Introduction to Chiral Perturbation Theory – II

One loop: renormalization

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Why go beyond $O(p^2)$? Why loops?

- Why not? Chiral Symmetry forbids $O(p^0)$ interactions between pions, but allows for all higher orders.

- Unitarity requires that if an amplitude at order p^2 is purely real, at order p^4 its imaginary part is nonzero.
  Take the $\pi\pi$ scattering amplitude. The elastic unitarity relation for the partial waves t_ℓ^I of isospin I and angular momentum ℓ reads:

$$Im\, t_\ell^I = \sqrt{1 - \frac{4M_\pi^2}{s}} |t_\ell^I|^2 \quad . \tag{1}$$

- The correct imaginary parts are generated automatically by loops.

- The divergences occuring in the loops can be disposed of just like in a renormalizable field theory.

Effective field theory

The method of effective quantum field theory provides a rigorous framework to compute Green functions that respect all the good properties we require: symmetry, analyticity, unitarity.

The method yields a systematic expansion of the Green functions in powers of momenta and quark masses.

In the following two lectures I will discuss in detail how this works when you consider loops:

- In lecture 2 I will consider the divergent part of the loops and discuss how the renormalization program works.
- In lecture 3 I will consider the finite, analytically nontrivial part of the loops and discuss in detail its physical meaning.

Outline of Lecture 2

- One–loop graphs in the scalar form factor of the pion.
- Renormalization of the scalar form factor to one loop.
- IR divergences: chiral logs.
- Generating functional.
- Chiral invariant renormalization.

Scalar form factor of the pion

$$\langle \pi^i(p_1)\pi^j(p_2)|\hat{m}(\bar{u}u+\bar{d}d)|0\rangle =: \delta^{ij}\Gamma(t) \quad , \qquad t=(p_1+p_2)^2 \quad ,$$

At tree level:

$$\Gamma(t) = 2\hat{m}B = M_\pi^2 + O(p^4) \quad ,$$

in agreement with the Feynman–Hellman theorem:
the expectation value of the perturbation in an eigenstate of the total Hamiltonian determines the derivative of the energy level with respect to the strength of the perturbation:

$$\hat{m}\frac{\partial M_\pi^2}{\partial \hat{m}} = \langle \pi|\hat{m}\bar{q}q|\pi\rangle = \Gamma(0) \quad .$$

This matrix element is relevant for the decay $h \to \pi\pi$, which, for a light higgs wold have been the main decay mode.

One loop graphs

$$\sim \int \frac{d^4 l}{(2\pi)^4} \frac{\{p^2, p\cdot l, l^2\}}{(l^2 - M^2)((p-l)^2 - M^2)} \quad , \qquad p = p_1 + p_2$$

$$\sim \underbrace{\int \frac{d^4 l}{(2\pi)^4} \frac{1}{(l^2 - M^2)}}_{T(M^2)} + p^2 \underbrace{\int \frac{d^4 l}{(2\pi)^4} \frac{1}{(l^2 - M^2)((p-l)^2 - M^2)}}_{J(p^2)}$$

$$\begin{aligned} T(M^2) &= a + bM^2 + \bar{T}(M^2) \\ J(t) &= J(0) + \bar{J}(t) \end{aligned}$$

$\bar{T}(M^2)$ and $\bar{J}(t)$ are finite (show this explicitly by deriving a sufficient number of times).

$$\Gamma(t) \sim M^2 \left[1 + \underbrace{bM^2 + tJ(0)}_{\text{divergent part}} + \bar{T}(M^2) + \bar{J}(t) \right]$$

Counterterms

$$\mathcal{L}_2 \sim \langle \chi_+ \rangle \sim M^2 s \phi^2$$

$$\mathcal{L}_4 \sim \underbrace{\langle \chi_+ \rangle^2 \sim M^4 s \phi^2}, \underbrace{\langle \chi_+ \rangle \langle u_\mu u^\mu \rangle \sim M^2 s \partial_\mu \phi^i \partial^\mu \phi^i}$$

$$\Downarrow \qquad\qquad \Downarrow$$

$$\Gamma(t) \sim M^4 \quad , \quad M^2 t$$

To remove the divergences one only needs to properly define the couplings in the lagrangian at order $O(p^4)$.

Quote from Weinberg's book on QFT, vol. I:

"(...) as long as we include every one of the infinite number of interactions allowed by symmetries, the so–called non–renormalizable theories are actually just as renormalizable as renormalizable theories."

Chiral logarithms

The square root of the linear term in the Taylor expansion of the form factor gives a measure of the physical extension ("radius") of the particle in question:

$$\Gamma(t) = \Gamma(0)\left[1 + \frac{1}{6}\langle r^2\rangle_S^\pi t + O(t^2)\right] \quad .$$

$$\langle r^2\rangle_S^\pi \sim J(0) = \int \frac{d^4l}{(2\pi)^4}\frac{1}{(l^2 - M^2)^2} \sim \ln\frac{M^2}{\Lambda^2} \quad .$$

The integral is UV divergent, but also IR divergent if $M \to 0$. While UV divergences are removed by counterterms, IR divergences are physical and are not cancelled by other contributions:

$$\lim_{M^2\to 0} \langle r^2\rangle_S^\pi \sim \ln M^2 \quad ,$$

the extension of the cloud of pions surrounding a pion (or any other hadron) goes to infinity if pions become massless (Li and Pagels '72).

Exercises:

Calculate $\Gamma(0)$ and $\langle r^2\rangle_S^\pi$ to one loop.

1. Draw all graphs that contribute.

2. Calculate the graphs after a Taylor expansion.

3. Calculate the contribution of the relevant counterterms:

$$\frac{l_3}{16}\langle\chi_+\rangle^2 \quad , \frac{il_4}{4}\langle u_\mu \chi_-^\mu\rangle \quad .$$

   What is the divergent part of l_3 and l_4?

4. Use two different parametrizations to do the calculations: Exponential: $U = \exp(i\phi/F)$;
   σ model: $U = \sigma + i\frac{\phi}{F}, \sigma^2 + \frac{\phi^2}{F^2} = 1$.
   $\phi = \phi^i \tau^i$.

Chiral symmetry

To remove the divergent part in $\Gamma(t)$ we have to fix the divergent part of chiral–invariant operator of order $O(p^4)$, like $\langle u_\mu u^\mu \rangle \langle \chi_+ \rangle$.

$$\langle u_\mu u^\mu \rangle \langle \chi_+ \rangle \sim M^2 \phi^2 \partial_\mu \phi^4 \partial^\mu \phi^6 + \ldots$$

Chiral symmetry implies that after calculating the divergent part of $\Gamma(s)$ I also know the divergent part of the $6\pi \to 6\pi$ scattering amplitude.

Questions:

1. Do we have a proof that quantum effects do not introduce violations of the chiral symmetry? Or that one can build a chiral invariant generating functional only with a path integral over a chiral invariant classical action?

2. Is there a tool that allows one to calculate the divergences keeping chiral invariance explicit in every step of the calculation?

Generating functional

- Consider a system with a spontaneously broken symmetry G. Define the generating functional as:

$$e^{iZ\{f\}} = \sum_{n=0} \frac{i^n}{n!} \int dx_1 \dots dx_n f^{i_1}_{\mu_1} \dots f^{in}_{\mu n} \langle 0 | T J^{\mu_1}_{i_1} \dots J^{\mu n}_{in} | 0 \rangle \quad ,$$

where J^i_μ are the Noether's currents associated to the spontaneously broken symmetry G of the system, and f^μ_i external fields coupled to them.

- The generating functional is invariant under gauge transformations of the external fields:

$$Z\{T(g)f\} = Z\{f\} \quad ,$$

where:

$$T(g)f_\mu = D(g_x)f_\mu(x)D^{-1}(g_x) - i\partial_\mu D(g_x)D^{-1}(g_x)$$

- What is the most general way of constructing an invariant generating functional out of a path integral over the Goldstone boson degrees of freedom?

Leutwyler's theorem

For Lorentz–invariant theories in 4 dimensions, a path integral constructed with gauge–invariant lagrangians is a necessary and sufficient condition to obtain a gauge–invariant generating functional.

The theorem also includes the case in which the symmetry is anomalous (like the $U(1)_A$ symmetry), and the case in which the symmetry is spontaneously broken.

Background field method – I

Take a $O(N)$ symmetric lagrangian of scalar fields:

$$\mathcal{L} = \mathcal{L}_0 + \sum_{n=1}^{\infty} \hbar^n \mathcal{L}_n$$

$$\mathcal{L}_0 = \frac{1}{2}\left(\partial_\mu \phi^i \partial^\mu \phi^i - M^2 \phi^i \phi^i\right) - \frac{g}{4}\left(\phi^i \phi^i\right) - \phi^i f^i$$

The generating functional is constructed via the path integral:

$$e^{iZ\{f\}/\hbar} = N \int [d\phi] e^{iS/\hbar} \quad , \qquad S = \int dx \mathcal{L}$$

$$Z = Z_0 + \hbar Z_1 + \hbar^2 Z_2 + O(\hbar^3) \quad , Z\{0\} = 0$$

The classical equations of motion for the field ϕ^i:

$$\frac{\delta S_0}{\delta \phi^i} = 0 \qquad \Rightarrow \qquad (M^2 + \Box)\bar{\phi}^i + g\bar{\phi}^2\bar{\phi}^i + f^i = 0$$

Background field method – II

Shift in the integration variable:

$$\phi^i = \bar{\phi}^i + \xi^i \ , \qquad [d\phi] = [d\xi] \ , \qquad \xi = O(\hbar^{1/2})$$

After the shift the path integral becomes:

$$e^{iZ\{f\}/\hbar} = N e^{i\bar{S}/\hbar} \int [d\xi] \exp\left\{ \frac{i}{\hbar} \int dx \left[\frac{1}{2} \xi^i D_{ij} \xi^j + O(\xi^3) \right] \right\}$$

$$\begin{aligned} D_{ij} &= -\Box \delta_{ij} + \sigma_{ij} \\ \sigma_{ij} &= -\left(M^2 + g\bar{\phi}^2 \right) \delta_{ij} - 2g\bar{\phi}^i \bar{\phi}^j \end{aligned}$$

The first quantum correction to the generating functional has a compact explicit expression:

$$Z_1 = \int dx \left[\frac{i}{2} \ln \left(D_{ij} {D_0}^{-1}_{ij} \right) + \mathcal{L}_1 \right] \ ,$$

where $D_0 = D_{|f=0}$.

Heat kernel

The UV divergent part of $\ln(DD_0^{-1})$ also has a compact explicit expression, obtained with the heat kernel formalism:

Given a differential operator of the form:

$$D^2 = -d^2 + \sigma \quad , \qquad d_\mu = \partial_\mu + \gamma_\mu \quad ,$$

its divergent part has the following expression:

$$\int dx \ln\left(D_{ij} D_{0}{}_{ij}^{-1}\right) = \frac{i}{(4\pi)^2(d-4)} \int dx \left[\frac{1}{6}\gamma_{\mu\nu}\gamma^{\mu\nu} + \sigma^2\right] + \ldots$$

In the case of the $O(N)$ theory that we are considering $\gamma_\mu = 0$ and:

$$\sigma^2 = 2(N+2)gM^2\phi^2 + (N+8)g^2\phi^4$$

Application to CHPT

$$U = e^{i\phi/F} \quad , \qquad \phi = \bar{\phi} + \xi$$

It is convenient to expand around the classical solution as follows:

$$U = \bar{u} e^{i\xi/F} \bar{u}$$

Transformation properties:

$$\begin{aligned} U &\rightarrow g_R U g_L^\dagger = g_R u h^\dagger h u g_L^\dagger \\ e^{i\xi/F} &\rightarrow h e^{i\xi/F} h^\dagger \end{aligned}$$

Expansion of $\mathcal{L}_2$ around the classical solution:

$$\begin{aligned} \int dx \mathcal{L}_2 &= \int dx \frac{F^2}{4} \langle u_\mu u^\mu + \chi_+ \rangle \\ &= \int dx \bar{\mathcal{L}}_2 + \int dx \frac{1}{2} \xi^i \Delta_{ij} \xi^j + O(\xi^3) \end{aligned}$$

Exercises

1. Calculate Δ_{ij}.

Solution:

$$
\begin{aligned}
\Delta &= -d^2 + \sigma \\
d_{\mu,\,ij} &= \partial_\mu \delta_{ij} + \gamma_{\mu,\,ij} \\
\gamma_{\mu,\,ij} &= -\frac{1}{2}\langle [\lambda_i, \lambda_j] \Gamma_\mu \rangle \\
\Gamma_\mu &= \frac{1}{2}\left\{ u^\dagger(\partial_\mu - i r_\mu) u + u(\partial_\mu - i l_\mu) u^\dagger \right\} \\
\sigma_{ij} &= -\frac{1}{8}\langle [u_\mu, \lambda_i][\lambda_j, u^\mu] + \{\lambda_i, \lambda_j\}\chi_+ \rangle
\end{aligned}
$$

2. Calculate the divergent part of $\ln \Delta\Delta_0^{-1}$.

CHPT 2 – Summary

- Including loops is necessary if we want to respect unitarity.
- The UV divergences encountered in loop integrals can be removed according to standard renormalization methods.
- Some loop integrals have also an IR singular behaviour which has a very clear physical meaning, and again shows the necessity of taking loop effects into account.
- As an illustration of these concepts I have considered the scalar form factor of the pion at one loop level.
- I have introduced the generating functional of Green functions of Noether currents. This is a central object in the theoretical analysis of systems that undergo a spontaneous symmetry breaking.
- Leutwyler has proved that doing a path integral over an effective lagrangian is the most general way to construct a symmetric generating functional.
- Finally, I have introduced some technical tools (background field method and heat kernel) to perform an explicitly chiral invariant renormalization of the theory.