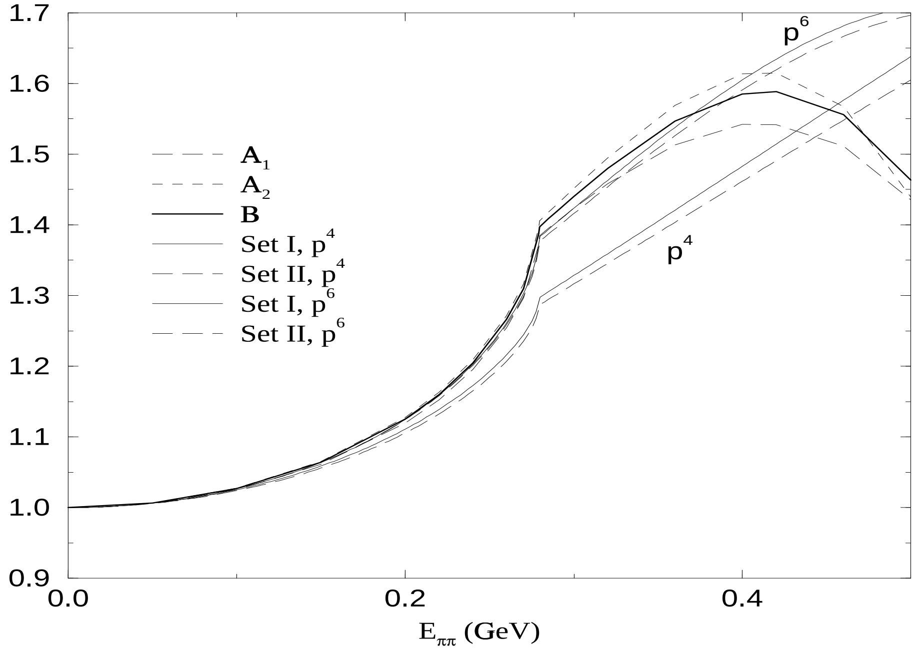
(Yes, in the previous transparency I was cheating!)

2. Show that without a constraint like $\bar{\Gamma}(0) = 1$, I do have an extra freedom in choosing the subtraction point t_0:

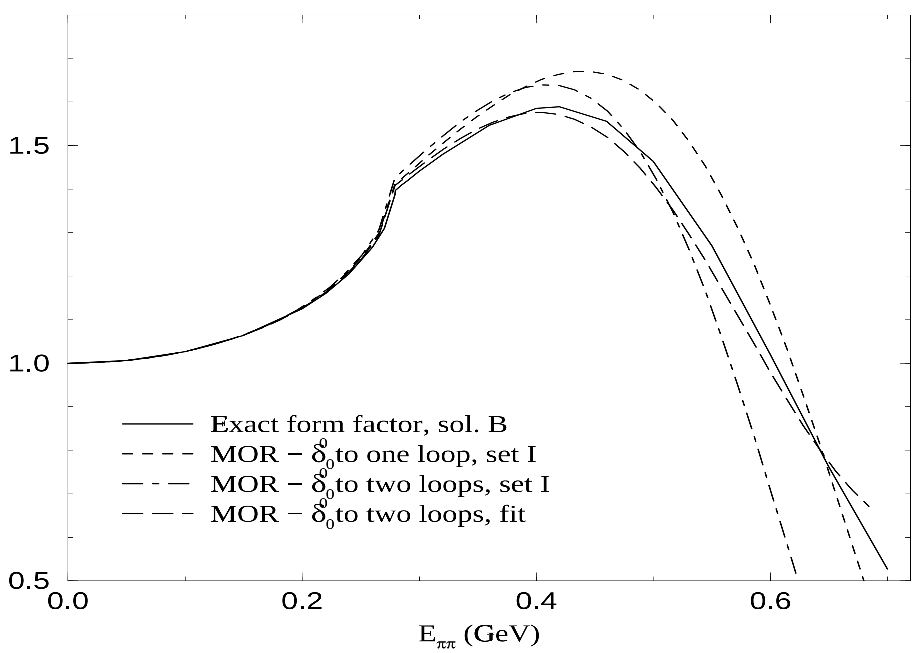
$$F(s) = (a+bt)\Omega(t) \neq (a+bt)\Omega(t,t_0) \quad .$$

3. And that a shift in t_0 generates an effect of higher chiral order in the polynomial $(a+bt)$.

$\mathrm{Re}(F^S_\pi(s)/F^S_\pi(0))$



$\mathrm{Re}(F^S_\pi(s)/F^S_\pi(0))$



Determination of the low energy constants – I

The degree of the polynomial $P(t)$ in the Omnès solution is determined by the behaviour of $F(t)$ at infinity. Suppose we knew a constant would be enough for the scalar form factor:

$$\bar{\Gamma}(t) = \Omega(t)$$

Comparing this to the chiral representation we get:

$$\begin{aligned} \frac{1}{6}\langle r^2\rangle_S^\pi &= \frac{1}{F_\pi^2}\left[l_4^r(\mu) - \frac{1}{16\pi^2}\left(\ln\frac{M_\pi^2}{\mu^2} + \frac{13}{12}\right) + O(M_\pi^2)\right] \\ &= \frac{1}{\pi}\int_{4M_\pi^2}^\infty dt' \frac{\delta(t')}{t'^2} \end{aligned}$$

A sum rule for $l_4^r(\mu)$.

Determination of the low energy constants – II

Consider the vector form factor of the pion:

$$\langle \pi^i(p_1)|\bar{q}\tau^k\gamma_\mu q|\pi^j(p_2)\rangle = i\epsilon^{ikj}F_V(t) \quad , \qquad q = \begin{pmatrix} u \\ d \end{pmatrix}$$

It satisfies a dispersion relation:

$$F_V(t) = 1 + \frac{t}{\pi}\int_{4M_\pi^2}^{\infty} \frac{dt'}{t'} \frac{\mathrm{Im}\, F_V(t')}{t'-t}$$

Which implies that the charge radius of the pion is given by the following sum rule:

$$\begin{aligned} \frac{1}{6}\langle r^2\rangle_V^\pi &= -\frac{1}{F_\pi^2}\left[l_6^r(\mu) - \frac{1}{96\pi^2}\left(\ln\frac{M_\pi^2}{\mu^2}+1\right) + O(M_\pi^2)\right] \\ &= \frac{1}{\pi}\int_{4M_\pi^2}^{\infty} dt' \frac{\mathrm{Im}\, F_V(t')}{t'^2} \end{aligned}$$

Resonance contribution to the sum rule

The ρ resonance will certainly contribute to the integral. In the narrow width approximation:

$$\text{Im}\, F_V(t) \sim \pi\delta(t - M_\rho^2)$$

we can evaluate:

$$\frac{1}{6}\langle r^2\rangle_V^\pi = \frac{f F_\rho}{F_\pi^2}\frac{1}{M_\rho^2}$$

Using the following estimate for the two coupling constants:

$$F_\rho = 144\,\text{MeV} \quad , \qquad f = 69\,\text{MeV}$$

we obtain

$$l_6^r(M_\rho) = -13.3 \times 10^{-3}$$

whereas if we extract it from the measured value of the charge radius, we obtain:

$$l_6^r(M_\rho) = -(13.5 \pm 2.5) \times 10^{-3}$$

Resonance saturation

Table 1: Contributions of the resonances V, A, S, S_1 and η_1 to the constants L_i^r in units of 10^{-3}.

i	$10^3 L_i^r(M_\rho)$	V	A	S	S_1	η_1	Total
1	0.7 ± 0.5	0.6	0	-0.2	$0.2^{b)}$	0	0.6
2	1.3 ± 0.7	1.2	0	0	0	0	1.2
3	-4.4 ± 2.5	-3.6	0	0.6	0	0	-3.0
4	-0.3 ± 0.5	0	0	-0.5	$0.5^{b)}$	0	0.0
5	1.4 ± 0.5	0	0	$1.4^{a)}$	0	0	1.4
6	-0.2 ± 0.3	0	0	-0.3	$0.3^{b)}$	0	0.0
7	-0.4 ± 0.2	0	0	0	0	-0.3	-0.3
8	0.9 ± 0.3	0	0	$0.9^{a)}$	0	0	0.9
9	6.9 ± 0.7	$6.9^{a)}$	0	0	0	0	6.9
10	-5.5 ± 0.7	-10.0	4.0	0	0	0	-6.0

$^{a)}$ Input. $^{b)}$ Estimate based on the limit $N_c \to \infty$.

Ecker, Gasser, Pich, de Rafael ('89)

CHPT 3 – Summary

- The finite, analytically nontrivial part of the one loop integrals automatically generates the correct imaginary parts, as required by unitarity.
- Effective quantum field theory is a systematic method to generate a perturbative solution of dispersion relations.
- In some cases it may be convenient to generate a better approximation to the exact solution of the dispersion relation, and merge that with CHPT. If done properly the combination of the two approaches is very powerful.
- Dispersion relations offer a perfect framework to take into account the contributions of resonances: these give a good quantitative explanation of the values of the low energy constants.