Effective Field Theories

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Abstract. These lectures introduce some of the basic ideas of effective field theories. The topics discussed include: relevant and irrelevant operators and scaling, renormalization in effective field theories, decoupling of heavy particles, power counting, and naive dimensional analysis. Effective Lagrangians are used to study the $\Delta S = 2$ weak interactions and chiral perturbation theory.

1 Introduction

An important idea that is implicit in all descriptions of physical phenomena is that of an effective theory. The basic premise of effective theories is that dynamics at low energies (or large distances) does not depend on the details of the dynamics at high energies (or short distances). As a result, low energy physics can be described using an effective Lagrangian that contains only a few degrees of freedom, ignoring additional degrees of freedom present at higher energies. One of the main purposes of these lectures is to make these qualitative statements quantitative.

First a simple example: The energy levels of the Hydrogen atom are calculated in textbooks using the Schrödinger equation for an electron bound to a proton by a Coulomb potential. To a good approximation, the only properties of the proton that are relevant for the computation are its mass and charge. An understanding of the quark substructure of the proton (let alone quantum gravity) is not necessary to compute the energy levels of the Hydrogen states. This is true provided an answer which has some theoretical uncertainty is sufficient. A more accurate calculation of the energy levels, for example including the hyperfine splitting, requires that we also know that the proton has spin-1/2, and a magnetic moment of 2.793 nuclear magnetons. An even more accurate calculation of the energy levels requires some knowledge of the proton charge radius, etc. More details of the proton structure are needed as we require a more accurate answer for the energy levels.

When we discuss effective theories, we will frequently talk about momentum scales characteristic of a given problem. The typical length scale characteristic of the Hydrogen atom is the Bohr radius $a_0 = 1/(m_e \alpha)$, and the typical momentum scale is of order $\hbar/a_0 \sim 1/a_0 = m_e \alpha$, using units in which $\hbar = 1$. The typical energy scale characteristic of Hydrogen is the Rydberg $\sim m_e \alpha^2$, and the typical time scale is $1/(m_e \alpha^2)$. The Hydrogen atom is more complicated than

Lectures at the Schladming Winter School, March 1996, UCSD/PTH 96-04

many relativistic bound states because it has two characteristic scales, $m_e\alpha$ and $m_e\alpha^2$. We can now give a quantitative estimate of the error caused by neglected interactions: the energy levels of Hydrogen can be computed by ignoring all dynamics on momentum scales Λ much larger than $m_e\alpha$, with an error of order $m_e\alpha/\Lambda$. As the desired accuracy increases, the scale Λ of the interactions that can be ignored, also increases.

The relevant interactions in an effective theory also depend on the question being studied. In the Hydrogen atom, the energy levels can be computed to an accuracy $(m_e\alpha/M_W)^2$ while ignoring the weak interactions, but if we are interested in atomic parity violation, the weak interactions are the leading contribution since strong and electromagnetic interactions conserve parity. Atomic parity violation will still be a very small effect, because the weak scale is much larger than the atomic scale.

An effective field theory describes low energy physics in terms of a few parameters. These low energy parameters can be computed in terms of (hopefully fewer) parameters in a more fundamental high energy theory. This computation can be done explicitly when the high energy theory is weakly coupled. In QED, for example, one can predict low energy parameters such as the magnetic moment of the electron which can be used in the Schrödinger equation. If the high energy theory is strong coupled, as in QCD, one usually treats the low energy parameters (such as the magnetic moment of the proton) as free parameters that are fit to experiment. We will deal with both cases in these lectures when we study the Fermi theory of weak interactions, and chiral perturbation theory.

We have said that high energy dynamics can be ignored in the study of processes at low energies. The precise form of this statement is subtle. It is not true that parameters in the high energy theory do not affect the low energy dynamics in any way. The precise statement is that the only effect of the high energy theory is to modify coupling constants in the low energy theory, or to put symmetry constraints on the low energy theory. The energy levels of Hydrogen should not depend on the masses of heavy particles such as the top quark. This is not true: changing the top quark mass while keeping the electromagnetic coupling constant at high energies fixed, changes the electromagnetic coupling constant at low energies,

$$m_t \frac{d}{dm_t}\left(\frac{1}{\alpha}\right) = -\frac{1}{3\pi}. \tag{1}$$

The proton mass also depends on the top quark mass,

$$m_p \propto m_t^{2/27}. \tag{2}$$

Despite this dependence, the value of m_t is irrelevant for studying the Hydrogen atom. The reason is that α and m_p are parameters of the Schrödinger equation for the Hydrogen atom. Fitting to the observed energy levels determines the value of α at low energies to be 1/137.036, and the proton mass to be 938.27 MeV. *The value of m_t is irrelevant for atomic physics if the Schrödinger equation is treated as a low energy theory whose parameters α, m_e, m_p are determined from* *low energy experiments*. The value of m_t *is* relevant if one studies how atomic physics changes as a function of m_t while keeping the high energy parameters constant.

High energy dynamics places non-trivial symmetry constraints on a low energy effective theory. An interesting example of such a constraint is the spin-statistics theorem. Non-relativistic quantum mechanics is a perfectly satisfactory theory, regardless of whether electrons are quantized using Bose, Fermi or Boltzmann statistics. However, a consistent relativistic formulation of the theory requires that electrons obey Fermi statistics, which is a constraint on non-relativistic quantum mechanics that follows from causality in quantum electrodynamics. The spin-statistics theorem is a statement about symmetry, and holds regardless of whether there is a simple connection between the high energy and low energy theories. In low energy QCD, the spin-statistics theorem implies that baryons are fermions and mesons are bosons.

The effective field theory technique is powerful precisely because one can compute low energy dynamics without any knowledge of the details of high energy interactions. This also has an unfortunate consequence – information about high energy interactions cannot be obtained using low energy measurements. Luckily, the last statement is not quite true. There are some vestiges of the high energy interactions in the symmetry constraints on the low energy theory, and in small corrections to low energy dynamics. Thus high precision low energy experiments can be used to probe high energy dynamics, and provide an alternative to high energy experiments.

2 The Renormalization Group and Scaling

Effective actions were used by Wilson, Fisher, and Kadanoff to study critical phenomena in condensed matter systems, and many of the ideas of effective theories were developed in this context. Consider the classic example of an Ising spin system on a square lattice with lattice spacing a, in an external magnetic field. The partition function is

$$Z = \sum_{s_i=\pm} \exp\left[K \sum_{\langle ij \rangle} s_i s_j + B \sum_i s_i\right], \tag{3}$$

where $\langle ij \rangle$ is a sum over nearest neighbors. At a second order phase transition, the correlation length ξ of the system becomes infinite. Intuitively, one expects that the properties of the Ising system near its critical point should not depend on the details of the system on the scale of the lattice spacing a. It took a decade of inspired work to convert this intuitive statement into equations.

To study the Ising model at its critical point, it is not necessary to retain all the information in the partition function (3). The idea of Kadanoff was to reduce the degrees of freedom by introducing a block spin. Divide the lattice of spins into blocks of four spins each (see Fig. 1). The block spin s' is defined for each block to be the average of the spins at the four corners of the block

$$s'_B = \frac{s_{B1} + s_{B2} + s_{B3} + s_{B4}}{4}, \tag{4}$$

where s'_B is the block spin for the block B, and s_{Bi} are the original spins at the four corners of block B . One can write the partition function Z as

$$\begin{aligned} Z &= \sum_{s_i=\pm} \exp\left[K \sum_{\langle ij \rangle} s_i s_j + B \sum_i s_i \right], \\ &= \int \sum_{s_i=\pm} \prod_B ds'_B \delta\left(s'_B - \frac{s_{B1} + s_{B2} + s_{B3} + s_{B4}}{4} \right) \\ &\qquad \times \exp\left[K \sum_{\langle ij \rangle} s_i s_j + B \sum_i s_i \right], \end{aligned} \tag{5}$$

where the product is over all blocks B. Performing the sum over s_i leads to

$$Z = \int \prod_B ds'_B \; e^{S\left[s'_B\right]}, \tag{6}$$

where

$$e^{S\left[s'_B\right]} = \sum_{s_i=\pm} \prod_B \delta\left(s'_B - \frac{s_{B1} + s_{B2} + s_{B3} + s_{B4}}{4} \right) \exp\left[K \sum_{\langle ij \rangle} s_i s_j + B \sum_i s_i \right]. \tag{7}$$

This is an exact renormalization group transformation (called a Kadanoff block spin transformation), that expresses the partition function in terms of a new action $S\left[s'_B\right]$ with a quarter the number of degrees of freedom and twice the lattice spacing as the original action.

The new variable s'_B is the average of four spins with values ± 1 (4), and can have the values ± 1, $\pm 1/2$, 0. The new action $S\left[s'_B\right]$ is much more complicated than the original action $K \sum_{\langle ij \rangle} s_i s_j + B \sum_i s_i$, but can in principle be computed using (7). Now repeat the block spin transformation an infinite number of times. At each step, the number of degrees of freedom is reduced by four. The block spin s eventually becomes a continuous variable, which is usually denoted by ϕ. The only problem is that the action becomes more and more complicated, and more and more non-local at each step. This was the difficulty that prevented the Kadanoff block spin method from being used for a long time. What is needed is a way to truncate the effective action in a systematic and controlled manner. It is also important (particularly in field theory) to have an effective action that is local.

A technical difficulty with the block spin transformation is that it is discrete; it is much easier to deal with continuous transformations. Wilson suggested studying the Ising model in momentum space. The variables in momentum space are Fourier transformed variables $s(k)$, where the momentum k is restricted to the Brillouin zone, $|k| \le k_{\text{max}} = \pi/a$. The Kadanoff block spin transformation

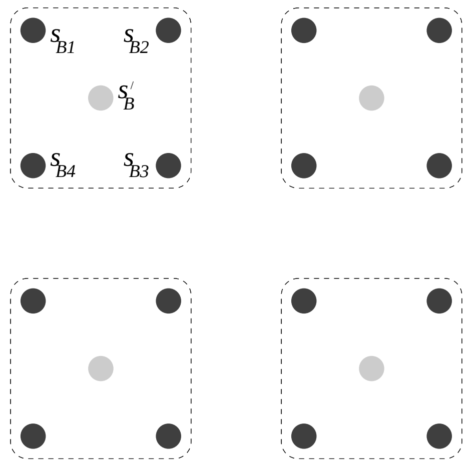

Fig. 1. The Kadanoff block spin transformation. The four spins at the corner of each block are replaced by an average spin at the center

takes $a \to 2a \to 4a$, etc., which corresponds to letting $k_{\max} \to k_{\max}/2 \to k_{\max}/4$, etc. k is a continuous variable, so instead consider decreasing $k_{\max}$ continuously. The partition function transformation formula becomes (using ϕ, Λ instead of $s, k_{\max}$)

$$Z = \int_{k \le \Lambda} \mathcal{D}\phi_k \; e^{-S_\Lambda[\phi_k]} = \int_{k \le \Lambda'} \mathcal{D}\phi'_k \; e^{-S'_{\Lambda'}[\phi'_k]}, \tag{8}$$

which is the momentum space analog of (7). The original action $S_\Lambda\,[\phi_k]$ contains all momentum modes up to some maximum value Λ, whereas the new action $S'_{\Lambda'}\,[\phi'_k]$ contains momentum modes up to Λ', where $\Lambda' < \Lambda$. The idea is to take $\Lambda' = \Lambda - \delta\Lambda$ infinitesimally different from Λ, so that S' is infinitesimally different from S. In the limit $\delta\Lambda \to 0$, the effective action satisfies a differential equation,

$$\frac{\partial S_\Lambda}{\partial \Lambda} = F\left[S_\Lambda\right], \tag{9}$$

where F is a functional of the action that can be determined from (8). Think of S_Λ as a set of actions, so that (9) gives the change in action as a function of cutoff. This is usually referred to as the renormalization group flow of the action. The action can be written as

$$\sum_i c_i \; \mathcal{O}_i, \tag{10}$$

in terms of coefficients c_i and some operator basis $\mathcal{O}_i$. The differential equation (9) is then a differential equation for the couplings,

$$\frac{\partial c_i}{\partial \Lambda} = F\left[\{c_i\}\right], \tag{11}$$

so that the renormalization group equation gives a flow in coupling constant space.

Finally, an extremely important point: the renormalization group equations are obtained by integrating out variables with momenta between $\Lambda - \delta\Lambda$ and Λ. There is both an infrared ($\Lambda - \delta\Lambda$) and ultraviolet (Λ) cutoff on the integration, so the renormalization group equations are local and non-singular.

Free Field Theory

To explicitly study the renormalization group equations, it is helpful to consider first a free scalar field in D dimensions, with action

$$S = \int d^D x \; \frac{1}{2}\partial_\mu \phi \partial^\mu \phi - \frac{1}{2} m^2 \phi^2. \tag{12}$$

The action S is dimensionless, so the dimension of $\phi(x)$ is determined from the kinetic term to be $[\phi] = (D-2)/2$, and the dimension of m^2 is $\left[m^2\right] = 2$.

We would like to study correlation functions

$$G_n\left(x_1, \ldots, x_n\right) = \left\langle \phi\left(x_1\right) \ldots \phi\left(x_n\right)\right\rangle_S, \tag{13}$$

computed using the action S at long distances (i.e. low momentum). It is convenient to make the change of variables,

$$x = s x', \qquad \phi(x) = s^{(2-D)/2} \phi'(x'), \tag{14}$$

so that

$$S' = \int d^D x' \; \frac{1}{2}\partial'_\mu \phi'(x') \partial'^\mu \phi'(x') - \frac{1}{2} m^2 s^2 \phi'(x')^2. \tag{15}$$

Correlation functions of $\phi(x)$ with action S are related to correlation functions of $\phi'(x')$ with action S' by

$$\left\langle \phi(s x_1) \ldots \phi(s x_n)\right\rangle_S = s^{n(2-D)/2} \left\langle \phi'(x_1) \ldots \phi'(x_n)\right\rangle_{S'}. \tag{16}$$

The long distance (low momentum) limit of correlation functions with action S is obtained by letting $s \to \infty$. These can be obtained by studying correlation functions at a fixed distance (fixed momentum) of the action S' as $s \to \infty$. The mass term in S' is $s^2 m^2$. Clearly, in the limit $s \to \infty$, the mass term becomes more and more important. The mass m^2 is called a relevant coupling, and dominates the long distance behavior of the correlation functions. Equivalently, ϕ^2 is called a relevant operator.

What about integrating out momentum shells to lower the cutoff? The original action had an implicit cutoff Λ. We should have integrated out momentum modes and lowered the cutoff to Λ/s, so that the rescaling transformation (14) restored the cutoff to its original value Λ. In free field theory, there is no coupling between the different modes. Thus integrating out a momentum shell produces an overall multiplicative factor in Z, i.e. an additive constant to the effective action. This shifts the cosmological constant, but does not affect the dynamics of ϕ.

Interactions

Next, add the interaction terms $\lambda\phi^4/4! + \lambda_6\phi^6/6!$ to the free Lagrangian. The dimensions of the coefficients are $[\lambda] = 0$, $[\lambda_6] = -2$. Rescaling the field as before (and ignoring, for the moment, integrating out momenta between Λ and Λ/s) gives the rescaled action

$$S' = \int d^D x' \; \frac{1}{2}\partial'_\mu \phi' \partial'^\mu \phi' - \frac{1}{2} m^2 s^2 \phi'^2 - \frac{\lambda}{4!}\phi'^4 - \frac{\lambda_6}{6! s^2}\phi'^6, \tag{17}$$

with the implicit rescaled cutoff Λ. In the limit $s \to \infty$, the ϕ^6 term vanishes as $1/s^2$, so ϕ^6 is called a irrelevant operator, and λ_6 is called an irrelevant coupling. The ϕ^4 term remains unchanged under rescaling, so it is equally important at all length scales. For this reason, ϕ^4 is known as a marginal operator, and λ_4 is called a marginal coupling.

In effective field theories, we are usually interested in studying the dynamics at low energies, but not exactly at zero energy. For example, we will be studying hadron dynamics at a scale of order 1 GeV, which is much smaller than the weak interaction scale of $M_W \sim 80$ GeV. In this case, the scale factor s between the weak and strong scales is $s = 80$, which is large but finite. Irrelevant operators (despite their name) then produce small corrections. In our scalar field theory example, the ϕ^6 operator produces corrections of order $1/s^2$, ϕ^8 produces corrections of order $1/s^4$, etc.

The alert reader will have noticed that the above results follow from dimensional analysis. The counting can trivially be generalized to an arbitrary Lagrangian:

1. Determine the canonical dimensions of the fields using the kinetic term.
2. Determine the (mass) dimensions of all the couplings.
3. Terms with a coupling constant with dimension d scale as s^d, so that the coupling is relevant, irrelevant or marginal depending on whether $d > 0$, $d < 0$ or $d = 0$. Equivalently, the operator is relevant, irrelevant, or marginal depending on whether its dimension is less than, greater than, or equal to the space-time dimension D.
4. To include all corrections up to order $1/s^r$, one should include all operators with dimension $\leq D + r$, i.e. all terms with coefficients of dimension $\geq -r$.

Let us now turn on the interactions. The first problem is that there are divergences in the quantum theory. These are handled by the standard regularization and renormalization procedure. In scalar field theory, for example, one can introduce a cutoff Λ to regulate the functional integral. In the presence of a cutoff, the relation (16) between correlation functions becomes

$$G_n\left(\{sx\}; m^2, \lambda_4, \lambda_6; \Lambda\right) = s^{n(2-D)/2} G_n\left(\{x\}; s^2 m^2, \lambda_4, s^{-2}\lambda_6; s\Lambda\right), \tag{18}$$

where we have explicitly included the cutoff dependence, and $\{x\}$ denotes $x_1, \ldots, x_n$. The left hand side is the desired correlation function. To get the infrared behavior of the left hand side, we need to replace the cutoff $s\Lambda$ by Λ on the right hand side. This is the hard part of the calculation which we have ignored so far, but one that you have all seen before – it is the standard renormalization group equation of quantum field theory:

$$\left[\Lambda\frac{\partial}{\partial\Lambda}+\beta_i\frac{\partial}{\partial c_i}+n\gamma_\phi\right]G=0. \tag{19}$$

Here c_i are the couplings, m^2, λ, λ_6, etc., and the β-functions β_i and anomalous dimension γ_ϕ are functions of c_i. The solution of this equation is also standard. Define running couplings which are solutions of the differential equation

$$\Lambda\frac{\partial}{\partial\Lambda}c_i\left(\Lambda\right)=\beta_i\left(c_i\left(\Lambda\right)\right). \tag{20}$$

Then

$$G_n\left(\{x\}, c_i\left(\Lambda_1\right), \Lambda_1\right)=e^{-n\int_{\Lambda_1}^{\Lambda_2}\gamma(\Lambda)d\log\Lambda}\,G_n\left(\{x\}, c_i\left(\Lambda_2\right), \Lambda_2\right). \tag{21}$$

Equation (18) can be combined with (21) to give

$$G\left(\{sx\}, c_i\left(\Lambda\right), \Lambda\right)=s^{n(2-D)/2}e^{-n\int_{\Lambda}^{\Lambda/s}\gamma\left(\Lambda'\right)d\log\Lambda'}G\left(\{x\}, s^{d_i}c_i\left(\Lambda/s\right), \Lambda\right) \tag{22}$$

where d_i is the dimension of coupling c_i. The only difference from (16) is the exponential prefactor, and that c_i is now the running coupling at Λ/s.

It is instructive to look at some examples of renormalization group equations before continuing with our general analysis. In QCD, the renormalization group equation for the dimensionless coupling constant g is

$$\mu\frac{\partial g}{\partial\mu}=-\frac{g^3}{16\pi^2}b_0+\mathcal{O}\left(g^5\right), \tag{23}$$

where $b_0=11N_c/3-2N_f/3$, N_c is the number of colors, and N_f is the number of flavors. Equation (23) is the β-function in any mass independent scheme, such as $\overline{MS}$, and μ is the dimensionful parameter that plays the role of Λ in such a scheme. The anomalous dimension for a field (such as the fermion field ψ) has the form

$$\gamma_\psi=\gamma_\psi^0\frac{g^2}{16\pi^2}+\mathcal{O}\left(g^4\right). \tag{24}$$

Other operators added to the Lagrangian, such as four-Fermi weak decay operators have renormalization group equations of the form

$$\mu\frac{\partial c_i}{\partial\mu}=\gamma_{ij}^0\frac{g^2}{16\pi^2}c_j+\mathcal{O}\left(g^4\right). \tag{25}$$

Let us neglect operator mixing for simplicity, so that γ_{ij} is a diagonal matrix, with elements $\gamma_i\delta_{ij}$. The solutions of the renormalization group equations are[1]

[1] The general case where γ_{ij} is not diagonal can be solved by finding the eigenvalues and eigenvectors of γ_{ij}.

$$
\begin{aligned}
\frac{1}{\alpha_s(\mu_1)} - \frac{1}{\alpha_s(\mu_2)} &= \frac{b_0}{2\pi}\log\frac{\mu_1}{\mu_2}, \\
\frac{c_i(\mu_1)}{c_i(\mu_2)} &= \left[\frac{\alpha_s(\mu_1)}{\alpha_s(\mu_1)}\right]^{-\gamma_i^0/2b_0}, \\
\exp\left[-\int_{\mu_2}^{\mu_1}\gamma_\psi(\mu)\, d\log\mu\right] &= \left[\frac{\alpha_s(\mu_1)}{\alpha_s(\mu_2)}\right]^{\gamma_\psi^0/2b_0}.
\end{aligned}
\tag{26}
$$

These equations show that the quantum scaling behavior differs from the classical one by logarithms. A more interesting case is a field theory in which the β-function for a dimensionless coupling such as g has the form shown in Fig. 2. The renormalization group equation for g, (23), shows that $g \to g^*$ as $\mu \to 0$ if g starts out in some neighborhood of g^*. For this reason g^* is known as an attractive (or stable) infrared fixed point for g. In this case, the renormalization group scaling in the limit $s \to \infty$ is dominated by $g \approx g^*$, so that

$$
\begin{aligned}
\mu\frac{\partial c_i}{\partial\mu} &= \gamma_{ij}(g^*)\, c_j, \\
\gamma_\phi(\mu) &\to \gamma_\phi(g^*).
\end{aligned}
\tag{27}
$$

Denote the fixed point values $\gamma_{ij}(g^*)$ and $\gamma(g^*)$ by γ_{ij}^* and γ^*, and assume for simplicity that $\gamma_{ij}^* = \gamma_i^*\delta_{ij}$. (As above, the general case is solved by finding the eigenvalues an eigenvectors of γ_{ij}^*.) The solutions of the renormalization group equations in the neighborhood of the fixed point become

$$
\begin{aligned}
\frac{c_i(\mu_1)}{c_i(\mu_2)} &= \left[\frac{\mu_1}{\mu_2}\right]^{\gamma_i^*}, \\
\exp\left[-\int_{\mu_2}^{\mu_1}\gamma_\phi(\mu)\, d\log\mu\right] &= \left[\frac{\mu_1}{\mu_2}\right]^{-\gamma^*}.
\end{aligned}
\tag{28}
$$

so that (22) becomes

$$
G_n(\{sx\}, c_i(\Lambda), \Lambda) = s^{n(2-D)/2} s^{n\gamma^*} G_n\left(\{x\}, s^{d_i-\gamma_i^*} c_i(\Lambda/s), \Lambda\right) \tag{29}
$$

This equation shows that scale invariance is recovered in the quantum theory at an infrared stable fixed point, but the quantum dimensions of fields and operators differ from their classical values. Operators now have dimension $D-d_i+\gamma_i^*$, their coefficients have dimension $d_i-\gamma_i^*$, and fields have dimension $(D-2)/2-\gamma^*$. This is the reason why γ, γ_{ij} are called anomalous dimensions. The classification into relevant, irrelevant and marginal operators is the same as before, except that one should use the quantum dimension of the operator which includes the anomalous dimension.

In weakly coupled theories, operator anomalous dimensions can be computed in perturbation theory, and are small. Thus quantum corrections cannot affect which operators are relevant or irrelevant, since the classical dimensions of operators are restricted to be integers or half-integers. The only effect of quantum

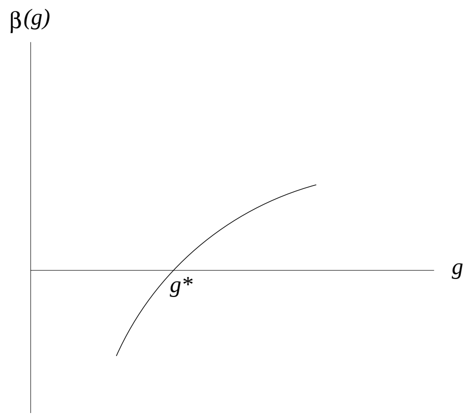

Fig. 2. An infrared stable fixed point of the β function

corrections is to turn marginal operators into relevant or irrelevant operators, depending on whether their anomalous dimension is negative or positive. In strongly coupled theories, more interesting effects can occur. For example, in walking technicolor theories it is believed that a composite operator $\overline{\psi}\psi$ with classical dimension 3 behaves in the quantum theory as a scalar field with dimension 1, i.e. $\overline{\psi}\psi$ has anomalous dimension -2. A solvable example of this kind exists in two dimensions. The two dimensional Thirring model with a fundamental fermion field

$$\mathcal{L} = \overline{\psi}\left(i\partial\!\!\!/ - m\right)\psi - \frac{1}{2}g\left(\overline{\psi}\gamma^\mu\psi\right)^2, \tag{30}$$

is dual to the sine-Gordon model with a fundamental scalar field

$$\mathcal{L} = \frac{1}{2}\partial_\mu\phi\partial^\mu\phi + \frac{\alpha}{\beta^2}\cos\beta\phi, \tag{31}$$

where the coupling constants g and β are related by

$$\frac{\beta^2}{4\pi} = \frac{1}{1+g/\pi}. \tag{32}$$

The fermion of the Thirring model is the sine-Gordon soliton, and the boson of the sine-Gordon model is a fermion-antifermion bound state in the Thirring model. The mapping (32) shows that the strongly coupled sine-Gordon model with $\beta^2 \approx 4\pi$ can be mapped onto a weakly coupled Thirring model with $g \approx 0$. There are two alternate descriptions of the same theory: (a) A strongly interacting boson theory with large anomalous dimensions[2] (b) A weakly interacting

[2] For example, the operator $\cos\beta\phi$ gets mapped to the Fermion mass term $\overline{\psi}\psi$. In two dimensions, the canonical dimensions of $\cos\beta\phi$ and $\overline{\psi}\psi$ are zero and one, respectively.

fermion theory with small anomalous dimensions. Formally, both descriptions are identical, but clearly (b) is better for doing practical calculations.

The scaling dimensions of fields was determined from the free Lagrangian. That is because one assumes that the effective Lagrangian can be written as a weakly coupled field theory in terms of correctly chosen degrees of freedom at low energies. If the degrees of freedom are strongly coupled, the scaling dimension of the fields may change from their canonical value, as we saw in the sine-Gordon model at $\beta^2 = 4\pi$. Often, the most difficult task in writing down an effective theory is *choosing the right degrees of freedom*. In the sine-Gordon model, it is better to use a weakly coupled soliton field ψ instead of the fundamental field ϕ if $\beta^2 \approx 4\pi$, i.e. the effective Lagrangian for the sine-Gordon model with $\beta^2 \approx 4\pi$ is the Thirring model. Low energy QCD is a weakly coupled theory when written in terms of pion fields, but not when written in terms of quark and gluon fields.[1] At low energies, the Goldstone boson fields scale with canonical dimension zero (they are like angles), which is different from $\overline{q}q$, which has dimension 3 in free field theory. There are many examples of this kind in condensed matter physics. For example, in Landau Fermi liquid theory, the degrees of freedom are weakly interacting quasiparticles, not the strongly interacting electrons.

SUMMARY

We can now summarize the results of this section.

1. Find a good set of variables to describe the dynamics.
2. Write down the effective action as a sum of operators, $\sum_i c_i \mathcal{O}_i$.
3. The scaling rule is that $c_i \to s^{d_i - \gamma_i} c_i$, where d_i is the naive dimension and γ_i is the anomalous dimension. The most important operators are those of lowest dimension. Hopefully, a good choice has been made in (1), so that the anomalous dimensions are small.
4. To include all corrections up to order $1/s^r$, one should include all operators with dimension $\leq D + r$, i.e. all terms with coefficients of dimension $\geq -r$.

There are a finite number of operators that contribute to a given order in $1/s$. In four dimensions, the dimensions of scalar, spinor and vector fields is

$$[\phi] = 1, \qquad [\psi] = 3/2, \qquad [A_\mu] = 1. \tag{33}$$

The allowed Lorentz invariant and gauge invariant operators of dimension ≤ 4 are $\phi^n, n \leq 4$, $\overline{\psi}\psi$, $\partial_\mu \phi \partial^\mu \phi$, $\overline{\psi} \not{D} \psi$, $\overline{\psi}\psi\phi$, $F_{\mu\nu} F^{\mu\nu}$.

[1] This is obvious in the large N_c limit, where one has a weakly interacting theory of mesons and baryons, with a coupling constant $1/N_c$.

3 Renormalizable Theories vs Effective Theories

Field theory textbooks argue that a quantum field theory should be renormalizable, i.e. that the Lagrangian contain only terms with dimension $\leq D$. Otherwise one needs an infinite number of counterterms, hence an infinite number of unknown parameters, and the theory has no predictive power.

An effective field theory Lagrangian contains an infinite number of terms. Let us write the Lagrangian in the form

$$\mathcal{L}_{\text{eft}} = \mathcal{L}_{\leq D} + \mathcal{L}_{D+1} + \mathcal{L}_{D+2} + \ldots , \tag{34}$$

where $\mathcal{L}_{\leq D}$ contains all terms with dimension $\leq D$, $\mathcal{L}_{D+1}$ contains terms with dimension $D+1$, $\mathcal{L}_{D+2}$ contains terms with dimension $D+2$, and so on. The usual renormalizable Lagrangian is just the first term, $\mathcal{L}_{\leq D}$. There are an infinite number of terms in $\mathcal{L}_{\text{eft}}$, but one still has *approximate predictive power*. The effective Lagrangian is used to compute processes at some scale Λ/s, where Λ is the scale of (possibly unknown) high energy interactions. One can compute with an error of $1/s$ by retaining only $\mathcal{L}_{\leq D}$. Furthermore, one can extend the approximation in a systematic way – to compute with an error of order $1/s^{r+1}$, one needs to retain terms up to $\mathcal{L}_{D+r}$. There are only a finite number of parameters to compute to a given order in $1/s$, so the theory has predictive power.

A non-renormalizable theory is just as good as a renormalizable theory for computations, provided one is satisfied with a finite accuracy.

The usual renormalizable field theory result is recovered if one takes the separation of scales $s \to \infty$. In this case, one can compute using a renormalizable Lagrangian $\mathcal{L}_{\leq D}$ with no errors. While exact computations are nice, they are irrelevant. Nobody knows the exact theory up to infinitely high energies. Thus any realistic calculation is done using an effective field theory. The standard "exact" textbook analysis of QED is really an approximate calculation in which terms suppressed by powers of $1/s$ have been neglected.

4 Two Simple Examples

We now consider two simple examples that illustrates the utility of the effective field theory method.

Rayleigh Scattering

The first example is Rayleigh scattering, the scattering of photons off atoms at low energies. Here low energies means energies small enough that one does not excite the internal states of the atom, or cause it to ionize. The atom can be treated as a particle of mass M, interacting with the electromagnetic field. Let $\psi(x)$ denote a field operator that creates an atom at the point x. Then the effective Lagrangian for the atom is

$$\mathcal{L} = \psi^\dagger \left(i\partial_t - \frac{p^2}{2M} \right) \psi + \mathcal{L}_{\text{int}}, \tag{35}$$

where $\mathcal{L}_{\text{int}}$ is the interaction term. Since the atom is neutral, the interaction term is a function of the electromagnetic field strength $F_{\mu\nu} = (\mathbf{E}, \mathbf{B})$. Gauge invariance forbids terms which depend only on the vector potential A_μ. At low energies, the dominant interaction is one which involves the smallest number of derivatives, and the smallest number of photon fields, and has the form

$$\mathcal{L}_{\text{int}} = a_0^3 \; \psi^\dagger \psi \left(c_1 E^2 + c_2 B^2 \right) \tag{36}$$

The electromagnetic field strength has mass dimension two, ψ has mass dimension 3/2 ($\psi^\dagger i\partial_t \psi$ has dimension four), so that $c_1 a_0^3$ and $c_2 a_0^3$ have mass dimension -3. The typical momentum scale is set by the size of the atom a_0, so one expects $c_{1,2}$ to be of order unity. The interaction (36) gives the scattering amplitude $\mathcal{A} \sim c_i a_0^3 \omega^2$, since the electric and magnetic fields are gradients of the vector potential, so each factor of $\mathbf{E}$ or $\mathbf{B}$ produces a factor of ω. The scattering cross-section is proportional to $|c_i|^2 \, a_0^6 \omega^4$. This has the correct dimensions to be a cross-section, so the phase-space is dimensionless, and one finds that

$$\sigma \propto a_0^6 \; \omega^4. \tag{37}$$

This reproduces the well-know ω^4 dependence of the Rayleigh scattering cross-section, which explains why the sky is blue. One can actually do better, and determine the factors of 4π in (37), but I won't discuss that here. Equation (37) has corrections of order ω/a_0 from higher dimension operators which have been neglected in (36).

The Euler-Heisenberg Lagrangian

The Euler-Heisenberg effective Lagrangian is the effective Lagrangian for photon-photon scattering at energies much lower than the electron mass m_e. The leading order Lagrangian is the free Maxwell theory,

$$\mathcal{L} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu}. \tag{38}$$

The first interactions that can occur are from higher dimension operators. The lowest non-trivial operators must contain four factors of the field strength $F_{\mu\nu}$ and hence must be of dimension eight,

$$\mathcal{L} = \frac{\alpha^2}{m_e^4} \left[c_1 \left(F_{\mu\nu} F^{\mu\nu} \right)^2 + c_2 \left(F_{\mu\nu} \widetilde{F}^{\mu\nu} \right)^2 \right]. \tag{39}$$

(Terms with only three field strengths are forbidden by charge conjugation symmetry.) The effective interaction (39) is generated from the box diagram of Fig. 3. The box diagram contains four factors of the electric charge e, and one factor of $1/16\pi^2$ for the loop. In addition, the only dimensionful parameter other than the external momenta is the electron mass m_e. This allows us to write the Lagrangian in the form (39), where $c_{1,2}$ are dimensionless constants. An explicit computation gives

$$c_1 = \frac{1}{90}, \qquad c_2 = \frac{7}{90}. \tag{40}$$

The low energy cross-section for $\gamma\gamma \to \gamma\gamma$ is obtained from the graph in the effective theory, Fig. 3(b). The scattering amplitude is $\mathcal{A} \sim \alpha^2\omega^4/m_e^4$, since each gradient of the photon field in (39) produces one factor of ω. This produces a cross-section of order

$$\sigma \sim \left(\frac{\alpha^2\omega^4}{m_e^4}\right)^2 \frac{1}{\omega^2}. \tag{41}$$

The phase space factor $1/\omega^2$ is obtained using dimensional analysis. The cross-section must have dimensions of area, so the phase space must have dimension -2. The only dimensionful parameter in the effective theory is the photon energy ω, so the phase space must be proportional to $1/\omega^2$. Thus we find $\sigma \sim \alpha^4\omega^6/m_e^8$, with an error of order ω^2/m_e^2 from neglected higher order interactions in (39).

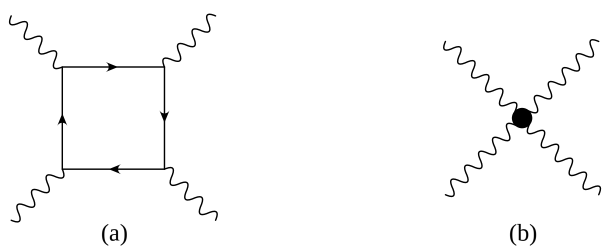

Fig. 3. Light by light scattering in (a) QED and (b) in the Euler-Heisenberg effective theory. The solid dot represents the four-photon interaction from the effective Lagrangian (39)

5 Weak Interactions at Low Energies: Tree Level

The classic example of an effective field theory is the Fermi theory of weak interactions. We first discuss how to obtain the Fermi theory as the low-energy limit of the renormalizable $SU(2) \times U(1)$ electroweak theory at tree level. The use of effective field theories for the tree level weak interactions will seem at first like applying a lot of unnecessary formalism to a trivial problem; the usefulness of the effective field theory method will only become apparent after we study the $\Delta S = 2$ weak interactions, which involve loop corrections in field theory. Finally, we will discuss the weak interactions including the leading logarithmic QCD corrections, for which the effective field theory method is indispensable.

The basic flavor changing vertex in the quark sector is the W coupling to the quark current

$$-\frac{ig}{\sqrt{2}}\, V_{ij}\, \overline{q}_i\, \gamma^\mu\, P_L\, q_j, \tag{42}$$

where V_{ij} is the Kobayashi-Maskawa mixing matrix, and $P_L = (1-\gamma_5)/2$ is the left-handed projection operator. The lowest order $\Delta S = 1$ amplitude arises from single W exchange (Fig. 4),

$$\mathcal{A} = \left(\frac{ig}{\sqrt{2}}\right)^2 V_{us} V_{ud}^*\, \left(\overline{u}\, \gamma^\mu\, P_L\, s\right) \left(\overline{d}\, \gamma^\nu\, P_L\, u\right) \left(\frac{-ig_{\mu\nu}}{p^2 - M_W^2}\right), \tag{43}$$

where the W boson propagator is in 't Hooft-Feynman gauge, p is the momentum transferred by the W, and u, d, s are quark spinors. The exchange of unphysical scalars $\phi^\pm$ can be neglected, since their Yukawa couplings to the light quarks are very small. The amplitude (43) produces a non-local four-quark interaction, because of the factor of $p^2 - M_W^2$ in the denominator. However, if the momentum transfer p is small compared with M_W, the non-local interaction can be approximated by a local interaction using the Taylor series expansion

$$\frac{1}{p^2 - M_W^2} = -\frac{1}{M_W^2}\left(1 + \frac{p^2}{M_W^2} + \frac{p^4}{M_W^4} + \ldots\right), \tag{44}$$

and retaining only a *finite* number of terms. To lowest order, the amplitude is

$$\mathcal{A} = \frac{i}{M_W^2}\left(\frac{ig}{\sqrt{2}}\right)^2 V_{us} V_{ud}^*\, \left(\overline{u}\, \gamma^\mu\, P_L\, s\right) \left(\overline{d}\, \gamma_\mu\, P_L\, u\right) + \mathcal{O}\left(\frac{1}{M_W^4}\right). \tag{45}$$

The amplitude (45) can be obtained using the effective Lagrangian

$$\mathcal{L} = -\frac{4G_F}{\sqrt{2}} V_{us} V_{ud}^*\, \left(\overline{u}\, \gamma^\mu\, P_L\, s\right) \left(\overline{d}\, \gamma_\mu\, P_L\, u\right) + \mathcal{O}\left(\frac{1}{M_W^4}\right), \tag{46}$$

where u, d and s are now the quark fields, and we have used the definition

$$\frac{G_F}{\sqrt{2}} \equiv \frac{g^2}{8M_W^2}. \tag{47}$$

The effective Lagrangian (46) can be used to study the weak decays of quarks at low energies. The basic interaction is a local four-Fermion vertex, as shown in Fig. 5. To avoid complications with hadronic matrix elements and QCD corrections (which will be discussed later), consider instead the effective Lagrangian for μ decay

$$\mathcal{L} = -\frac{4G_F}{\sqrt{2}}\, \left(\overline{e}\, \gamma^\mu\, P_L\, \nu_e\right) \left(\overline{\nu}_\mu\, \gamma^\mu\, P_L\, \mu\right) + \mathcal{O}\left(\frac{1}{M_W^4}\right), \tag{48}$$

whose derivation is almost identical to that of (46). Using (48), neglecting the $1/M_W^4$ terms, and integrating over phase space gives the standard result for the muon lifetime at lowest order,

$$\Gamma_\mu = \frac{G_F^2 m_\mu^5}{192\pi^3}. \tag{49}$$

This calculation is well known, and will not be repeated here.

To summarize: at lowest order, the "full theory," which is the $SU(2) \times U(1)$ electroweak theory, can be replaced by the "effective theory," which is QED plus the effective Lagrangian (46) (or (48)), up to corrections of order $1/M_W^4$. The effective theory can be used to compute physical processes such as the muon lifetime. So far, the effective field theory method is a fancy way of saying that we have approximated the W boson propagator in Fig. 4 by $1/M_W^2$. The real advantage of the effective field theory method will be apparent after we have discussed the one-loop $\Delta S = 2$ amplitude including QCD radiative corrections.

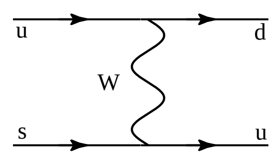

Fig. 4. W exchange diagram for the $\Delta S = 1$ weak interactions

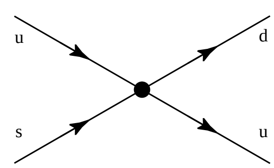

Fig. 5. The effective four-Fermi interaction of (46). This interaction reproduces the results of Fig. 4 to order $1/M_W^2$

6 Renormalization in Effective Field Theories

In quantum field theory, knowing the Lagrangian is not sufficient to compute results for physical quantities. In addition, one needs to specify a way to get finite, unambiguous answers for physical quantities. In perturbation theory, this corresponds to a choice of renormalization scheme which (i) regulates the integrals and (ii) subtracts the infinities in a systematic way. The effective Lagrangian

(46) that we have constructed is non-renormalizable, since it contains an operator of dimension six, times a coefficient G_F which is of order $1/M_W^2$. The neglected $1/M_W^4$ term contains operators of dimension eight, and so on. To use the effective Lagrangian beyond tree level, it is necessary to give a renormalization scheme as part of the definition of the effective field theory. Without this additional information, the effective Lagrangian (46) is meaningless.

It is important to keep in mind that the effective field theory is a *different theory* from the full theory. The full theory of the weak interactions is a renormalizable field theory. The effective field theory is a non-renormalizable field theory, and has a different divergence structure from the full theory. The effective field theory is constructed to correctly reproduce the low-energy effects of the full theory to a given order in $1/M_W$. The effective Lagrangian includes more terms as one works to higher orders in $1/M_W$. The effective field theory method is useful only for computing results to a certain order in $1/M_W$. If one is interested in the answer to all orders in $1/M_W$, it is obviously much simpler to use the full theory.

The renormalization scheme must be carefully chosen to give a sensible effective field theory. To see what the possible problems might be, consider the flavor diagonal effective Lagrangian from W and Z exchange

$$\mathcal{L} = -\frac{4G_F}{\sqrt{2}} V_{ui} V_{ui}^* \; (\overline{u}\,\gamma^\mu\, P_L\, q_i)\,(\overline{q}_i\,\gamma_\mu\, P_L\, u) + (\mathrm{Z}-\mathrm{exchange}) + \mathcal{O}\left(\frac{1}{M_W^4}\right), \quad (50)$$

where $i = d, s, b$. At tree level, the W and Z exchange graphs contribute to flavor diagonal parity violating u-quark interactions at order $G_F \sim 1/M_W^2$. At one loop, the interaction (50) induces a $Z\overline{u}u$ vertex from the graph in Fig. 6 which is of the form

$$I \sim \frac{1}{M_W^2} \int d^4k \; \frac{1}{k^2}, \quad (51)$$

neglecting the γ-matrix structure. The $1/k^2$ factor is from the two fermion propagators in the loop, and G_F has been rewritten as $G_F \sim 1/M_W^2$. Since the effective field theory is valid up to energies of order M_W, one can estimate the integral using a momentum space cutoff Λ of order M_W,

$$I \sim \frac{1}{M_W^2} \Lambda^2 \sim \mathcal{O}\left(1\right). \quad (52)$$

Thus the interaction (50) produces a one loop correction to the $Z\overline{u}u$ vertex of order one. Similarly, one can show that higher order terms, such as the dimension eight operators, are all equally important. A loop graph of the form Fig. 6 (where the vertex is now a dimension eight operator) is of order

$$I' \sim \frac{1}{M_W^4} \int d^4k \; \frac{1}{k^2} k^2 \sim \frac{\Lambda^4}{M_W^4} \sim \mathcal{O}\left(1\right), \quad (53)$$

etc. The additional k^2 in the integral (53) is from the extra ∂^2 at the four-quark vertex in the dimension eight operator arising from the order p^2 term in the expansion of (46). The loop graph with an insertion of the dimension eight operator is just as important as the loop graph with an insertion of the dimension six operator; both are of order unity and cannot be neglected. Similarly, all the higher order terms in the effective Lagrangian are equally important, and the entire expansion breaks down. A similar problem also occurs in the flavor changing $\Delta S = 1$ weak interactions that we have been studying, but the analysis is more subtle because of the GIM mechanism, which is why we considered the $Z\overline{u}u$ vertex.

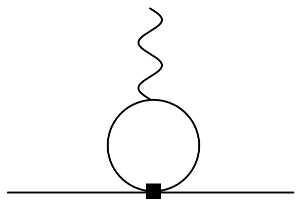

Fig. 6. One loop correction to the $Z\overline{u}u$ vertex. The solid square represents either the dimension six four-quark interaction of eq. (50), or the dimension eight four-quark operator discussed in the text

The effective field theory expansion breaks down if one introduces a mass-dependent subtraction scheme such as a momentum space cutoff.[3] This problem can be cured if one uses a mass-independent subtraction scheme, such as dimensional regularization and minimal subtraction, in which the dimensional parameter μ only appears in logarithms, and never as explicit powers such as μ^2. In such a subtraction scheme β-functions and anomalous dimensions of composite operators are mass independent. If one estimates the integrals (51) and (53) in a mass-independent subtraction scheme, one finds

$$
\begin{aligned}
I &= \frac{1}{M_W^2} \int d^4k \; \frac{1}{k^2} \sim \frac{m^2}{M_W^2} \log \mu, \\
I' &= \frac{1}{M_W^4} \int d^4k \; \frac{1}{k^2} k^2 \sim \frac{m^4}{M_W^4} \log \mu,
\end{aligned}
\tag{54}
$$

where m is some dimensionful parameter that is *not* the renormalization scale μ. It must be some other dimensionful scale that enters the loop graph of Fig. 6, such as the quark mass or the external momentum. This completely changes the estimate of the integrals. The integrals are no longer of order one, but are small provided $m \ll M_W$. As a result:

[3] One way to solve this problem is to use a cutoff $\Lambda \ll M_W$. This method does not allow one to easily match between the full and effective theories, or to include QCD corrections.

1. The effective Lagrangian produces a well-defined expansion of the weak amplitudes in powers of m/M_W, where m is some low scale such as the quark mass or the external momentum (or $\Lambda_{\rm QCD}$ when one includes QCD effects). One has a systematic expansion in powers of some low scale over M_W. This makes precise what is meant by neglecting $1/M_W^4$ terms in (45).
2. Loop integrals do not have a power law dependence on $\mu \sim M_W$, so one can count powers of $1/M_W$ directly from the effective Lagrangian. Graphs with one insertion of terms in $\mathcal{L}_{\rm eff}$ of order $1/M_W^2$ produces amplitudes of order $1/M_W^2$. Graphs with one insertion of terms of order $1/M_W^4$ or two insertions of terms of order $1/M_W^2$ produce amplitudes of order $1/M_W^4$, etc.
3. The effective field theory behaves for all practical purposes like a renormalizable field theory if one works to some fixed order in $1/M_W$. This is because there are only a finite number of terms in $\mathcal{L}_{\rm eff}$ that are allowed to a given order in $1/M_W$. Terms of higher order in $1/M_W$ can be safely neglected because they can never be multiplied by positive powers of M_W to produce effects comparable to lower order terms.

It is well-known that different renormalization schemes lead to equivalent answers for all physical quantities. In an effective field theory, a mass-independent subtraction scheme is particularly convenient, since it provides an efficient way of keeping only a few operators in $\mathcal{L}_{\rm eff}$, and in deciding which Feynman graphs are important. Nevertheless, one must be able to obtain the same results in a mass-dependent scheme such as a momentum space cutoff. This is true in principle: a mass dependent scheme has an infinite number of contributions that are of leading order (from the dimension four, six, eight, ..., operators). If one resums this contribution, then the remaining effects (again from an infinite number of terms) will be of order $1/M_W^2$. Resumming the latter leaves a contribution of $1/M_W^4$, etc. The net result of this procedure is to reproduce the same answer as that obtained much more simply using a mass-independent renormalization scheme. The connection between different renormalization schemes is much more complicated in an effective field theory (which is non-renormalizable), than in a renormalizable field theory.

7 Decoupling of Heavy Particles

There is one important drawback to using a mass-independent subtraction scheme – heavy particles do not decouple. [4] This must obviously be true since the contribution of particles to β-functions does not depend on the particle mass. For example, a 1 TeV charged lepton makes the same contribution as an electron to the QED β-function at 1 GeV.

It is instructive to look at the contribution of a charged fermion to the β-function in QED. Evaluating the diagram of Fig. 7 in dimensional regularization gives

[4] A mass independent subtraction scheme does not satisfy the conditions for the Appelquist-Carazzone theorem.

$$i\frac{e^2}{2\pi^2}\left(p_\mu p_\nu - p^2 g_{\mu\nu}\right)\left[\frac{1}{6\epsilon} - \frac{\gamma}{6} - \int_0^1 dx\ x(1-x)\ \log\frac{m^2 - p^2 x(1-x)}{4\pi\mu^2}\right], \quad (55)$$

where p is the external momentum, m is the fermion mass, γ is Euler's constant, and μ is the scale parameter of dimensional regularization.

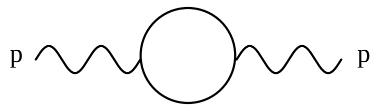

Fig. 7. One loop contribution to the QED β-function from a fermion of mass m

Mass-Dependent Scheme

In a mass-dependent scheme, such as an off-shell momentum space subtraction scheme, one subtracts the value of the graph at a Euclidean momentum point $p^2 = -M^2$, to get

$$-i\frac{e^2}{2\pi^2}\left(p_\mu p_\nu - p^2 g_{\mu\nu}\right)\left[\int_0^1 dx\ x(1-x)\ \log\frac{m^2 - p^2 x(1-x)}{m^2 + M^2 x(1-x)}\right]. \quad (56)$$

The fermion contribution to the QED β-function is obtained by acting with $(e/2)M d/dM$ on the coefficient of $i\left(p_\mu p_\nu - p^2 g_{\mu\nu}\right)$,

$$\begin{aligned}\beta\left(e\right) &= -\frac{e}{2}M\frac{d}{dM}\frac{e^2}{2\pi^2}\left[\int_0^1 dx\ x(1-x)\ \log\frac{m^2 - p^2 x(1-x)}{m^2 + M^2 x(1-x)}\right] \\ &= \frac{e^3}{2\pi^2}\int_0^1 dx\ x(1-x)\ \frac{M^2 x(1-x)}{m^2 + M^2 x(1-x)}.\end{aligned} \quad (57)$$

The fermion contribution to the β-function is plotted in Fig. 8. When the fermion mass m is small compared with the renormalization point M, $m \ll M$, the β-function contribution is

$$\beta\left(e\right) \approx \frac{e^3}{2\pi^2}\int_0^1 dx\ x(1-x) = \frac{e^3}{12\pi^2}. \quad (58)$$

As the renormalization point passes through m, the fermion decouples, and for $M \ll m$, its contribution to β vanishes as

$$\beta\left(e\right) \approx \frac{e^3}{2\pi^2}\int_0^1 dx\ x(1-x)\frac{M^2 x(1-x)}{m^2} = \frac{e^3}{60\pi^2}\frac{M^2}{m^2}. \quad (59)$$

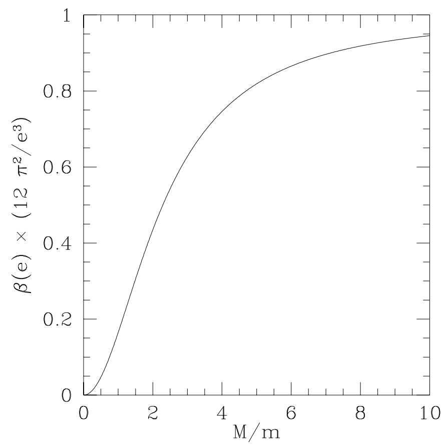

Fig. 8. Contribution of a fermion of mass m to the QED β-function. The result is given for the momentum-space subtraction scheme, with renormalization scale M. The β-function does not attain its limiting value of $e^3/12\pi^2$ until $M \gtrsim 10m$. The fermion decouples for $M \ll m$

The $\overline{MS}$ Scheme

In the $\overline{MS}$ scheme, one subtracts the $1/\epsilon$ pole and redefines $4\pi\mu^2 e^{-\gamma} \to \mu^2$, to give

$$-i\frac{e^2}{2\pi^2}\left(p_\mu p_\nu - p^2 g_{\mu\nu}\right)\left[\int_0^1 dx\ x(1-x)\log\frac{m^2 - p^2 x(1-x)}{\mu^2}\right]. \tag{60}$$

The fermion contribution to the QED β-function is obtained by acting with $(e/2)\mu d/d\mu$ on the coefficient of $i\left(p_\mu p_\nu - p^2 g_{\mu\nu}\right)$,

$$\begin{aligned}\beta\left(e\right) &= -\frac{e}{2}\mu\frac{d}{d\mu}\frac{e^2}{2\pi^2}\left[\int_0^1 dx\ x(1-x)\log\frac{m^2 - p^2 x(1-x)}{\mu^2}\right] \\ &= \frac{e^3}{2\pi^2}\int_0^1 dx\ x(1-x) = \frac{e^3}{12\pi^2},\end{aligned} \tag{61}$$

which is independent of the fermion mass and μ.

The fermion contribution to the β-function in the $\overline{MS}$ scheme does not vanish as $m \gg \mu$, so the fermion does not decouple as it should. There is another problem: the finite part of the Feynman graph in the $\overline{MS}$ scheme at low momentum is

$$-i\frac{e^2}{2\pi^2}\left(p_\mu p_\nu - p^2 g_{\mu\nu}\right)\left[\int_0^1 dx\ x(1-x)\log\frac{m^2}{\mu^2}\right], \tag{62}$$

from (60). For $\mu \ll m$ the logarithm becomes large, and perturbation theory breaks down. These two problems are related. The large finite parts correct for the fact that the value of the running coupling used at low energies is incorrect, because it was obtained using the "wrong" β-function. The two problems can be solved at the same time by integrating out heavy particles. One uses a theory including the fermion when $m < \mu$, and a theory without the fermion when $m > \mu$. Effects of the heavy particle in the low energy theory are included via higher dimension operators, which are suppressed by inverse powers of the heavy particle mass. The matching condition of the two theories at the scale of the fermion mass is that S-matrix elements for light particle scattering in the low-energy theory without the heavy particle must match those in the high-energy theory with the heavy particle.

For the case of a spin-1/2 fermion at one loop, this implies that the running coupling is continuous at $m = \mu$. The β-function is discontinuous at $m = \mu$, since the fermion contributes $e^3/12\pi^2$ to β above m and zero below m. The β-function is a step-function, instead of having a smooth crossover between $e^3/12\pi^2$ and zero, as in the momentum-space subtraction scheme. Decoupling of heavy particles is implemented by hand in the $\overline{MS}$ scheme by integrating out heavy particles at $\mu \sim m$. One calculates using a sequence of effective field theories with fewer and fewer particles. The main reason for using the $\overline{MS}$ scheme and integrating out heavy particles is that it is much easier to use in practice than the momentum-space subtraction scheme. Virtually all radiative corrections beyond one-loop are evaluated in practice using the $\overline{MS}$ scheme.

There are some instances in which heavy particle effects are important in the low energy effective theory. An example of this is the top quark in the standard model. The reason is that the top quark has a mass $m_t = g_t v/\sqrt{2}$, where g_t is the top quark Yukawa coupling, and v is the vacuum expectation value of the Higgs field. Taking m_t large while keeping v fixed is equivalent to taking g_t large. Diagrams involving top quarks and scalars (either the Higgs boson or the longitudinal parts of the W and Z) can be large, because they involve factors of g_t which can cancel any $1/m_t$ suppression. We will see an example of this in the next section, where the $\Delta S = 2$ amplitude is shown to grow with m_t. One can still integrate out the heavy top quark, but the low energy theory contains operators with coefficients which grow with m_t.

8 Weak Interactions at Low Energies: One Loop

The ideas discussed so far can now be applied to the weak interactions at one loop. The amplitude for the $\Delta S = 2$ amplitude for K^0-$\overline{K}^0$ mixing is of order G_F^2. The leading contribution to this amplitude in the standard model is from the box diagram of Fig. 9, where one sums over quarks $i, j = u, c, t$ in the intermediate states. The sum of the W and unphysical scalar exchange graphs is

$$\mathcal{A}^{\text{box}} = \frac{g^4}{128\pi^2 M_W^2} \sum_{i,j} \xi_i \xi_j \; \overline{E}(x_i, x_j) \; \left(\overline{d}\, \gamma^\mu \, P_L \, s\right) \left(\overline{d}\, \gamma_\mu \, P_L \, s\right) , \tag{63}$$

where

$$x_i = \frac{m_i^2}{M_W^2}, \tag{64}$$

$$\xi_i = V_{is} V_{id}^*, \tag{65}$$

$$\begin{aligned}\overline{E}(x,y) = -xy \Big\{ & \frac{1}{x-y}\left[\frac{1}{4} - \frac{3}{2}\frac{1}{x-1} - \frac{3}{4}\frac{1}{(x-1)^2}\right]\log x \\ & + \frac{1}{y-x}\left[\frac{1}{4} - \frac{3}{2}\frac{1}{y-1} - \frac{3}{4}\frac{1}{(y-1)^2}\right]\log y - \frac{3}{4}\frac{1}{(x-1)(y-1)}\Big\},\end{aligned} \tag{66}$$

and

$$\overline{E}(x,x) = -\frac{3}{2}\left(\frac{x}{x-1}\right)^3 \log x - x\left[\frac{1}{4} - \frac{9}{4}\frac{1}{x-1} - \frac{3}{2}\frac{1}{(x-1)^2}\right]. \tag{67}$$

In the limit $m_u = 0$ and $m_{c,t} \ll M_W$,[5]

$$\mathcal{A}^{\rm box} = -\frac{G_F^2}{4\pi^2}\ \left(\overline{d}\,\gamma^\mu\, P_L\, s\right)\left(\overline{d}\,\gamma_\mu\, P_L\, s\right)\left[\xi_c^2\, m_c^2 + \xi_t^2\, m_t^2 + 2\xi_c\xi_t\, m_c^2 \log\frac{m_t^2}{m_c^2}\right], \tag{68}$$

using (47). The $\Delta S = 2$ amplitude is of order $1/M_W^4$, rather than $1/M_W^2$ as one might naively expect, because of the GIM mechanism: The quark mass independent piece of the $\Delta S = 2$ amplitude is proportional to

$$\xi_u + \xi_c + \xi_t = \sum_i V_{id} V_{is}^* = 0, \tag{69}$$

which vanishes because the KM matrix is unitary.

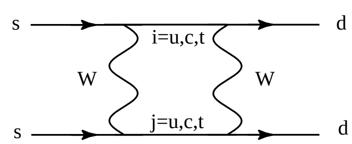

Fig. 9. The box diagram for the $\Delta S = 2$ $K^0 - \overline{K}^0$ mixing amplitude

[5] The $\Delta S = 2$ amplitude is considered in the limit $m_t \ll M_W$. This was the approximation used in the original calculations, and makes it easier for the reader to compare with the literature. It also simplifies the discussion somewhat, because the t-quark and c-quark can be treated in a similar fashion.

Matching at M_W

We will now reproduce (68) using an effective field theory calculation to one loop. At the scale M_W, the $\Delta S=2$ amplitude in the full theory is given by a loop graph in the effective theory involving two insertions of the $\Delta S=1$ interaction, plus a local four-Fermi $\Delta S=2$ interaction. The sum of the loop graph and the local $\Delta S=2$ interaction must reproduce the $\Delta S=2$ interaction in the full theory to order $1/M_W^4$, as shown schematically in Fig. 10. The tree level graphs of Figs. 4 and 5 are chosen to be the same in the full and effective theory to order $1/M_W^2$, but this does not imply that the loop graphs in the full and effective theory are equal to order $1/M_W^4$. The two loop graphs in Fig. 10 would be equal to order $1/M_W^4$ if the loop graphs in the full and effective theory were finite. However, in general, the graphs are infinite, and need subtractions. There is no simple relation between the renormalization prescriptions in the full and effective theories and one needs to add a local $\Delta S=2$ counterterm at the scale M_W, which is the difference between the loop graphs in the full and effective theories. The graphs in the effective theory are more divergent than in the full theory. In our example, the box diagram in the full theory is convergent by naive power counting.

$$I_{\text{full}} \sim \int d^4k \left(\frac{1}{k}\right)^2 \left(\frac{1}{k^2}\right)^2, \tag{70}$$

whereas the graph in the effective theory is quadratically divergent,

$$I_{\text{eff}} \sim \int d^4k \left(\frac{1}{k}\right)^2, \tag{71}$$

where we have used a factor of $1/k$ for each internal fermion line, and $1/k^2$ for each internal boson line. In the case of the standard model, the graph in the effective theory is more convergent than the naive estimate because of the GIM mechanism. As we have seen, the fermion mass-independent part of the diagram is proportional to $\xi_u+\xi_c+\xi_t$, which vanishes. Thus the non-vanishing parts of the graphs in the full and effective theory must involve a factor of the internal fermion mass. In fact, there have to be two factors of the fermion mass because the $\Delta S=1$ vertex only involves left-handed fields, and a fermion mass changes a left-handed fermion to a right-handed fermion. Thus in the effective theory, the non-zero part of the diagram must have two mass insertions on each of the fermion lines (there is a separate GIM mechanism for each line because of the independent sums over i and j in (63)), as represented in Fig. 10. This increases the degree of convergence of the diagram by two for each internal quark line, and converts it from a diagram that diverges like k^2 to a diagram that converges like $1/k^2$. Since the diagrams in the full and effective theory are both finite, the local $\Delta S=2$ vertex induced at the scale M_W vanishes.

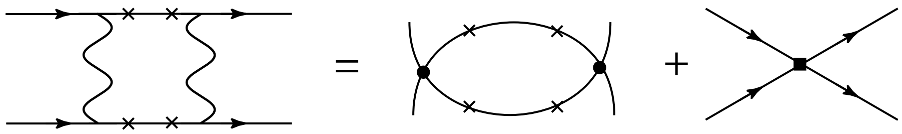

Fig. 10. Box diagram for the $\Delta S = 2$ amplitude in the full and effective theories. The crosses represent fermion mass insertions. The solid circle is a $\Delta S = 1$ vertex, and the solid square is a local $\Delta S = 2$ vertex

Matching at m_t

The effective Lagrangian remains unchanged down to the scale $\mu = m_t$, if one neglects QCD radiative corrections. At the scale $\mu = m_t$, one integrates out the top quark. The "full theory" is now the effective Lagrangian including six quarks, and the "effective theory" is the effective Lagrangian including only five quarks. The $\Delta S = 1$ interactions in the five-quark theory are trivially obtained from those in the six-quark theory, by dropping all terms that contain the t-quark. The $\Delta S = 2$ interactions in the five- and six-quark theories are given in Fig. 11, where the intermediate states in the six-quark theory are the u, c and t quarks, and in the five-quark theory are the u and c quarks. There is no GIM cancellation once the top quark has been integrated out of the theory, so the loop graph in the five-quark theory is divergent, and there will (in principle) be a non-zero counterterm induced at the scale m_t. The value of the counterterm is the difference in the diagrams in the theories above and below m_t, and so is given by the graphs in the theory above m_t that involve at least one t-quark in the loop, as shown in Fig. 12. All other graphs in the six-quark theory are identical to the corresponding graphs in the five-quark theory. The loop graphs in the theory above m_t can be calculated quite simply, and lead to the matching condition

$$c_2(\mu = m_t - 0) = \frac{G_F^2}{4\pi^2}\left[\xi_t^2\, m_t^2 + 2\xi_c\xi_t\left(m_t^2 + m_c^2\right) + 2\xi_u\xi_t\left(m_t^2 + m_u^2\right)\right], \qquad (72)$$

where the contributions come from the finite part of Fig. 12, and c_2 is the coefficient of the $\Delta S = 2$ operator $\left(\overline{d}\,\gamma^\mu\, P_L\, s\right)\left(\overline{d}\,\gamma_\mu\, P_L\, s\right)$. Using the relation (69) and neglecting m_u gives

$$c_2(\mu = m_t - 0) = \frac{G_F^2}{4\pi^2}\left[-\xi_t^2\, m_t^2 + 2\xi_c\xi_t\, m_c^2\right]. \qquad (73)$$

Equation (73) is really the difference of two calculations at the scale m_t – one in the full theory and one in the effective theory. Both calculations are sensitive to infrared effects, such as confinement. However, all infrared effects cancel in the difference, and $c_2\left(\mu = m_t + 0\right) - c_2\left(\mu = m_t - 0\right)$ is not sensitive to infrared effects. An arbitrary infrared regulator can be used if the diagrams

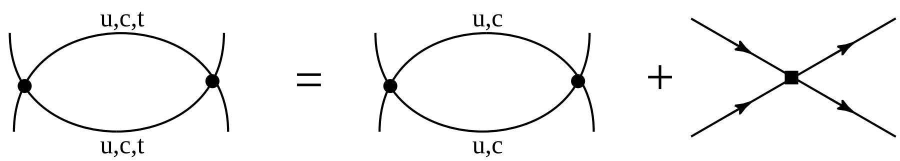

Fig. 11. Matching condition at the t-quark scale. The solid square is the $\Delta S = 2$ counterterm induced at $\mu = m_t$

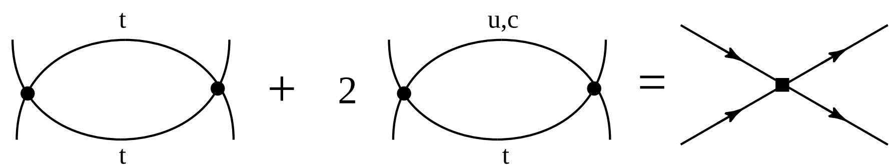

Fig. 12. Graphs to be computed to evaluate the $\Delta S = 2$ counterterm induced at $\mu = m_t$

are infrared divergent. The loop graphs will depend on the choice of regulator, but the matching condition will not. The matching condition is only sensitive to momenta of order $\mu = m_t$, so mass parameters such as m_u are short distance parameters such as the $\overline{MS}$ mass renormalized at $\mu = m_t$.

Scaling from m_t to m_c

The next step is to scale from the scale m_t to m_c. The loop graph Fig. 13 is divergent, because there is no longer a GIM mechanism in the five-quark theory, and c_2 is renormalized proportional to c_1^2, where c_1 is the coefficient of the $\Delta S = 1$ operator. This implies that there is a renormalization group equation for c_2,

$$\mu \frac{d}{d\mu} c_2 = \frac{1}{8\pi^2}\ c_1^2\, m_c^2\ \xi_c \xi_t, \tag{74}$$

where the anomalous dimension is computed using the infinite part of Fig. 13. Integrating this equation from m_t to m_c gives

$$\begin{aligned} c_2(m_c) &= c_2(m_t) + \frac{1}{8\pi^2} c_1^2\ m_c^2\ \xi_c \xi_t\ \log \frac{m_c}{m_t}, \\ &= c_2(m_t) - \frac{G_F^2}{2\pi^2}\ \xi_c \xi_t\ m_c^2\ \log \frac{m_t^2}{m_c^2}, \end{aligned} \tag{75}$$

substituting $c_1 = -4G_F/\sqrt{2}$.

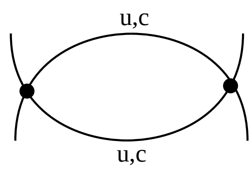

Fig. 13. The infinite part of this graph contributes to the renormalization group scaling of the $\Delta S = 2$ amplitude

Matching at m_c

Finally, one integrates out the c quark. This is virtually identical to the matching condition at the t-quark scale, and gives

$$c_2(\mu = m_c - 0) = c_2(\mu = m_c + 0) + \frac{G_F^2}{4\pi^2}\left[\xi_c^2 m_c^2 + 2\xi_c\xi_u m_c^2\right]. \tag{76}$$

Combining (72)–(76) reproduces the the box diagram computation (68).

There are some important features of the $\Delta S = 2$ computation which are generic to any effective field theory computation. (i) The contributions proportional to the heaviest mass scale m_t arise from matching conditions at that scale. (ii) contributions proportional to lower mass scales (such as m_c) arise from matching at the scale m_c, and also from from matching at scales larger than m_c (such as m_t). (iii) Contributions proportional to logarithms of two scales arise from renormalization group evolution between the two scales.

It seems that the effective field theory method is much more complicated than directly computing the original box diagram in Fig. 9. The effective theory method has broken the computation of the box diagram into several steps. The computations involved at each step in the effective field theory are much simpler than the box diagram calculation. The box diagram involves several different mass scales in the internal propagators, which leads to complicated Feynman parameter integrals that must be evaluated. The matching condition computations in the effective field theory each involve only a single mass scale, and are much simpler. One can contrast the full answer (68) with the individual pieces of the effective field theory calculation in (72)–(76). Furthermore, in the effective field theory calculation it is trivial to include the leading logarithmic QCD corrections to the $\Delta S = 2$ amplitude. The corresponding computation in the full theory is far more difficult, and involves computing two loop diagrams such as the one in Fig. 14.

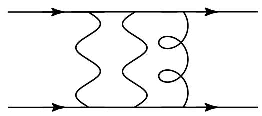

Fig. 14. A QCD radiative correction to the box diagram

QCD Corrections

The leading logarithmic corrections to the $\Delta S = 2$ amplitude sum all corrections of the form $(\alpha_s \log r)^n$, where r is large ratio of scales such as M_W/m_c, but neglect corrections of the form $\alpha_s(\alpha_s \log r)^n$. The QCD corrections to the matching condition only involve a single scale, and do not have any large logarithms. For example, the matching condition at the scale m_t only involves corrections that depend on $\alpha_s(\mu)$ and $\log m_t/\mu$. Evaluating these corrections by setting the $\overline{MS}$ parameter $\mu = m_t$ implies that there is no leading logarithmic correction to the matching condition. The only leading logarithmic QCD corrections arise from renormalization group scaling between different scales. This computation is straightforward, and only involves the infinite parts of one loop diagrams. The renormalization group equation (74) is replaced by

$$\mu \frac{d}{d\mu}\, c_2(\mu) = \frac{1}{8\pi^2}\, m_c^2(\mu)\, c_1^2(\mu)\ \xi_c \xi_t + \gamma_2(\mu)\ c_2(\mu), \tag{77}$$

where m_c has been replaced by the running mass, c_1 has been replaced by the running coupling $c_1(\mu)$, and γ_2 is the anomalous dimension

$$\gamma_2 = \frac{\alpha_s(\mu)}{\pi}, \tag{78}$$

of the $\Delta S = 2$ operator $(\overline{d}\, \gamma^\mu\, P_L\, s)\, (\overline{d}\, \gamma_\mu\, P_L\, s)$, which can be obtained from the infinite part of Fig. 15. The running mass $m_c(\mu)$ satisfies the renormalization group equation

$$\mu \frac{d}{d\mu}\, m_c(\mu) = \gamma_m\ m_c(\mu) = -\frac{2\alpha_s(\mu)}{\pi}\ m_c(\mu). \tag{79}$$

If c_1 satisfies a simple renormalization group equation of the form

$$\mu \frac{d}{d\mu}\, c_1(\mu) = \gamma_1\ c_1(\mu), \tag{80}$$

one can solve (77)–(80) to obtain the QCD corrected value for $c_2(\mu)$. At one loop, it is convenient to define b and $\hat{\gamma}_i$

$$\mu \frac{d}{d\mu}\, g = \beta(g) = -b \frac{g^3}{16\pi^2} + \ldots, \tag{81}$$

and

$$\gamma_i = \hat{\gamma}_i \, \frac{g^2}{16\pi^2} + \ldots, \tag{82}$$

for $i = 1, 2, m$. One can then solve (79) and (80),

$$\begin{aligned} m_c(\mu) &= m_c(\mu') \left[\frac{g(\mu)}{g(\mu')}\right]^{-\hat{\gamma}_m/b} = \left[\frac{\alpha_s(\mu')}{\alpha_s(\mu)}\right]^{\hat{\gamma}_m/2b}, \\ c_1(\mu) &= c_1(\mu') \left[\frac{g(\mu)}{g(\mu')}\right]^{-\hat{\gamma}_1/b} = \left[\frac{\alpha_s(\mu')}{\alpha_s(\mu)}\right]^{\hat{\gamma}_1/2b}. \end{aligned} \tag{83}$$

Substituting (83) into (77) and integrating gives

$$\begin{aligned} c_2(m_c) &= c_2(m_t) \left[\frac{\alpha_s(m_t)}{\alpha_s(m_c)}\right]^{\hat{\gamma}_2/2b} \\ &\quad + \frac{m_c^2(m_t)\; c_1^2(m_t)}{g(m_t)^2\,(2 + 2\hat{\gamma}_1/b + 2\hat{\gamma}_m/b - \hat{\gamma}_2/b)} \left[\left(\frac{\alpha_s(m_t)}{\alpha_s(m_c)}\right)^{2+2\hat{\gamma}_1/b+2\hat{\gamma}_m/b} \right. \\ &\qquad \left. - \left(\frac{\alpha_s(m_t)}{\alpha_s(m_c)}\right)^{\hat{\gamma}_2/2b}\right]. \end{aligned} \tag{84}$$

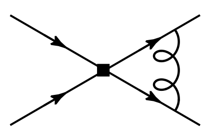

Fig. 15. Graph contributing to the anomalous dimension of the $\Delta s = 2$ operator $(\overline{d}\,\gamma^\mu\,P_L\,s)\,(\overline{d}\,\gamma_\mu\,P_L\,s)$

The actual computation of these effects in the standard model is more involved, because the $\Delta S = 1$ Lagrangian does not satisfy a simple renormalization group equation of the form (80). There is operator mixing, and (80) is replaced by a matrix equation. Nevertheless, it is possible to compute the results using an effective field theory method, though the final form of the answer is more complicated than (84). The reader is referred to the papers by Gilman and Wise for details. The computation of QCD corrections in the full theory is far more complicated, and has never been done.

To compare the advantages and disadvantages of the full and effective theory computation, let us concentrate only on the m_t part of the $\Delta S = 2$ amplitude. The effective field theory computation gives the $\Delta S = 2$ amplitude as an expansion in powers of m_t/M_W, and we have computed the leading term in (68). The general form of the effective field theory result is

$$\text{answer} = \left(\frac{m_t}{M_W}\right)^2 \left(\frac{\alpha_s(M_W)}{\alpha_s(m_t)}\right)^{\gamma_2/2b} + \left(\frac{m_t}{M_W}\right)^4 \left(\frac{\alpha_s(M_W)}{\alpha_s(m_t)}\right)^{\gamma_4/2b} + \dots \quad (85)$$

where γ_i are the anomalous dimensions of the dimension six, eight, etc. operators. (For example, compare with (84).) Evaluating each of these anomalous dimension is a separate computation. Equation (85) is useful if there is a large ratio of scales, $m_t/M_W \ll 1$, so that one only needs a few terms in the expansion (85). The full theory computation (63) sums up the entire series, and gives an answer of the form

$$\text{answer} = f(m_t/M_W), \quad (86)$$

which is valid for any value of the ratio m_t/M_W. The computations involved in (86) are necessarily more complicated than those for the effective field theory, because one obtains the entire functional form of the answer, rather than the first few terms in a series expansion. However, it is not possible to compute the leading logarithmic QCD corrections to (86), since each term in the expansion has a different anomalous dimension. For the c quark, it is more important to sum the leading QCD corrections, than to include higher order terms in $m_c/M_W \sim 1/50$, and the effective theory method is useful. The recently measured value of the top quark mass indicates that the ratio $m_t/M_W \sim 2$. In this case, it is more important to retain the entire form of the m_t/M_W dependence, than to include the QCD radiative corrections. The way the calculation is done in practice is to integrate out the t-quark and W-boson together at some scale μ which is comparable to both m_t and M_W, and then use an effective theory to scale down to m_c so as to include the QCD corrections between $\{M_W, m_t\}$ and m_c. Clearly, the ideal procedure would be to retain the entire functional form (86), as well as the entire QCD radiative correction. This has been done in a toy model using a non-local effective Lagrangian, but it is not known how to do this in general.

A very different example where an infinite set of anomalous dimensions can be computed is the QCD evolution of parton structure functions. In QCD, the Altarelli-Parisi splitting functions for the parton distribution functions contain the same information as the infinite set of anomalous dimensions of the twist-two operators. The distribution functions can be written as matrix elements of non-local operators, and the one-loop anomalous dimension is a function, whose moments give the anomalous dimensions of the infinite tower of twist two operators.

9 The Non-linear Sigma Model

The previous results discussed effective field theories in the perturbative regime, where one could compute the effective Lagrangian from the full theory in a systematic perturbative expansion. One can also apply effective field theory ideas to situations where one can not derive the effective Lagrangian from the full theory directly. The classic example of this is the use of non-linear sigma models to study spontaneously broken global symmetries, and in particular, the use of chiral Lagrangians to study pion interactions in QCD.

Consider first the linear sigma model with Lagrangian

$$\mathcal{L} = \tfrac{1}{2}\partial_\mu \boldsymbol{\phi} \cdot \partial^\mu \boldsymbol{\phi} - \lambda(\boldsymbol{\phi}\cdot\boldsymbol{\phi} - v^2)^2, \tag{87}$$

where $\boldsymbol{\phi} = (\phi_1, \ldots, \phi_N)$ is a real N-component scalar field. This theory will illustrate some ideas which will be needed for the study of chiral symmetry breaking in QCD. The Lagrangian (87) has a global $O(N)$ symmetry under which $\boldsymbol{\phi}$ transforms as an $O(N)$ vector. The potential has been chosen so that it is minimized for $|\boldsymbol{\phi}| = v$. The set of field configurations where $|\boldsymbol{\phi}| = v$ is known as the vacuum manifold, and in our example, it is the set of points $\boldsymbol{\phi} = (\phi_1, \ldots, \phi_N)$, with $\phi_1^2 + \phi_2^2 + \ldots + \phi_N^2 = v^2$, i.e. it is the $N-1$ dimensional sphere S^{N-1}. The $O(N)$ symmetry can be used to rotate the vector $\langle\boldsymbol{\phi}\rangle$ to a standard direction, which can be chosen to be $(0, 0, \ldots, v)$, the north pole of the sphere. The vacuum of the Lagrangian has spontaneously broken the $O(N)$ symmetry down to the $O(N-1)$ subgroup which acts on the first $N-1$ components. The other generators of $O(N)$ do not leave $(0, 0, \ldots, v)$ invariant. $O(N)$ has $N(N-1)/2$ generators, so the number of Goldstone bosons is equal to the number of broken generators, $N(N-1)/2 - (N-1)(N-2)/2 = N-1$. The $N-1$ Goldstone bosons correspond to rotations of the vector $\boldsymbol{\phi}$, which leave its length unchanged. The potential energy V is unchanged under rotations of $\boldsymbol{\phi}$, so these modes are massless. The remaining mode is a radial excitation which changes the length of $\boldsymbol{\phi}$, and produces a massive excitation, with mass $m_H = \sqrt{8\lambda}\, v$.

It is convenient to switch to "polar coordinates", and define

$$\boldsymbol{\phi} = (\rho + v)\ \ e^{i\sum_s X^s\cdot\pi^s} \begin{pmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 1 \end{pmatrix}, \tag{88}$$

where $X^s, s = 1, \ldots, N-1$ are $N-1$ broken generators, and π^s and ρ are a new basis for the N fields. This change of variables is only well-defined for small angles π^s. The Lagrangian in terms of the new fields is

$$\mathcal{L} = \frac{1}{2}\partial_\mu\rho\partial^\mu\rho - \lambda\left(\rho^2 + 2\rho v\right)^2 + \frac{1}{2}\left(\rho + v\right)^2 \left[\partial_\mu e^{-i\sum_s X^s\cdot\pi^s}\partial^\mu e^{i\sum_s X^s\cdot\pi^s}\right]_{NN}, \tag{89}$$

where $[\ ]_{NN}$ is the NN element of the matrix. At energies small compared to the radial excitation mass $\sqrt{8\lambda}\, v$, the ρ field can be neglected, and the Lagrangian reduces to

$$\mathcal{L} = \frac{1}{2}v^2 \left[\partial_\mu e^{-i\sum_s X^s\cdot\pi^s}\partial^\mu e^{i\sum_s X^s\cdot\pi^s}\right]_{NN}, \tag{90}$$

which describes the self-interactions of the Goldstone bosons.

There are some generic features of Goldstone boson interactions that are easy to understand:

1. The Goldstone boson fields are derivatively coupled. The Goldstone bosons describe the local orientation of the $\boldsymbol{\phi}$ field. A constant Goldstone boson field is a $\boldsymbol{\phi}$ field that has been rotated by the same angle everywhere in spacetime, and corresponds to a vacuum that is equivalent to the standard vacuum $\langle \boldsymbol{\phi} \rangle = (0, 0, \ldots, 1)$. Thus the Lagrangian must be independent of π^s when π^s is a constant, so only gradients of π^s appear in the Lagrangian.
2. The effective Lagrangian describes a theory of *weakly* interacting Goldstone bosons at low energy. The Goldstone boson couplings are proportional to their momentum, and so vanish for low-momentum Goldstone bosons.
3. The Goldstone boson Lagrangian is non-linear in the Goldstone boson fields. The Goldstone boson Lagrangian describes the dynamics of fields constrained to live on the vacuum manifold. The constraint equation, $\phi_1^2 + \phi_2^2 + \ldots + \phi_N^2 = v^2$, is non-linear, and leads to a non-linear Lagrangian.
4. The vacuum manifold is generically curved (like our sphere S^{N-1}), and does not have a set of global coordinates. The π^s coordinates defined in (88) only make sense for small fluctuations of the Goldstone boson fields about the north pole, which is adequate for perturbation theory. For studying non-perturbative effects or global properties, it is better not to introduce the angular coordinates, but to write the Lagrangian directly in terms of fields that take values on the vacuum manifold, $\pi(x) \in S^{N-1}$.
5. The amplitude for the broken symmetry currents to produce a Goldstone boson from the vacuum is proportional to the symmetry breaking strength v.

10 The CCWZ Formalism

The general formalism for effective Lagrangians for spontaneously broken symmetries was worked out by Callan, Coleman, Wess, and Zumino. Consider a theory in which a global symmetry group G is spontaneously broken to a subgroup H. The vacuum manifold is the coset space G/H. In our example, $G = O(N)$, $H = O(N-1)$, and $G/H = O(N)/O(N-1) = S^{N-1}$.

We would like to choose a set of coordinates which describe the local orientation of the vacuum for small fluctuations about the standard vacuum configuration. Let $\Xi(x) \in G$ be the rotation matrix that transforms the standard vacuum configuration to the local field configuration. The matrix Ξ is not unique: Ξh, where $h \in H$, gives the same field configuration, since the standard vacuum is invariant under H transformations. In our example, one can describe the direction of the vector $\boldsymbol{\phi}$ by giving the $O(N)$ matrix Ξ, where

$$\boldsymbol{\phi}(x) = \Xi(x) \begin{pmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ v \end{pmatrix} . \tag{91}$$

The same configuration $\boldsymbol{\phi}(x)$ can also be described by $\Xi(x)h(x)$, where $h(x)$ is a matrix of the form

$$h(x) = \begin{pmatrix} h'(x) & 0 \\ 0 & 1 \end{pmatrix}, \tag{92}$$

with $h'(x)$ an arbitrary $O(N-1)$ matrix, since

$$\begin{pmatrix} h'(x) & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ v \end{pmatrix}. \tag{93}$$

The CCWZ prescription is to pick a set of broken generators X, and choose

$$\Xi(x) = e^{iX\cdot\pi(x)}. \tag{94}$$

Consider the $O(N)$ theory for $N=3$, which is the theory of a vector $\boldsymbol{\phi}$ in three-dimensions, and so is easy to visualize. The symmetry group G is the group $G = O(3)$ of rotations in three-space. The standard vacuum configuration $\langle\boldsymbol{\phi}\rangle$ can be chosen to be $\boldsymbol{\phi}$ pointing towards the north pole N, and the unbroken symmetry group $H = O(2) = U(1)$ is rotations about the axis ON, where O is the center of the sphere (see Fig. 16). The group generators are J_1, J_2, J_3, and the unbroken generator is J_3, where J_k generate rotations about the kth axis. The CCWZ prescription is to choose

$$\Xi(x) = e^{i[J_1\pi(x)+J_2\pi_2(x)]} \tag{95}$$

to represent $\boldsymbol{\phi}$ along OA. The matrix Ξ rotates a vector pointing along the ON axis to $\boldsymbol{\phi} = OA$ by rotating along a line of longitude.

Under a global symmetry transformation g, the matrix $\Xi(x)$ is transformed to the new matrix $g\Xi(x)$, since $\boldsymbol{\phi}(x) \to g\boldsymbol{\phi}(x)$. (Note that g is a global transformation, and does not depend on x.) The new matrix $g\Xi(x)$ is no longer in standard form, (94), but can be written as

$$g\ \Xi = \Xi'\ h, \tag{96}$$

since two matrices $g\,\Xi$ and Ξ' which describe the same field configuration differ by an H transformation. That h is non-trivial is a well-known property of rotations in three dimensions. Take an object and rotate it from N to A and then to B. This transformation is not the same as a direct rotation from N to B, but can be written as a rotation about ON, followed by a rotation from N to B. The transformation h in (96) is non-trivial because the Goldstone boson manifold G/H is curved.

The transformation (96) is usually written as

$$\Xi(x) \to g\ \Xi(x)\ h^{-1}(g, \Xi(x)), \tag{97}$$

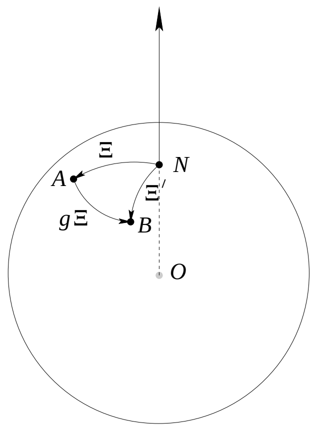

Fig. 16. The vacuum manifold for the $O(3)$ sigma model. The standard configuration $\boldsymbol{\phi}$ is along ON. Under the transformation g, A gets mapped to B

where we have made clear the implicit dependence of h on x through its dependence on g and $\Xi(x)$. Equations (94) and (97) give the CCWZ choice for the Goldstone boson field, and its transformation law. Any other choice gives the same results for all observables, such as the S-matrix, but does not give the same off-shell Green functions.

11 The QCD Chiral Lagrangian

The CCWZ formalism can now be applied to QCD. In the limit that the u, d and s quark masses are neglected, the QCD Lagrangian has a $SU(3)_L \times SU(3)_R$ chiral symmetry under which the left- and right-handed quark fields transform independently,

$$\psi_L(x) \to L\ \psi_L(x), \qquad \psi_R(x) \to R\ \psi_R(x), \tag{98}$$

where

$$\psi = \begin{pmatrix} u \\ d \\ s \end{pmatrix} . \tag{99}$$

The $SU(3)_L \times SU(3)_R$ chiral symmetry is spontaneously broken to the vector $SU(3)$ subgroup by the $\langle \overline{\psi}\psi \rangle$ condensate. The symmetry group is $G = SU(3)_L \times SU(3)_R$, the unbroken group is $H = SU(3)_V$, and the Goldstone boson manifold is the coset space $SU(3)_L \times SU(3)_R / SU(3)_V$ which is isomorphic to $SU(3)$. The generators of G are T_L^a and T_R^a which act on left and right handed quarks respectively, and the generators of H are the flavor generators $T^a = T_L^a + T_R^a$.

There are two commonly used bases for the QCD chiral Lagrangian, the ξ-basis and the Σ-basis, and we will consider them both. There are many simplifications that occur for QCD because the coset space G/H is isomorphic to a Lie group. This is not true in general; in the $O(N)$ model, the space S^{N-1} is not isomorphic to a Lie group for $N \neq 4$.

The ξ-basis

The unbroken generators of H plus the broken generators X span the space of all symmetry generators of G. One choice of broken generators is to pick $X^a = T^a_L - T^a_R$. Let the $SU(3)_L \times SU(3)_R$ transformation be represented in block diagonal form,

$$g = \begin{bmatrix} L & 0 \\ 0 & R \end{bmatrix}, \tag{100}$$

where L and R are the $SU(3)_L$ and $SU(3)_R$ transformations, respectively. The unbroken transformation have the form (100) with $L = R = U$,

$$h = \begin{bmatrix} U & 0 \\ 0 & U \end{bmatrix}. \tag{101}$$

The Ξ field is then defined using the CCWZ prescription (94)

$$\Xi(x) = e^{iX\cdot\pi(x)} = \exp i \begin{bmatrix} T\cdot\pi & 0 \\ 0 & -T\cdot\pi \end{bmatrix} = \begin{bmatrix} \xi(x) & 0 \\ 0 & \xi^\dagger(x) \end{bmatrix}, \tag{102}$$

where

$$\xi = e^{iT\cdot\pi} \tag{103}$$

denotes the upper block of $\Xi(x)$. The transformation rule (97) gives

$$\begin{bmatrix} \xi(x) & 0 \\ 0 & \xi^\dagger(x) \end{bmatrix} \to \begin{bmatrix} L & 0 \\ 0 & R \end{bmatrix} \begin{bmatrix} \xi(x) & 0 \\ 0 & \xi^\dagger(x) \end{bmatrix} \begin{bmatrix} U^{-1} & 0 \\ 0 & U^{-1} \end{bmatrix}. \tag{104}$$

This gives the transformation law for ξ,

$$\xi(x) \to L\ \xi(x)\ U^{-1}(x) = U(x)\ \xi(x)\ R^\dagger, \tag{105}$$

which defines U in terms of L, R, and ξ.

The Σ basis

The Σ-basis is obtained from the CCWZ prescription using $X^a = T^a_L$ for the broken generators. In this case, (94) gives

$$\Xi(x) = e^{iX\cdot\pi(x)} = \exp i \begin{bmatrix} T\cdot\pi & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} \Sigma(x) & 0 \\ 0 & 1 \end{bmatrix} \tag{106}$$

where

$$\Sigma = e^{iT\cdot\pi} \tag{107}$$

denotes the upper block of $\Xi(x)$. The transformation law (97) is

$$\begin{bmatrix} \Sigma(x) & 0 \\ 0 & 1 \end{bmatrix} \to \begin{bmatrix} L & 0 \\ 0 & R \end{bmatrix} \begin{bmatrix} \Sigma(x) & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} U^{-1} & 0 \\ 0 & U^{-1} \end{bmatrix}, \tag{108}$$

which gives $U = R$, and

$$\Sigma(x) \to L\ \Sigma(x)\ R^\dagger. \tag{109}$$

Comparing with (105), one sees that Σ and ξ are related by

$$\Sigma(x) = \xi^2(x). \tag{110}$$

The Lagrangian

The Goldstone boson fields are angular variables, and are dimensionless. When writing down effective Lagrangians in field theory, it is convenient to use fields which have mass dimension one, as for any other spin-zero boson field. The standard choice is to use

$$\xi = e^{iT\cdot\pi/f}, \qquad \Sigma = e^{2iT\cdot\pi/f}, \tag{111}$$

where $f \sim 93$ MeV is the pion decay constant. The π matrix is

$$\boldsymbol{\pi} = \pi^a T^a, \tag{112}$$

where the group generators have the usual normalization $\mathrm{tr}\, T^a T^b = \delta^{ab}/2$,

$$\boldsymbol{\pi} = \frac{1}{\sqrt{2}} \begin{bmatrix} \frac{1}{\sqrt{2}}\pi^0 + \frac{1}{\sqrt{6}}\eta & \pi^+ & K^+ \\ \pi^- & -\frac{1}{\sqrt{2}}\pi^0 + \frac{1}{\sqrt{6}}\eta & K^0 \\ K^- & \overline{K}^0 & -\frac{2}{\sqrt{6}}\eta \end{bmatrix}. \tag{113}$$

The low energy effective Lagrangian for QCD is the most general possible Lagrangian consistent with spontaneously broken $SU(3) \times SU(3)$ symmetry. Unlike our weak interaction example, one cannot simply compute the effective Lagrangian directly from the original QCD Lagrangian. The connection between the original and effective theories is non-perturbative. The effective Lagrangian has an infinite set of unknown parameters, but we will see that it can still be used to obtain non-trivial predictions for experimentally measured quantities.

It is easy to construct the most general Lagrangian invariant under the transformation $\Sigma \to L\Sigma R^\dagger$. The most general invariant term with no derivatives must be the product of terms of the form $\mathrm{Tr}\, \Sigma\Sigma^\dagger \ldots \Sigma\Sigma^\dagger$, where Σ and $\Sigma^\dagger$'s alternate. However, $\Sigma\Sigma^\dagger = 1$, so all such terms are constant, and independent of the pion fields. This is just our old result that all Goldstone bosons are derivatively coupled. The only invariant term with two derivatives is

$$\mathcal{L}_2 = \frac{f^2}{4}\, \mathrm{Tr}\ \partial_\mu \Sigma \partial^\mu \Sigma^\dagger. \tag{114}$$

Expanding Σ in a power series in the pion field gives

$$\mathcal{L}_2 = \text{Tr } \partial_\mu \pi \partial^\mu \pi + \frac{1}{3f^2} \text{Tr } [\pi, \partial_\mu \pi]^2 + \dots \quad . \tag{115}$$

The coefficient of the two-derivative term in (114) is fixed by requiring that the kinetic term for the pions in (115) has the standard normalization for scalar fields. The Lagrangian (114) only has terms with an even number of pions, since the pion is a pseudoscalar. The Lagrangian (114) determines all the multi-pion scattering amplitudes to order p^2 in terms of a single constant f. For example, the $\pi - \pi$ scattering amplitude is given by the term $\text{Tr}\,[\pi, \partial_\mu \pi]^2 / 3f^2$, etc.

The Chiral Currents

Noether's theorem can be used to compute the $SU(3)_L$ and $SU(3)_R$ currents. If a Lagrangian $\mathcal{L}$ is invariant under an infinitesimal global symmetry transformation with parameter ϵ, the current j_μ is given by computing the change of the Lagrangian when one makes the same transformation, with ϵ a function of x,

$$\delta \mathcal{L} = \partial_\mu \epsilon(x)\; j^\mu(x). \tag{116}$$

The infinitesimal form of the $SU(3)_L$ transformation $\Sigma \to L\Sigma$ is

$$\Sigma \to \Sigma + i\epsilon_L^a\, T^a \Sigma, \tag{117}$$

where $L = \exp i\epsilon_L^a T^a \approx 1 + i\epsilon_L^a T^a + \dots$. The change in (114) under (117) is

$$\delta \mathcal{L} = \partial_\mu \epsilon_L^a \;\; \text{Tr } T^a \Sigma \partial^\mu \Sigma^\dagger \tag{118}$$

so that the $SU(3)_L$ currents are

$$j_L^\mu = \frac{i}{2} f^2 \;\text{Tr } T^a \Sigma \partial^\mu \Sigma^\dagger. \tag{119}$$

The right handed currents are obtained by applying the parity transformation, $\pi(x) \to -\pi(-x)$ or by making an infinitesimal $SU(3)_R$ transformation, so that

$$j_R^\mu = \frac{i}{2} f^2 \;\text{Tr } T^a \Sigma^\dagger \partial^\mu \Sigma. \tag{120}$$

The axial current has the expansion

$$j_A^{\mu a} = j_R^{\mu a} - j_L^{\mu a} = -f \partial^\mu \pi^a + \dots \tag{121}$$

The matrix element $\langle 0 | \, j_A^{\mu a} \left| \pi^b \right\rangle = i f p^\mu \delta^{ab}$, so that f is the pion decay constant. The experimental value of the π decay rate, $\pi \to \mu \overline{\nu}$ determines $f \approx 93$ MeV.

The low-energy effective theory of the weak interactions is an expansion in some low mass scale (such as m_c or Λ_{QCD}) over M_W. The QCD chiral Lagrangian is an expansion in derivatives, and so is an expansion in p/Λ_χ. The pion couplings are weak, as long as the pion momentum is small compared with Λ_χ. There are two important questions that have to be answered before one can use the effective Lagrangian: (i) What terms in the effective Lagrangian are required to compute to a given order in p/Λ_χ? (ii) What is the value of Λ_χ? Then one has an estimate of the neglected higher-order terms in the expansion, and the energy at which the effective theory breaks down.

It is useful to eliminate all redundant terms in the effective Lagrangian. One can often eliminate many terms in the effective Lagrangian by making suitable field redefinitions. Field redefinitions are not very useful in renormalizable field theories, because they make renormalizable Lagrangians look superficially non-renormalizable. For example, a field redefinition

$$\phi \to \phi + \epsilon \phi^2, \tag{122}$$

turns

$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi \, \partial^\mu \phi - \frac{1}{2} m^2 \phi^2 - \lambda \phi^4 \tag{123}$$

into

$$\mathcal{L} = \frac{1}{2}\partial_\mu \phi \, \partial^\mu \phi - \frac{1}{2} m^2 \phi^2 - \lambda \phi^4 + \epsilon \left(2\phi \, \partial_\mu \phi \, \partial^\mu \phi - m^2 \phi^3 - 4\lambda \phi^5 \right) + \mathcal{O}\left(\epsilon^2\right), \tag{124}$$

which looks superficially like a non-renormalizable interaction. Equations (124) and (123) define identical theories, and the field redefinition (122) has turned a simple Lagrangian into a more complicated one. However, in the case of non-renormalizable theories which contain an infinite number of terms, one can use field redefinitions to eliminate many higher-order terms in the effective Lagrangian (see Ref. 7). The way this is usually done in practice is to use the equations of motion derived from the lowest-order terms in the effective Lagrangian to simplify or eliminate higher order terms.

Weinberg's Power Counting Argument

The QCD chiral Lagrangian is

$$\mathcal{L} = \sum_k \mathcal{L}_k, \tag{125}$$

where $\mathcal{L}_2$, $\mathcal{L}_4$, etc. are the terms in the Lagrangian with two derivatives, four derivatives, and so on. Consider an arbitrary loop graph, such as the one in Fig. 17. It contains m_2 interaction vertices that come from terms in $\mathcal{L}_2$, m_4 interaction vertices from terms in $\mathcal{L}_4$, etc. The general form of the diagram is

$$\mathcal{A} = \int \left(d^4 p\right)^L \frac{1}{\left(p^2\right)^I} \prod_k \left(p^k\right)^{m_k}, \tag{126}$$

where L is the number of loops, I is the number of internal lines, and p represents a generic momentum. The factors are easy to understand: there is a d^4p integral for each loop, each internal boson propagator is $1/p^2$, and each vertex in $\mathcal{L}_k$ gives a factor of p^k. In a mass-independent subtraction scheme, the only dimensional parameters are the momenta p. Thus the amplitude $\mathcal{A}$ must have the form $\mathcal{A} \sim p^D$, where

$$D = 4L - 2I + \sum_k \ k \ m_k, \tag{127}$$

from (126). For any Feynman graph, one can show that

$$V - I + L = 1, \tag{128}$$

where V is the number of vertices, I is the number of internal lines, and L is the number of loops. Combining (127),(128), and using $V = \sum_k m_k$, one gets

$$D = 2 + 2L + \sum_k (k-2)\, m_k. \tag{129}$$

The chiral Lagrangian starts at order p^2, so $k \geq 2$, and all terms in (129) are non-negative. As a result, only a finite number of terms in the effective Lagrangian are needed to work to a fixed order in p, and the chiral Lagrangian acts like a renormalizable field theory. For example, to compute the scattering amplitudes to order p^4, one needs

$$4 = 2 + 2L + \sum_k (k-2)\, m_k, \tag{130}$$

which has the solutions $L = 0$, $m_4 = 1$, $m_{k>4} = 0$, or $L = 1$ and $m_{k>2} = 0$. That is, one only needs to consider tree level diagrams with one insertion of $\mathcal{L}_4$, or one-loop graphs with the lowest order Lagrangian $\mathcal{L}_2$ to compute all scattering amplitudes to order p^4.

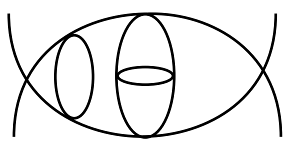

Fig. 17. A loop graph for multipion interactions

Naive Dimensional Analysis

Consider the $\pi - \pi$ scattering amplitude to order p^4. The power counting argument implies that there are two contributions to this: a tree level graph with one insertion of $\mathcal{L}_4$, and a loop graph using $\mathcal{L}_2$. The loop graph is of order

$$I \sim \int \frac{d^4k}{(2\pi)^4} \ \frac{k^2}{f^2} \ \frac{k^2}{f^2} \ \frac{1}{k^4}, \tag{131}$$

where $1/k^4$ is from the two internal propagators, and each four-pion interaction vertex is of order k^2/f^2, from (115). Here k denotes a generic internal momentum in the Feynman diagram. Estimating this integral gives

$$I \sim \frac{p^4}{16\pi^2}\frac{1}{f^4}\ \log\mu, \tag{132}$$

where μ is the $\overline{MS}$ renormalization scale, and p represents a generic external momentum. A four derivative operator in the Lagrangian of the form

$$a\,\mathrm{tr}\ \partial_\mu \Sigma \partial^\mu \Sigma^\dagger \partial_\nu \Sigma \partial^\nu \Sigma^\dagger, \tag{133}$$

produces a four-pion interaction of order ap^4/f^4 when one expands the Σ field in a power series in π/f. The total four-pion amplitude, which is the sum of the tree and loop graphs, is μ-independent. A shift in the renormalization scale μ is compensated for by a corresponding shift in a. A change in μ of order one produces a shift in a of order $\delta a \sim 1/16\pi^2$. Generically, a must be at least as big as δa,

$$|a| \gtrsim |\delta a| \sim \frac{1}{16\pi^2}, \tag{134}$$

because a shift in the renormalization point of order one produces a shift in a of this size. Write the effective Lagrangian as

$$\mathcal{L} = \frac{f^2}{4}\left[\mathrm{tr}\,\partial_\mu \Sigma \partial^\mu \Sigma^\dagger + \frac{1}{\Lambda_\chi^2}\mathcal{L}_4 + \frac{1}{\Lambda_\chi^4}\mathcal{L}_6 + \ldots\right] \tag{135}$$

where $1/\Lambda_\chi$ is the expansion parameter of the effective Lagrangian, i.e. (135) gives an expansion for scattering amplitudes in powers of p/Λ_χ. The estimate (134) for the size of the four derivative term implies that

$$\Lambda_\chi \lesssim 4\pi f. \tag{136}$$

One can show that a similar estimate holds for all the higher order terms in $\mathcal{L}$, i.e. the six derivative term has a coefficient of order $1/\Lambda_\chi^4$, etc. Numerous calculations suggest that in QCD, the inequality (136) can be replaced by the estimate

$$\Lambda_\chi \sim 4\pi f \sim 1\ \mathrm{GeV}, \tag{137}$$

for the expansion parameter of the effective Lagrangian. This parameter is large enough that one can apply chiral Lagrangians to low energy processes involving pions and kaons. If the expansion parameter were f instead of $4\pi f$, chiral Lagrangians would not be useful even for pions, since $m_\pi > f$.

The naive dimensional analysis estimate equivalent to (135) is that a term in the Lagrangian has the form

$$f^2 \Lambda_\chi^2 \left(\frac{\pi}{f}\right)^n \left(\frac{\partial}{\Lambda_\chi}\right)^m, \tag{138}$$

as can be seen by expanding the (135) in the pion fields. For example, the kinetic term Tr $\partial_\mu \Sigma\, \partial^\mu \Sigma^\dagger$ has a coefficient of order

$$f^2 \Lambda_\chi^2 \left(\frac{\partial}{\Lambda_\chi}\right)^2 \sim f^2, \tag{139}$$

the four derivative term Tr $\partial_\mu \Sigma \partial^\mu \Sigma^\dagger \partial_\nu \Sigma \partial^\nu \Sigma^\dagger$ has a coefficient of order

$$f^2 \Lambda_\chi^2 \left(\frac{\partial}{\Lambda_\chi}\right)^4 \sim \frac{f^2}{\Lambda_\chi^2} \sim \frac{1}{16\pi^2}, \quad \text{etc.,} \tag{140}$$

which agrees with the earlier estimates.

12 Explicit Symmetry Breaking

The light quark masses explicitly break the chiral $SU(3)_L \times SU(3)_R$ symmetry of the QCD Lagrangian. The quark mass term in the QCD Lagrangian is

$$\mathcal{L}_m = -\overline{\psi}_L M \psi_R + h.c., \tag{141}$$

where

$$M = \begin{bmatrix} m_u & 0 & 0 \\ 0 & m_d & 0 \\ 0 & 0 & m_s \end{bmatrix}, \tag{142}$$

is the quark mass matrix. The mass term $\mathcal{L}_m$ can be treated as chirally invariant if M is an external field that transforms as

$$M \to L M R^\dagger \tag{143}$$

under chiral $SU(3)_L \times SU(3)_R$. The symmetry breaking terms in the chiral Lagrangian are terms that are invariant when M has the transformation rule (143). The symmetry is then explicitly broken when M is fixed to have the value (142). The lowest order term in the effective Lagrangian to first order in M is

$$\mathcal{L}_m = \mu \frac{f^2}{2} \operatorname{tr}\left(\Sigma^\dagger M + M^\dagger \Sigma\right), \tag{144}$$

which breaks the degeneracy of the vacuum and picks out a particular orientation for Σ. All vacua Σ = constant are no longer degenerate, and $\Sigma = 1$ is the lowest energy state. Expanding in small fluctuations about $\Sigma = 1$ gives

$$\mathcal{L}_m = -2\mu \operatorname{tr} M\pi^2. \tag{145}$$

Substituting (142) and (113) for M and π and evaluating the trace gives

$$\begin{aligned} M^2_{\pi^\pm} &= \mu\left(m_u + m_d\right) + \Delta M^2, \\ M^2_{K^\pm} &= \mu\left(m_u + m_s\right) + \Delta M^2, \\ M^2_{K^0,\overline{K}^0} &= \mu\left(m_d + m_s\right), \end{aligned} \tag{146}$$

and the π^0, η mass matrix

$$\mu\begin{bmatrix} m_u+m_d & \frac{m_u-m_d}{\sqrt{3}} \\ \frac{m_u-m_d}{\sqrt{3}} & \frac{1}{3}\left(m_u+m_d+4m_s\right) \end{bmatrix}. \tag{147}$$

To first order in the isospin-breaking parameter $m_u - m_d$, the matrix (147) has eigenvalues

$$\begin{aligned} M^2_{\pi^0} &= \mu\left(m_u+m_d\right), \\ M^2_\eta &= \frac{\mu}{3}\left(m_u+m_d+4m_s\right). \end{aligned} \tag{148}$$

There is an isospin breaking electromagnetic contribution to the charged Goldstone boson masses ΔM^2 (included in (146)), which is comparable in size to the isospin breaking from $m_u - m_d$. To lowest order in $SU(3)$ breaking, ΔM^2 is equal for $\pi^\pm$ and $K^\pm$, and vanishes for the neutral mesons. The absolute values of the quark masses can not be determined from the meson masses, because they always occur in the combination μm, and μ is an unknown parameter. However, the meson masses can be used to obtain quark mass ratios. From (146)–(147) one gets

$$\frac{m_u}{m_d} = \frac{M^2_{K^+} - M^2_{K^0} + 2M^2_{\pi^0} - M^2_{\pi^+}}{M^2_{K^0} - M^2_{K^+} + M^2_{\pi^+}}, \tag{149}$$

$$\frac{m_s}{m_d} = \frac{M^2_{K^0} + M^2_{K^+} - M^2_{\pi^+}}{M^2_{K^0} - M^2_{K^+} + M^2_{\pi^+}}, \tag{150}$$

and the Gell-Mann–Okubo formula

$$4M^2_{K^0} = 3M^2_\eta + M^2_\pi. \tag{151}$$

Substituting the measured meson masses gives the lowest order values

$$\frac{m_u}{m_d} = 0.55, \qquad \frac{m_s}{m_d} = 20.1. \tag{152}$$

and $0.99\ \mathrm{GeV}^2 = 0.92\ \mathrm{GeV}^2$ for the Gell-Mann–Okubo formula.

There is an ambiguity in extracting the light quark masses at second order in M. The matrices M and $(\det M)M^{\dagger -1}$ both have the same $SU(3)_L \times SU(3)_R$ transformation properties, and are indistinguishable in the chiral Lagrangian. One has an ambiguity of the form

$$M \to M + \lambda\left(\det M\right) M^{\dagger -1} \tag{153}$$

in the quark mass matrix at second order in M. This transformation can be written explicitly as

$$\begin{bmatrix} m_u & 0 & 0 \\ 0 & m_d & 0 \\ 0 & 0 & m_s \end{bmatrix} \to \begin{bmatrix} m_u+\lambda m_d m_s & 0 & 0 \\ 0 & m_d+\lambda m_u m_s & 0 \\ 0 & 0 & m_s+\lambda m_u m_d \end{bmatrix}. \tag{154}$$

One cannot determine the light quark mass ratios to second order using chiral perturbation theory alone, because of the ambiguity (153). This ambiguity can be numerically significant for the ratio m_u/m_d, since it produces an effective u-quark mass of order $m_d m_s/\Lambda_\chi \sim m_d m_K^2/\Lambda_\chi^2 \sim 0.3 m_d$. The value of m_u is very important, because $m_u = 0$ solves the strong CP problem. The second order term $(\det M)M^{\dagger -1}$ produces an effective u-quark mass that is indistinguishable from m_u in the chiral Lagrangian. Various estimates of the ratio m_u/m_d from different processes (e.g. meson masses, baryon masses $\eta \to 3\pi$) all tend to give the ratio $m_u/m_d \sim 0.56$, so m_u can only be zero if second-order effects in M were of the same size in different processes. One way this could occur is if instantons effects at the scale $\mu \sim 1$ GeV were important. An instanton produces an effective operator of the form $(\det M)M^{\dagger -1}$. If instantons at $\mu \sim 1$ GeV are important, they would lead to an effective mass matrix in the chiral Lagrangian of the form $M_{\text{eff}} = M + \lambda \det M M^{-1}$, which would be the same for all processes. This produces $(m_u/m_d)_{\text{eff}} \sim 0.56$ in all processes, while still having $m_u = 0$. The only way to distinguish m_u from $(m_u)_{\text{eff}}$ is to do a reliable computation that relates the QCD Lagrangian directly to the chiral Lagrangian.

An on-shell particle has $p^2 = M^2$. Since the meson mass-squared is proportional to the quark mass M, the quark matrix M counts as two powers of p for chiral power counting, i.e. terms in $\mathcal{L}_2$ contain two powers of p or one power of M, terms in $\mathcal{L}_4$ contain four powers of p, two powers of p and one power of M, or two powers of M, etc. One can then show that the power counting arguments derived earlier still hold for the effective Lagrangian, including symmetry breaking.

13 π-π Scattering

We now have all the pieces necessary to compute the π-π scattering amplitude near threshold. The full chiral Lagrangian to order p^2 is

$$\mathcal{L}_2 = \frac{f^2}{4} \operatorname{tr} \partial_\mu \Sigma \partial^\mu \Sigma^\dagger + \mu \frac{f^2}{2} \operatorname{tr} \left(\Sigma^\dagger M + M^\dagger \Sigma \right) . \tag{155}$$

Expanding this to fourth order in the pion fields gives

$$\mathcal{L}_2 = \frac{1}{3f^2} \operatorname{Tr} \left[\pi, \partial_\mu \pi \right]^2 + \frac{2}{3} \mu \operatorname{Tr} M \pi^4 . \tag{156}$$

The π-π scattering amplitude has two contributions, one from the kinetic term and the other from the mass term. Adding the two contributions reproduces the result of Weinberg. The details are left as a homework problem.

The π-π scattering amplitude at order p^4 has two contributions, the one loop diagram Fig. 18(a) involving only the lowest order Lagrangian, and a tree graph Fig. 18(b) with terms from $\mathcal{L}_4$. The answer has the form

$$\frac{A}{16\pi^2}\, p^4 \,\log p^2/\mu^2 + L(\mu)\, p^4 , \tag{157}$$

where the first term is from the loop diagram, and the second term is the tree graph contribution from $\mathcal{L}_4$. The coefficient A of the loop graph is completely determined, since there are no unknown parameters in $\mathcal{L}_2$. The loop graph must have a logarithmic term, the so-called chiral logarithm. When $s > 4m_\pi^2$, the π-π scattering amplitude must have an imaginary part from the physical $\pi\pi$ intermediate state, by unitarity. The imaginary part is generated by the chiral logarithm. When $s > 4m_\pi^2$, the argument of the logarithm changes sign, and one gets an imaginary part since $\log(-|r|) = \log|r| + i\pi$. The imaginary part is completely determined by the tree level graph of order p^2, so that the chiral logarithm has a known coefficient. The tree level terms in $\mathcal{L}_4$ are known as low energy constants or counterterms.[6] The total scattering amplitude is μ independent, so the counterterms satisfy the renormalization group equation

$$\mu \frac{d}{d\mu} L(\mu) = \frac{A}{8\pi^2}. \tag{158}$$

The naive dimensional analysis argument discussed earlier is the statement that the counterterm $L(\mu)$ is typically at least as big as the anomalous dimension $A/8\pi^2$.

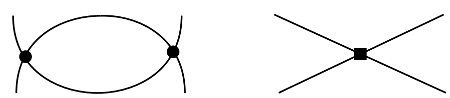

Fig. 18. Diagrams contributing to $\pi-\pi$ scattering to order p^4. The solid dot represents interaction vertices from $\mathcal{L}_2$, and the solid square represents interactions from $\mathcal{L}_4$

A generic chiral perturbation theory amplitude has the form (157). There is a chiral logarithm and some counterterms. If one works in a systematic expansion in powers of p, the chiral logarithm is determined completely in terms of lower order terms in the Lagrangian. The counterterms involve additional unknown parameters. There are three main approaches used in the literature to extract useful information from (157):

1. One can hope that the chiral logarithm is numerically more important than the counterterm, when one picks a reasonable renormalization point such as $\mu \sim 1$ GeV. This is formally correct, since $p^4 \log p^2/\mu^2 \gg p^4$ in the limit $p \to 0$. However, in practical examples, p^2 is of order m_π^2 or m_K^2, and the logarithm is -3.9 and -1.4, which is not very large (especially for the K).

[6] There are two parts to $\mathcal{L}_4$. There is the infinite part of the coefficient (of order $1/\epsilon$ in dimensional regularization), which is used to cancel divergences from loops using vertices in $\mathcal{L}_2$, and the finite part which affects measurable quantities. The infinite parts are usually never discussed explicitly, and the finite parts are called counterterms.

Nevertheless, the chiral logarithm provides useful information. For example, the correction to f_K/f_π has the form

$$\frac{f_K}{f_\pi} = 1 - \frac{3M_K^2}{64\pi^2 f^2} \log M_K^2/\mu^2 + L(\mu). \tag{159}$$

Setting $\mu \sim \Lambda_\chi$, and neglecting the counterterm gives $f_K/f_\pi = 1.19$, compared with the experimental value of 1.2. The chiral logarithm contribution alone gives a reasonable estimate of the size of the correction (this is just naive dimensional analysis at work), but it also gets the sign correct. The chiral logarithms are also useful in comparing numerical QCD calculations in the quenched approximation, which do not have the full chiral logarithms, with experimental data.

2. The systematic approach which has been used by Gasser and Leutwyler is to write down the most general Lagrangian to order p^4, which contains eight counterterms. This is used to compute $N > 8$ different processes, so that all the counterterms are determined, and one has non-trivial predictions for the remaining $N-8$ amplitudes. This procedure has been more-or-less completed in the meson sector to order p^4, and the results are in good agreement with experiment. At order p^6, there are over 100 terms in the Lagrangian.
3. The third method is to find a process for which there is no counterterm. Typically, this occurs for electromagnetic processes involving neutral particles, such as $K_S^0 \to \gamma\gamma$. Since there is no counterterm, the loop graph must be finite, but it can be non-zero. For example, the leading contribution to $K_S^0 \to \gamma\gamma$ is from the loop graph Fig. 19, and gives an amplitude of order p^2. There are no counterterms for this process at this order. The amplitude at order p^2 is in good agreement with the experimental branching ratio for this process.

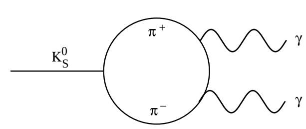

Fig. 19. Leading contribution to $K_S^0 \to \gamma\gamma$

14 Chiral Perturbation Theory for Matter Fields

Chiral perturbation theory can also be applied to the interactions of the Goldstone bosons with all other particles, which are generically referred to as matter fields. The matter fields (baryon, heavy mesons, etc.) transform as irreducible representations of $SU(3)_V$, but do not form representations of chiral $SU(3)_L \times SU(3)_R$. To discuss the interactions of matter fields, it is more convenient to use the ξ-basis of Sect. 11. We will consider the interactions of the pions with the spin-1/2 baryon octet. The generalization to other matter fields will be obvious. The CCWZ prescription for matter fields such as the baryon is that under a $SU(3)_L \times SU(3)_R$ transformation, the transformation law is

$$B \to U B U^\dagger, \tag{160}$$

where U is implicitly defined in terms of L and R in (105), and the octet of baryon fields is

$$\begin{bmatrix} \frac{1}{\sqrt{2}}\Sigma^0 + \frac{1}{\sqrt{6}}\Lambda & \Sigma^+ & p \\ \Sigma^- & -\frac{1}{\sqrt{2}}\Sigma^0 + \frac{1}{\sqrt{6}}\Lambda & n \\ \Xi^- & \Xi^0 & -\frac{2}{\sqrt{6}}\Lambda \end{bmatrix}. \tag{161}$$

Under a $SU(3)_V$ transformation, $L = R = U$, and the baryon transforms as an $SU(3)_V$ adjoint. Any transformation that reduces to the adjoint transformation law for $SU(3)_V$ transformations is acceptable. For example, one can choose

$$B \to L B L^\dagger, \qquad B \to U B R^\dagger, \qquad \textit{etc.} \tag{162}$$

The different choices are all equivalent, and correspond to redefining the baryon field. For example, if B has the transformation law (160), then $\xi B \xi^\dagger$ and $B\xi$ transform as $B \to LBL^\dagger$ and $B \to UBR^\dagger$ respectively.

The baryon chiral Lagrangian is the most general invariant Lagrangian written in terms of B and ξ. In writing the Lagrangian, it is convenient to introduce the definitions

$$\begin{aligned} A^\mu &= \tfrac{i}{2}\left(\xi \partial^\mu \xi^\dagger - \xi^\dagger \partial^\mu \xi\right) = \frac{\partial^\mu \pi}{f} + \ldots, \\ V^\mu &= \tfrac{1}{2}\left(\xi \partial^\mu \xi^\dagger + \xi^\dagger \partial^\mu \xi\right) = \frac{1}{2f^2}\left[\pi, \partial^\mu \pi\right] + \ldots, \end{aligned} \tag{163}$$

which transform as

$$A^\mu \to U A^\mu U^\dagger, \tag{164}$$

and

$$V^\mu \to U V^\mu U^\dagger - \partial^\mu U U^\dagger, \tag{165}$$

under $SU(3)_L \times SU(3)_R$. The covariant derivative on baryons is defined by

$$D^\mu B = \partial^\mu B + \left[V^\mu, B\right], \tag{166}$$

which transforms as

$$D^\mu B \to U\, D^\mu B\, U^\dagger. \tag{167}$$

The most general baryon Lagrangian to order p is

$$\mathcal{L} = \mathrm{tr}\,\overline{B}\left(i\not{D} - m_B\right)B + D\,\mathrm{tr}\,\overline{B}\gamma^\mu\gamma_5\left\{A_\mu, B\right\} + F\,\mathrm{tr}\,\overline{B}\gamma^\mu\gamma_5\left[A_\mu, B\right] + \mathcal{L}_\xi, \tag{168}$$

where $\mathcal{L}_\xi$ is the purely meson Lagrangian with $\Sigma = \xi^2$, m_B is the baryon mass, $\not{D}$ is the covariant derivative (166) and F and D are the usual axial vector coupling constants, with $g_A = F + D$.

The presence of the dimensionful parameter m_B in the Lagrangian ruins the power counting arguments necessary for a sensible effective field theory. Loop graphs in baryon chiral perturbation theory will produce corrections of order $m_B/\Lambda_\chi \sim 1$, so the entire chiral expansion breaks down. There is an alternative formulation of baryon chiral perturbation theory that avoids this problem. The idea is to expand the Lagrangian about nearly on-shell baryons, so that one has a Lagrangian that can be expanded in powers of $1/m_B$, and has no term of order m_B. The method used is similar to that used for heavy quark fields in HQET. Instead of using the Dirac baryon field B, one uses a velocity-dependent baryon field B_v, which is related to the original baryon field B by

$$B_v(x) = \frac{1+\not{v}}{2}\, B(x)\; e^{i m_B v\cdot x}, \tag{169}$$

where v is the velocity of the baryon. In the baryon rest frame, $v = (1,0,0,0)$, and

$$B_v(x) = \frac{1+\gamma^0}{2}\, B(x)\; e^{i m_B t}, \tag{170}$$

which corresponds to keeping only the particle part of the spinor, and subtracting the baryon mass m_B from all energies. In terms of the field B_v, the chiral Lagrangian is

$$\mathcal{L}_v = \mathrm{tr}\,\overline{B}\left(iv\cdot D\right)B + D\,\mathrm{tr}\,\overline{B}\gamma^\mu\gamma_5\left\{A_\mu, B\right\} + F\,\mathrm{tr}\,\overline{B}\gamma^\mu\gamma_5\left[A_\mu, B\right] + \mathcal{O}\left(\frac{1}{m_B}\right) + \mathcal{L}_\xi. \tag{171}$$

The baryon mass term is no longer present, and the baryon Lagrangian now has an expansion in powers of $1/m_B$. Note that the baryon chiral Lagrangian starts at order p, whereas the meson Lagrangian starts at order p^2.

A similar procedure can be applied to other matter fields, not just to baryons, provided one can factor the common mass (such as m_B) out of the Feynman graphs. For baryon chiral perturbation theory, this is possible because baryon number is conserved, so one can remove a common mass m_B from all baryons. A similar method also works for hadrons containing a heavy quark, such as the B and B^* mesons, because b-quark number is conserved by the strong interactions, and the B and B^* are degenerate in the heavy quark limit. It cannot be used for processes such as $\rho \to \pi\pi$, because the ρ mass m_ρ turns into the pion energy in the final state.

The velocity-dependent Lagrangian $\mathcal{L}_v$ has no dimensionful coefficients in the numerator. This implies that the power counting arguments of an effective field theory are valid. One has two expansion parameters, $1/m_B$ and $1/\Lambda_\chi$. The power counting rule (129) is now

$$D = 1 + 2L + \sum_k m_k\,(k-2) + \sum_k n_k\,(k-1)\,, \tag{172}$$

where m_k is the number of vertices from the p^k terms in the meson Lagrangian, and n_k is the number of vertices from the p^k terms in the baryon Lagrangian. The proof of this result is similar to (129), and will be omitted. The difference between the meson and baryon terms arises because the meson propagator is $1/k^2$, whereas the baryon propagator is $1/k \cdot v$.

The naive dimensional analysis estimate (138) is now

$$f^2 \Lambda_\chi^2 \left(\frac{\pi}{f}\right)^n \left(\frac{\partial}{\Lambda_\chi}\right)^m \left(\frac{B}{f\sqrt{\Lambda_\chi}}\right)^r \tag{173}$$

For example, the kinetic term $\overline{B}\,(iv\cdot D)\,B$ has a coefficient of order

$$f^2 \Lambda_\chi^2 \left(\frac{\partial}{\Lambda_\chi}\right)^1 \left(\frac{B}{f\sqrt{\Lambda_\chi}}\right)^2 \sim 1, \tag{174}$$

and the four-baryon term $\overline{B}B\,\overline{B}B$ has a coefficient of order

$$f^2 \Lambda_\chi^2 \left(\frac{B}{f\sqrt{\Lambda_\chi}}\right)^4 \sim \frac{1}{f^2}. \tag{175}$$

Similar power counting arguments hold for all strongly interacting gauge theories. For example, in tests for quark and lepton substructure, one uses the operator

$$\frac{4\pi}{\Lambda_{\rm ELP}^2}\,\overline{q}q\,\overline{q}q, \tag{176}$$

and places limits on $\Lambda_{\rm ELP}$. A quark field has the same power counting rules as a baryon field in baryon chiral perturbation theory. Comparing with (175), we see that

$$\Lambda_{\rm ELP} = \Lambda/\sqrt{4\pi}, \tag{177}$$

where Λ is the scale of the composite interactions defined by analogy with the chiral scale Λ_χ: i.e. scattering amplitudes vary on a momentum scale Λ.

πN Scattering

A simple application of the baryon chiral Lagrangian is the computation of $\pi - N$ scattering amplitude at threshold, to order p. From (172), the only graphs which contribute are tree graphs which involve terms from the meson Lagrangian at order p^2, and the baryon Lagrangian at order p. The two diagrams which contribute are shown in Fig. 20. The pion-nucleon vertex in Fig. 20(a) vanishes at threshold, since it is proportional to $\boldsymbol{p}$. The two-π nucleon vertex in Fig. 20(b) is

$$\frac{i}{2f^2}\overline{B}\left[\pi,\partial^\mu\pi\right]v^\mu B. \tag{178}$$

The amplitude can be rewritten using $\pi = \pi^a T_B^a$, where T_B^a are the flavor matrices in the baryon representation. The pions are in the adjoint representation of flavor, so the flavor matrices acting on pions can be written in terms of the structure constants

$$\left(T_\pi^c\right)_{ba} = i f_{abc}. \tag{179}$$

Using the commutation relations $\left[T_B^a, T_B^b\right] = i f_{abc} T_B^c$, and evaluating Fig. 20 using the interaction (178) gives the amplitude

$$\mathcal{A} = -\frac{i}{f^2}\ M_\pi\ T_\pi \cdot T_B, \tag{180}$$

where we have used $E = M_\pi$ for the energy of the pion. Changing from the non-relativistic normalization of baryon states to the relativistic normalization (where the states are normalized to $2E$) gives the Weinberg-Tomozawa formula for the pion-nucleon scattering amplitude

$$\mathcal{A} = -\frac{2i}{f^2}\ M_B M_\pi\ \left(T_\pi \cdot T_B\right). \tag{181}$$

$T_\pi \cdot T_B = 1/2\left[\left(T_\pi + T_B\right)^2 - T_\pi^2 - T_B^2\right] = 1/2\left[I(I+1) - 2 - 3/4\right]$, so that $T_\pi \cdot T_B$ is -1 in the isospin-1/2 channel and $1/2$ in the isospin-3/2 channel.

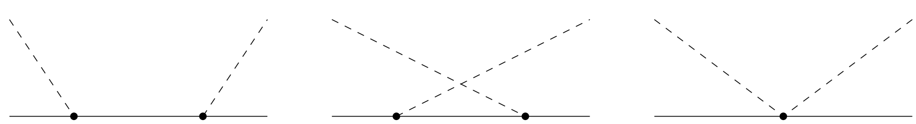

Fig. 20. Contributions to πN scattering at order p

Non-Analytic Terms

The chiral Lagrangian for matter fields can be used to compute loop corrections. The matter field Lagrangian has an expansion in powers of p, whereas the Goldstone boson Lagrangian had an expansion in powers of p^2. Consequently, loop integrals for matter field interactions can have either even or odd dimension. The even dimensional integrals have the same structure as for the mesons, and lead to non-analytic terms of the form $(M^{2r}/16\pi^2 f^2)\log M^2/\mu^2$, where M is a π, K or η mass, and r is an integer. The μ dependence is cancelled by a corresponding μ dependence in a higher order term in the Lagrangian. The odd dimensional integrals lead to non-analytic terms of the form $(M^{2r+1}/16\pi f^2)$, where r is an integer. These odd-dimensional terms do not have a multiplying logarithm, since the μ dependence cannot be absorbed by a higher dimension operator in the effective Lagrangian. The operator would have to have the form M^{2r+1} which is proportional to $m_q^{r+1/2}$, where m_q is the quark mass. Such an operator cannot exist in the Lagrangian, since it contains a square-root of the quark mass matrix. Also note that the odd-dimensional integrals have one less power of π in the denominator.

15 Chiral Perturbation Theory for Hadrons Containing a Heavy Quark

Chiral perturbation theory can also be applied to hadrons containing a heavy quark. The hadrons are treated as matter fields, and one writes down the most general possible Lagrangian consistent with the chiral symmetries, as for the spin-1/2 baryons. In additional, one can constrain some of the terms in the Lagrangian using heavy quark symmetry. As a simple example, consider the interaction of the pseudoscalar and vector mesons B and B^* (or D and D^*) with pions. These mesons can be treated using the velocity dependent formulation, since b-quark number is conserved by the strong interactions, and the B and B^* are degenerate in the heavy quark limit. It is conventional to combine B and B^* into a single field H defined by

$$H = \frac{1+\not{v}}{2}\left[B_\mu^* \gamma^\mu - B\gamma_5\right], \tag{182}$$

where B and B^* are column vectors which contain the states $b\overline{u}$, $b\overline{d}$, and $b\overline{s}$. The transformation law for H under heavy quark spin symmetry transformation S_Q and $SU(3)_V$ flavor symmetry transformation U is

$$H \to S_Q H U^\dagger. \tag{183}$$

The most general Lagrangian consistent with these symmetries to order p is

$$\mathcal{L} = \mathrm{tr}\,\overline{H}\left(iv\cdot D\right)H + g\,\mathrm{tr}\,\overline{H}H\gamma^\mu\gamma_5 A_\mu, \tag{184}$$

where

$$D^\mu H = \partial^\mu H - HV^\mu. \tag{185}$$

There is only a single coupling constant g which appears to this order, so the $BB^*\pi$ and $B^*B^*\pi$ couplings are related to each other by the heavy quark spin symmetry. The other possible interaction term,

$$\mathrm{tr}\,\overline{H}\gamma^\mu\gamma_5 H A_\mu \tag{186}$$

is forbidden by heavy quark spin symmetry, and is suppressed by one power of $1/m_Q$. This term splits the $BB^*\pi$ and $B^*B^*\pi$ couplings at order $1/m_Q$.

The chiral Lagrangian (184) can be used to compute corrections to various quantities for heavy hadrons. For example, one can show that

$$\frac{f_{B_s}}{f_B} = 1 - \tfrac{5}{6}\left(1+3g^2\right)\frac{M_K^2}{16\pi^2 f^2}\log\frac{M_K^2}{\mu^2}, \tag{187}$$

and

$$\frac{B_{B_s}}{B_B} = 1 - \tfrac{2}{3}\left(1-3g^2\right)\frac{M_K^2}{16\pi^2 f^2}\log\frac{M_K^2}{\mu^2}, \tag{188}$$

where f_B and B_B are the decay constant for B decay and the bag constant for $B^0 - \overline{B}^0$ mixing respectively. Further applications can be found in the literature.

16 Acknowledgments

I would like to thank Gabriela Barenboim, C. Glenn Boyd, Richard F. Lebed, Ira Z. Rothstein, and Oscar Vives for helpful comments on the manuscript. This work was supported in part by a Department of Energy grant DOE-FG03-90ER40546, by a Presidential Young Investigator award from the National Science Foundation, PHY-8958081, and by a "Profesor Visitante IBERDROLA de Ciencia y Tecnología" position at the Departmento de Física Teórica of the University of València.

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