Heavy Quark Effective Theory

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Reference

- A.V. Manohar and M.B. Wise,
  Heavy Quark Physics,
  Cambridge University Press (2000)

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Outline

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Introduction

QCD describes the dynamics of quarks, and has a non-perturbative scale $\Lambda_{\rm QCD} \sim 200\,$MeV.

Simplifications when $m_Q \gg \Lambda_{\rm QCD}$

A single heavy quark interacting with light particles can be described by an effective field theory known as HQET.

Applied to c and b quarks. The t-quark decays via $t \to bW$ before it forms hadrons. The width in the standard model is $\Gamma_t \approx 1.5\,$GeV

NRQCD

Systems with two heavy quarks (such as J/ψ, Υ or $\bar{t}t$ near threshold) are described by a completely different effective theory, NRQCD [non-relativistic QCD].

Velocity Superselection Rule

Consider a heavy quark Q interacting with light degrees of freedom, such as light quarks and gluons.

$$v^\mu = \frac{p^\mu}{m_Q}, \qquad \delta v^\mu = \frac{\delta p^\mu}{m_Q} \sim \frac{\Lambda_{\rm QCD}}{m_Q} \to 0$$

Quark has a constant velocity.

$$[x, v] = \frac{1}{m_Q}\,[x, p] = \frac{i\hbar}{m_Q} \to 0.$$

Cannot simultaneously have a well-defined position and momentum, but can have a well-defined position and velocity.

Spin-Flavor Symmetry

In the $m_Q \to \infty$ limit: static color $\mathbf{3}$ source in the rest frame $v^\mu = (1,0,0,0)$.

Strong interactions flavor blind
$\Rightarrow$ HQ flavor symmetry
Symmetry breaking $\propto 1/m_b - 1/m_c$

Color coupling is color electric charge. The magnetic interaction is $\propto 1/m_Q$ for a pointlike spin-1/2 fermion (not true for the proton). $\Rightarrow$ HQ spin symmetry
Symmetry breaking $\propto 1/m_Q$.

Combining gives HQ $SU(4)$ spin-flavor symmetry:
$b\uparrow$, $b\downarrow$, $c\uparrow$ and $c\downarrow$ transform as a $\mathbf{4}$

QCD Lagrangian

$$\mathcal{L} = \sum_{i=c,b,t} \bar{Q}_i \left(i \not{D} - m_{Q_i} \right) Q_i + \mathcal{L}_{\text{light}}$$

The Lagrangian has m_Q term, and no well-defined $m_Q \to \infty$ limit. It describes the interactions of Q at all energies, including those greater than m_Q.

Want to consider an effective theory valid for momenta smaller than m_Q, which makes the simplications of low momentum manifest.

Should have an expansion in $1/m_Q$

HQET Lagrangian

HQET Lagrangian:

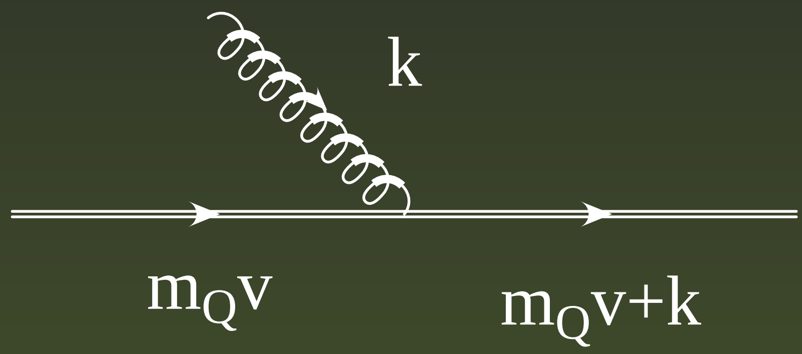
$$\mathcal{L}_{\rm HQET} = \mathcal{L}_0 + \frac{1}{m_Q}\mathcal{L}_1 + \frac{1}{m_Q^2}\mathcal{L}_2 + \ldots$$

$\mathcal{L}_0$ has spin-flavor symmetry,

$1/m_Q$ terms are symmetry breaking corrections.

Quark Propagator

Look at the quark propagator:



$$i\frac{\not{p}+m_Q}{p^2-m_Q^2+i\epsilon}$$

$p = mv + k$, where k is called the residual momentum, and is of order Λ_{QCD}.

HQET Propagator

$$i\frac{m_Q \not{v} + \not{k} + m_Q}{(m_Q v + k)^2 - m_Q^2 + i\epsilon}$$

Expanding this in the limit $k \ll m_Q$ gives

$$i\frac{1+\not{v}}{2k\cdot v + i\epsilon} + \mathcal{O}\left(\frac{k}{m_Q}\right) = i\frac{P_+}{k\cdot v + i\epsilon} + \mathcal{O}\left(\frac{k}{m_Q}\right),$$

with a well defined limit.

$$P_+ \equiv \frac{1+\not{v}}{2}$$

Projectors

$$P_+ = \frac{1+\not{v}}{2}, \qquad P_- = \frac{1-\not{v}}{2},$$

In the rest frame,

$$P_+ = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$P_+^2 = P_+,\ \ P_-^2 = P_-,\ \ P_+ + P_- = 1,\ \ P_+P_- = 0\ \ P_-P_+ = 0.$$

Gluon Vertex

The quark-gluon vertex

$$-igT^a\gamma^\mu \to -igT^a P_+ \gamma^\mu P_+ = -igT^a v^\mu,$$

using the identity

$$P_+\gamma^\mu = v^\mu + \gamma^\mu P_-$$

In the rest frame: the coupling is purely that of an electric charge.

HQET L_0

HQET Lagrangian:

$$\mathcal{L} = \bar{h}_v(x)\,(iD\cdot v)\,h_v(x)\,,$$

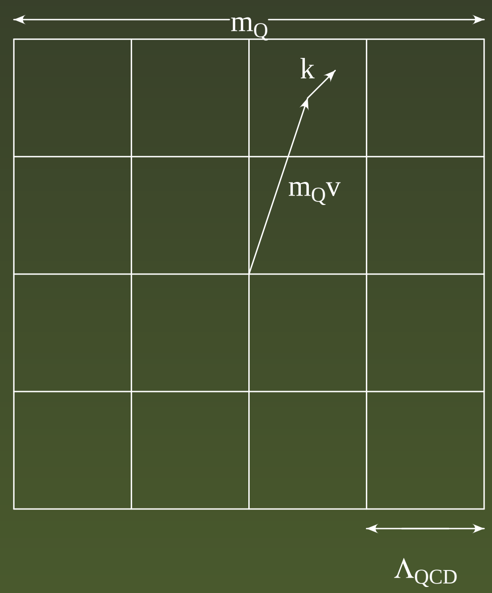
$h_v(x)$ is the quark field in the effective theory and satisfies

$$P_+ h_v(x) = h_v(x)\,.$$

h_v annihilates quarks with velocity v, but does not create antiquarks

Dividing up momentum space

v appears explictly in the HQET Lagrangian.
h_v describes quarks with velocity v, and momenta within Λ_{QCD} of $m_Q v$.



quarks with velocity $v' \neq v$ are far away in the EFT

Feynman rules

$$\mathcal{L} = \bar{h}_v \, (iD \cdot v) \, h_v$$

$$D_\mu = \partial_\mu + igT^a A^a_\mu$$

The $\partial \cdot v$ term gives a propagator

$$\frac{iP_+}{k \cdot v}$$

The $A \cdot v$ term gives a vertex

$$-igT^a v^\mu$$

Manifest Spin-Flavor Symmetry

$$\mathcal{L}_0 = \sum_{f=c,b,t} \bar{h}_{fv}\,(iD \cdot v)\,h_{fv},$$

has manifest spin-flavor symmetry, since $D \cdot v$ does not depend on the spin or the flavor of the heavy quark.

Light degrees of freedom

Hadrons containing a single heavy quark contain Q, and light quarks and gluons [light degrees of freedom ℓ].

D^+ meson has a c quark, $\bar{d}$ quark, plus $\bar{q}q$ pairs and gluons. Quantum numbers of ℓ are the same as the $\bar{d}$.

Total angular momentum $\mathbf{J}$ is conserved

$\mathbf{S}_Q$ is conserved as $m_Q \to \infty$

$$\text{Define} \qquad \mathbf{S}_\ell \equiv \mathbf{J} - \mathbf{S}_Q$$

Spin of the light degrees of freedom

Multiplet Structure

$$\mathbf{J}^2 = j(j+1), \quad \mathbf{S}_Q^2 = s_Q(s_Q+1), \quad \mathbf{S}_\ell^2 = s_\ell(s_\ell+1)$$

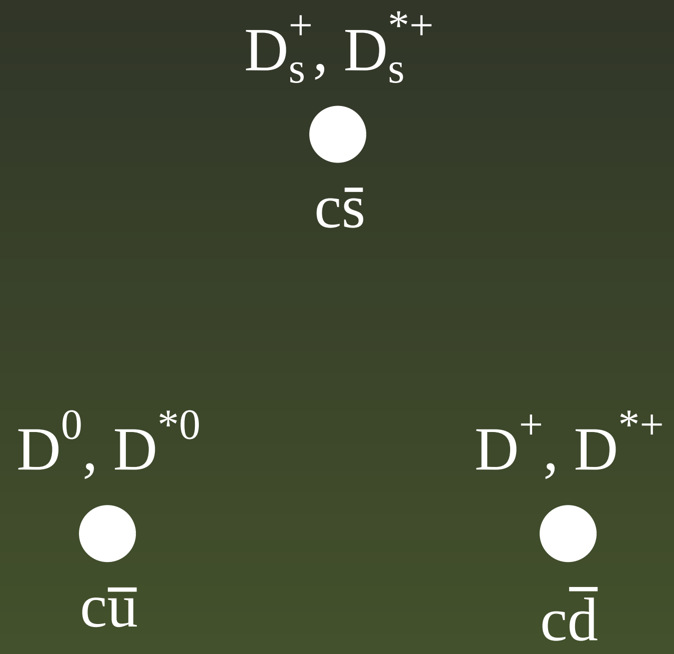
$s_Q = 1/2$, so heavy hadrons are in degenerate multiplets with $j = s_\ell \pm 1/2$, unless $s_\ell = 0$, in which case there is a single $j = 1/2$ multiplet.

Ground state mesons: Q and a light antiquark $\bar{q}$, so $s_\ell = 1/2$.
$j = 0 \oplus 1$ and negative parity, since quarks and antiquarks have opposite parity

Degenerate 0^- and 1^- mesons which form a flavor $\bar{\mathbf{3}}$

Called $H^{(Q)}$

Ground State Mesons



$\bar{B}_s^0$, $\bar{B}^-$, $\bar{B}^0$ (spin-0) $\bar{B}_s^{*0}$, $\bar{B}^{*-}$, $\bar{B}^{*0}$ (spin-1)

NOTE: $c \in D$, $b \in \bar{B}$

Excited Mesons

In the quark model, the first excitation has $L = 1$, with $s_\ell = 1/2$ and $s_\ell = 3/2$

$s_\ell = 1/2 \Rightarrow 0^+$ and 1^+ states D_0^* and D_1^*

$s_\ell = 3/2 \Rightarrow 1^+$ and 2^+ states D_1 and D_2^*

The $s_\ell = 1/2$ and $s_\ell = 3/2$ multiplets are not related by HQ symmetry, though they are related in a NR quark model

Quark Field

Write the quark field as

$$
\begin{aligned}
Q(x) &= e^{-im_Q v\cdot x}\left[h_v(x) + \mathcal{Q}_v(x)\right] \\
&= e^{-im_Q t}\begin{pmatrix} h_v(x) \\ \mathcal{Q}_v(x) \end{pmatrix}
\end{aligned}
$$

$p = m_Q v + k$, so the x dependence of $h_v(x)$ is k.

$$
\begin{aligned}
\not{v} h_v(x) &= h_v(x), \\
\not{v} \mathcal{Q}_v(x) &= -\mathcal{Q}_v(x)
\end{aligned}
$$

Tree-level Lagrangian

At tree-level, one does not have to worry about renormalization effects:

$$\begin{aligned}
\mathcal{L} &= \bar{Q}\left(i\not{D} - m_Q\right) Q \\
&= \left(\bar{h}_v + \bar{\mathcal{Q}}_v\right) e^{im_Q v\cdot x} \left(i\not{D} - m_Q\right) e^{-im_Q v\cdot x} \left(h_v + \mathcal{Q}_v\right) \\
&= \left(\bar{h}_v + \bar{\mathcal{Q}}_v\right) \left(i\not{D} - m_Q + m_Q \not{v}\right) \left(h_v + \mathcal{Q}_v\right) \\
&= \left(\bar{h}_v + \bar{\mathcal{Q}}_v\right) \left(i\not{D} h_v + \left[i\not{D} - 2m_Q\right] \mathcal{Q}_v\right)
\end{aligned}$$

The result can be simplified using

$$P_+\gamma^\mu P_+ = v^\mu, \;\; P_-\gamma^\mu P_- = -v^\mu, \;\; P_+\gamma^\mu P_- = \gamma_\perp^\mu, \;\; P_-\gamma^\mu P_+ = \gamma_\perp^\mu,$$

where the $\perp$ projector is defined for any vector A by

$$A_\perp^\mu \equiv A^\mu - v^\mu \;\, v\cdot A.$$

$$\mathcal{L} = \bar{h}_v \left(iv \cdot D \right) h_v - \mathcal{Q}_v \left(iv \cdot D + 2m_Q \right) \mathcal{Q}_v + \bar{h}_v i \not{D}_\perp \mathcal{Q}_v + \bar{\mathcal{Q}}_v i \not{D}_\perp h_v$$

Quadratic in $\mathcal{Q}_v$:

$$\mathcal{L} = \bar{h}_v \left(iv \cdot D \right) h_v + \bar{h}_v i \not{D}_\perp \frac{1}{2m_Q + iv \cdot D} i \not{D}_\perp h_v .$$

The last term can be expanded in a power series in $1/m_Q$,

$$\frac{1}{2m_Q + iv \cdot D} = \frac{1}{2m_Q} - \frac{1}{4m_Q^2} iv \cdot D + \ldots \quad .$$

$1/m_Q$ Lagrangian

The effective Lagrangian to order $1/m_Q$ is thus

$$\mathcal{L} = \bar{h}_v \left(iv \cdot D \right) h_v + \frac{1}{2m_Q} \bar{h}_v i \not{D}_\perp \, i \not{D}_\perp h_v .$$

This can be rewritten using the identities

$$\gamma^\alpha \gamma^\beta = g^{\alpha\beta} - i\sigma^{\alpha\beta}, \;\; \left[D^\alpha, D^\beta \right] = igG^{\alpha\beta}$$

$$\begin{aligned} \mathcal{L} &= \bar{h}_v \left(iv \cdot D \right) h_v + \frac{1}{2m_Q} \bar{h}_v \left(iD_\perp \right)^2 h_v - \frac{g}{4m_Q} \bar{h}_v \sigma_{\alpha\beta} G^{\alpha\beta} h_v \\ & \quad + \mathcal{O}\left(\frac{1}{m_Q^2} \right) \end{aligned}$$

$$\mathcal{L} = \bar{h}_v \left(iv \cdot D\right) h_v + \frac{1}{2m_Q} \bar{h}_v \left(iD_\perp\right)^2 h_v - c_F \frac{g}{4m_Q} \bar{h}_v \sigma_{\alpha\beta} G^{\alpha\beta} h_v$$

The $(iD_\perp)^2$ term violates flavor symmetry at order $1/m_Q$

$g\sigma_{\alpha\beta}G^{\alpha\beta}$ term violates spin and flavor symmetry at order $1/m_Q$

One can carry out the expansion to higher order in $1/m_Q$ to obtain the tree level HQET Lagrangian.

Field Redefinitions

Field redefinition

$$h_v \to \left[1 + \frac{a}{m_Q} iv \cdot D\right] h_v$$

changes the effective Lagrangian to

$$\begin{aligned}\mathcal{L} &= \bar{h}_v \left(iv \cdot D\right) h_v + \frac{1}{2m_Q} \bar{h}_v \left(iD_\perp\right)^2 h_v - \frac{g}{4m_Q} \bar{h}_v \sigma_{\alpha\beta} G^{\alpha\beta} h_v \\ &\quad + \frac{2a}{m_Q} h_v \left(iv \cdot D\right)^2 h_v + \mathcal{O}\left(\frac{1}{m_Q^2}\right)\end{aligned}$$

For $a = 1/2$, one can replace $D_\perp^2 \to D_\perp^2 + \left(v \cdot D\right)^2 = D^2$

Field redefinition change off-shell amplitudes, but not S-matrix elements.

The only thing that the effective theory and full theory have to agree on are S-matrix elements.

Convenient to eliminate t derivatives, i.e. $v \cdot D$ terms in $\mathcal{L}$

[Use the lowest order equation of motion $(v \cdot D)\, h_v = 0$ to remove time derivatives]

Masses

Hadron mass mass in effective theory is $M_H - m_Q$.

- Lowest order: all hadrons degenerate, mass m_Q
- Order one: Hadron mass

$$\langle H_Q | \mathcal{H}_0 | H_Q \rangle \equiv \bar{\Lambda}$$

where H_0 = Hamiltonian from lowest order Lagrangian (including the light degrees of freedom).
$\bar{\Lambda}$ has different values for each multiplet:
$\bar{\Lambda}$ for mesons, $\bar{\Lambda}_\Lambda$, $\bar{\Lambda}_\Sigma$

$\lambda_1, \ \lambda_2$

- Order $1/m_Q$:

$$\frac{\mathcal{H}_1}{m_Q} = -\frac{\mathcal{L}_1}{m_Q} = -\frac{1}{2m_Q}\bar{h}_v \left(iD_\perp\right)^2 h_v + c_F \frac{g}{4m_Q}\bar{h}_v \sigma_{\alpha\beta} G^{\alpha\beta} h_v .$$

Define two non-perturbative parameters

$$\begin{aligned} \lambda_1 &= \left\langle H_Q \right| \bar{h}_v \left(iD_\perp\right)^2 h_v \left| H_Q \right\rangle, \\ 8\left(\mathbf{S}_Q \cdot \mathbf{S}_\ell\right) \lambda_2 &= \left\langle H_Q \right| \bar{h}_v g \sigma_{\alpha\beta} G^{\alpha\beta} h_v \left| H_Q \right\rangle, \end{aligned}$$

In the rest frame:

$$\begin{aligned} \left(iD_\perp\right)^2 &= -\mathbf{p}^2 \\ \sigma_{\alpha\beta} G^{\alpha\beta} &= -2\boldsymbol{\sigma} \cdot \mathbf{B} \end{aligned}$$

Meson Masses

$$m_H = m_Q + \bar{\Lambda} - \frac{\lambda_1}{2m_Q} + c_F \frac{2\lambda_2 \mathbf{S}_Q \cdot \mathbf{S}_\ell}{m_Q}$$

$\mathbf{S}_Q \cdot \mathbf{S}_\ell = (J^2 - S_Q^2 - S_\ell^2)/2$, so

$$\begin{aligned}
m_B &= m_b + \bar{\Lambda} - \frac{\lambda_1}{2m_b} - \frac{3\lambda_2}{2m_b} \\
m_{B^*} &= m_b + \bar{\Lambda} - \frac{\lambda_1}{2m_b} + \frac{\lambda_2}{2m_b} \\
m_D &= m_c + \bar{\Lambda} - \frac{\lambda_1}{2m_c} - \frac{3\lambda_2}{2m_c} \\
m_{D^*} &= m_c + \bar{\Lambda} - \frac{\lambda_1}{2m_c} + \frac{\lambda_2}{2m_c}
\end{aligned}$$

Note that heavy quark symmetry implies that $\bar{\Lambda}$, λ_1 and λ_2 have the same value in the b and c systems (upto renormalization)

$$0.49\ \mathrm{GeV}^2 = m_{B*}^2 - m_B^2 = 4\lambda_2 = m_{D*}^2 - m_D^2 = 0.55\ \mathrm{GeV}^2,$$

up to corrections of order $1/m_{b,c}$.

$$90 \pm 3\mathrm{MeV} = m_{B_s} - m_{B_d} = \bar{\Lambda}_s - \bar{\Lambda}_d = m_{D_s} - m_{D_d} = 99 \pm 1\mathrm{MeV}$$
$$345 \pm 9\mathrm{MeV} = m_{\Lambda_b} - m_B = \bar{\Lambda}_\Lambda - \bar{\Lambda}_d = m_{\Lambda_c} - m_D = 416 \pm 1\mathrm{MeV}$$

$\bar{\Lambda}$, λ_1 and λ_2 will occur elsewhere

Meson Field

$Q\bar{q}$ mesons $\rightarrow$ field $H_v^{(Q)}$ (4×4 matrix, bispinor)

$$H_v^{(Q)}(x) \rightarrow D(\Lambda)\, H_{\Lambda^{-1}v}^{(Q)}\left(\Lambda^{-1}x\right) D(\Lambda)^{-1}$$

Pseudoscalar $P_v^{(Q)}(x)$ and vector $P_{v\mu}^{*(Q)}(x)$

Vector particles have a polarization vector ϵ_μ, with $\epsilon \cdot \epsilon = -1$, and $v \cdot \epsilon = 0$.

$$H_v^{(Q)} = \frac{1 + \not{v}}{2}\left[\not{P}_v^{*(Q)} + i P_v^{(Q)} \gamma_5\right].$$

$$H_v^{(Q)} = \frac{1+\not{v}}{2}\left[\not{P}_v^{*(Q)} + iP_v^{(Q)}\gamma_5\right].$$

Parity:

$$H_v^{(Q)}(x) \to \gamma^0 H_{v_P}^{(Q)}(x_P)\,\gamma^0,$$

where

$$x_P = \left(x^0, -\mathbf{x}\right), \;\; v_P = \left(v^0, -\mathbf{v}\right).$$

$$H_v^{(Q)} = \frac{1+\not{v}}{2}\left[\not{P}_v^{*(Q)} + iP_v^{(Q)}\gamma_5\right].$$

$$\not{v}H_v^{(Q)} = H_v^{(Q)}, \qquad H_v^{(Q)}\not{v} = -H_v^{(Q)}.$$

using $v \cdot P_v^{*(Q)} = 0$

Conjugate field:

$$\bar{H}_v^{(Q)} = \gamma^0 H_v^{(Q)\dagger}\gamma^0 = \left[\not{P}_v^{*(Q)^\dagger} + iP_v^{(Q)^\dagger}\gamma_5\right]\frac{1+\not{v}}{2},$$

which also transforms as a bispinor,

$$\bar{H}_v^{(Q)} \to D\left(\Lambda\right)\bar{H}_{v'}^{(Q)} D\left(\Lambda\right)^{-1}.$$

since

$$\gamma^0 D\left(\Lambda\right)^\dagger \gamma^0 = D\left(\Lambda\right)^{-1}.$$

Normalization of States

$$\langle H\left(p',\varepsilon'\right)|\,H\left(p,\varepsilon\right)\rangle = 2E_{\mathbf{p}}\left(2\pi\right)^{3}\delta^{(3)}\left(\mathbf{p}-\mathbf{p}'\right)\delta_{\varepsilon\varepsilon'},$$

Mass dimension -1

HQET states eigenstates of the $m_Q \to \infty$ theory and labelled by v and k, with $v \cdot k = 0$. They differ from full QCD states.

$$\langle H(v',k',\varepsilon')|H(v,k,\varepsilon)\rangle = 2v^{0}\left(2\pi\right)^{3}\delta_{vv'}\delta^{3}(\mathbf{k}-\mathbf{k}')\delta_{\varepsilon\varepsilon'}.$$

Usually take $k=0$. States have mass dimension $-3/2$

$$|H(p)\rangle = \sqrt{m_H}\left[|H(v)\rangle + \mathcal{O}\left(\frac{1}{m_Q}\right)\right]$$

Similarly

$$\begin{aligned}
\overline{u}(p,s)\gamma^\mu u(p,s) &= 2p^\mu \\
\overline{u}(v,s)\gamma^\mu u(v,s) &= 2v^\mu \\
u(p,s) &= \sqrt{m_H}\, u(v,s)
\end{aligned}$$

Meson Decay Constants

$$\langle 0|\,\bar{q}\gamma^\mu\gamma_5 Q\,|P\,(p)\rangle = -i\,f_P\,p^\mu,$$

where f_P has mass dimension one. ($f_\pi = 131$ MeV)

$$\langle 0|\,\bar{q}\,\gamma^\mu\,Q\,|P^*\,(p,\epsilon)\rangle = f_{P^*}\,\epsilon^\mu,$$

f_{P^*} has mass dimension two.

$$\bar{q}\,\Gamma\,Q = \bar{q}\,\Gamma\,Q_v + \mathcal{O}\,(\alpha_s) + \mathcal{O}\left(\frac{1}{m_Q}\right)$$

so we need

$$\langle 0|\,\bar{q}\,\Gamma\,Q_v\,|H(v)\rangle\,,$$

Spurion Analysis

$$\bar{q}\,\Gamma\,Q_v \to \bar{q}\,\Gamma\,D(R)_Q Q_v$$

- Pretend that Γ transforms as $\Gamma \to \Gamma D(R)_Q^{-1}$
- Write down operators which are invariant when $Q_v \to D(R)_Q Q_v$, $\Gamma \to \Gamma D(R)_Q^{-1}$, $H_v^{(Q)} \to D(R)_Q H_v^{(Q)}$.
- Set Γ to its fixed value γ^μ or $\gamma^\mu\gamma_5$ to obtain the operator with the correct transformation properties.

Decay Matrix Element

$$\langle 0|\, \bar{q}\, \Gamma\, Q_v\, |H(v)\rangle = \mathrm{Tr}\ \frac{a}{2}\Gamma H_v^{(Q)}$$

$$a = a_0(v^2) + a_1(v^2)\not{v}$$

$$a \times \begin{cases} -iv^\mu P_v^{(Q)} & \text{if } \Gamma = \gamma^\mu\gamma_5, \\ P_v^{*(Q)\mu} & \text{if } \Gamma = \gamma^\mu, \end{cases}$$

$$\begin{aligned} \langle 0|\, \bar{q}\, \gamma^\mu\gamma_5\, Q_v\, |P(v)\rangle &= -iav^\mu, \\ \langle 0|\, \bar{q}\, \gamma^\mu\, Q_v\, |P^*(v)\rangle &= a\epsilon^\mu. \end{aligned}$$

$$f_P = \frac{a}{\sqrt{m_P}}, \qquad f_{P^*} = a\sqrt{m_{P^*}}.$$

$$f_P = \frac{a}{\sqrt{m_P}}, \qquad f_{P^*} = m_P f_P,$$

so $f_P \propto m_P^{-1/2}$, $f_{P^*} \propto m_P^{1/2}$.

a has the same value for c and b:

$$\frac{f_B}{f_D} = \sqrt{\frac{m_D}{m_B}}, \qquad f_{D^*} = m_D f_D, \qquad f_{B^*} = m_B f_B.$$

Measure from the decays $D \to \bar{\ell}\nu_\ell$ and $\bar{B} \to \ell\bar{\nu}_\ell$

$$\Gamma = \frac{G_F^2 |V_{Qq}|^2}{8\pi} f_P^2 m_\ell^2 m_P \left(1 - \frac{m_\ell^2}{m_P^2}\right)^2 .$$

JLQCD Collaboration [S. Aoki et al., Phys. Rev. Lett. 80 (1998) 5711]

Decay Constant	Value in MeV
f_D	197 ± 2
f_{D_s}	224 ± 2
f_B	173 ± 4
f_{B_s}	199 ± 3

Note that this simulation suggests that there is a substantial correction to the heavy quark symmetry prediction $f_B/f_D = \sqrt{m_D/m_B} \simeq 0.6$.

$\bar{B} \to D^{(*)}$ Form Factors

Semileptonic $b \to c$ decays via the weak current $\bar{c}\,\gamma_\mu P_L\, b$
Decay form-factors are defined by:

$$
\begin{aligned}
\left\langle D(p')\right| V^\mu \left|\bar{B}(p)\right\rangle &= f_+(q^2)\,(p+p')^\mu + f_-(q^2)\,(p-p')^\mu\,, \\
\left\langle D^*(p',\epsilon)\right| V^\mu \left|\bar{B}(p)\right\rangle &= g(q^2)\epsilon^{\mu\nu\alpha\tau}\epsilon^*_\nu\,(p+p')_\alpha\,(p-p')_\tau\,, \\
\left\langle D^*(p',\epsilon)\right| A^\mu \left|\bar{B}(p)\right\rangle &= -i\,f(q^2)\epsilon^{*\mu} \\
& \quad -i\epsilon^*\cdot p\Big[a_+(q^2)\,(p+p')^\mu + a_-(q^2)\,(p-p')^\mu\Big],
\end{aligned}
$$

where $q = p - p'$
Six form-factors

w

Label states by v and v', and use

$$w = v \cdot v' = \frac{m_B^2 + m_{D^{(*)}}^2 - q^2}{2 m_B m_{D^{(*)}}}$$

The allowed kinematic range for w is

$$0 \leq w - 1 \leq \frac{\left(m_B - m_{D^{(*)}}\right)^2}{2 m_B m_{D^{(*)}}}$$

The zero-recoil point, at which $D^{(*)}$ is at rest in the $\bar{B}$ rest frame, is $w = 1$ (maximum q^2)

Better to use:

$$\frac{\langle D(p')|V^\mu|\bar{B}(p)\rangle}{\sqrt{m_B m_D}} = h_+(w)\,(v+v')^\mu + h_-(w)\,(v-v')^\mu,$$

$$\frac{\langle D^*(p',\epsilon)|V^\mu|\bar{B}(p)\rangle}{\sqrt{m_B m_{D^*}}} = h_V(w)\epsilon^{\mu\nu\alpha\beta}\epsilon^*_\nu v'_\alpha v_\beta,$$

$$\frac{\langle D^*(p',\epsilon)|A^\mu|\bar{B}(p)\rangle}{\sqrt{m_B m_{D^*}}} = -ih_{A_1}(w)(w+1)\epsilon^{*\mu} + ih_{A_2}(w)(\epsilon^*\cdot v)v^\mu$$

$$+ih_{A_3}(w)(\epsilon^*\cdot v)v'^\mu.$$

$$q^2_{\text{light}} \sim (\Lambda_{\text{QCD}} v - \Lambda_{\text{QCD}} v')^2 = 2\Lambda^2_{\text{QCD}}(1 - w)$$

HQ symmetry should hold if:

$$2\Lambda^2_{\text{QCD}}\,(w - 1) \ll m^2_{b,c}.$$

The heavy meson form factors are expected to vary on the scale $q^2_{\text{light}} \sim \Lambda^2_{\text{QCD}}$, i.e. on the scale $w \sim 1$.

QCD matrix elements are of the form:

$$\left\langle H^{(c)}(p') \right| \bar{c}\,\Gamma\, b \left| H^{(b)}(p) \right\rangle$$

At leading order in $1/m_{c,b}$ and $\alpha_s(m_{c,b})$:

$$\left\langle H^{(c)}(v') \right| \bar{c}_{v'}\,\Gamma\, b_v \left| H^{(b)}(v) \right\rangle$$

use trick as before $\Gamma \to D(R)_c \Gamma D(R)_b^{-1}$

$$\bar{c}_{v'}\,\Gamma\, b_v = \mathrm{Tr}\; X\,\bar{H}_{v'}^{(c)}\,\Gamma\, H_v^{(b)},$$

$$X = X_0 + X_1 \not{v} + X_2 \not{v}' + X_3 \not{v}\not{v}',$$

where the coefficients are functions of $w = v \cdot v'$.

Isgur-Wise Function

Use $X = -\xi(w)$:

$$\langle D(v') | \bar{c}_{v'}\, \gamma_\mu \, b_v \, | \bar{B}(v) \rangle = \xi(w)\,[v_\mu + v'_\mu],$$

$$\langle D^*(v', \epsilon) | \bar{c}_{v'}\, \gamma_\mu \gamma_5 \, b_v \, | \bar{B}(v) \rangle = -i\xi(w)\,[(1+w)\epsilon^*_\mu - (\epsilon^* \cdot v) v'_\mu],$$

$$\langle D^*(v', \epsilon) | \bar{c}_{v'}\, \gamma_\mu \, b_v \, | \bar{B}(v) \rangle = \xi(w)\,\epsilon_{\mu\nu\alpha\beta}\epsilon^{*\nu} v'^{\alpha} v^{\beta}.$$

Six form-factors in terms of one Isgur-Wise function

$$h_+(w) = h_V(w) = h_{A_1}(w) = h_{A_3}(w) = \xi(w),$$

$$h_-(w) = h_{A_2}(w) = 0$$

Normalization of $\xi(1)$

Consider the forward matrix element of the vector current $\bar{b}\gamma^\mu b$ between $\bar{B}$ meson states. Setting $v' = v$, and letting $c \to b$, $D \to \bar{B}$,

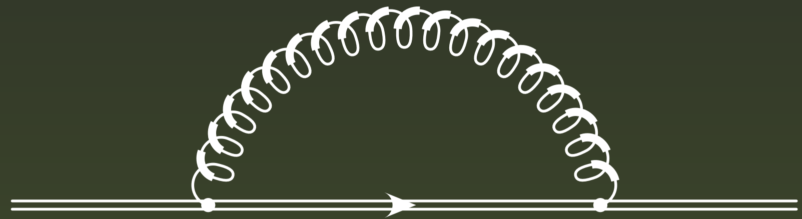
$$\frac{\langle \bar{B}(p)|\bar{b}\gamma_\mu b|\bar{B}(p)\rangle}{m_B} = \langle \bar{B}(v)|\bar{b}_v\gamma_\mu b_v|\bar{B}(v)\rangle = 2\ \xi(w=1)\, v_\mu .$$

where ξ for $b \to b$ is the same as for $b \to c$.

So $\xi(1) = 1$.
This fixes the absolute normalization and allows one to determine V_{cb}.

Radiative Corrections

Consider heavy quark wavefunction renormalization:



In Feynman gauge ($n = 4 - \epsilon$)

$$\int \frac{d^n q}{(2\pi)^n}(-igT^A\mu^{\epsilon/2})v_\lambda \frac{i}{(q+p)\cdot v}(-igT^A\mu^{\epsilon/2})v^\lambda \frac{(-i)}{q^2}$$

$$= \quad -\left(\frac{4}{3}\right) g^2\mu^\epsilon \int \frac{d^n q}{(2\pi)^n}\frac{1}{q^2 \;\; v\cdot(q+p)}$$

Add a gluon mass to regulate the IR divergence, so that one can isolate the UV divergence from the $1/\epsilon$ pole

$$-\left(\frac{4}{3}\right) g^2\mu^\epsilon \int \frac{d^nq}{(2\pi)^n}\frac{1}{(q^2-m^2)\, v\cdot(q+p)}$$

Use the identity

$$\frac{1}{a^r b^s} = 2^s\frac{\Gamma(r+s)}{\Gamma(r)\Gamma(s)}\int_0^\infty d\lambda \frac{\lambda^{s-1}}{[a+2b\lambda]^{r+s}},$$

to get

$$-\left(\frac{8}{3}\right) g^2\mu^\epsilon \int_0^\infty d\lambda \int \frac{d^nq}{(2\pi)^n}\frac{1}{\left[q^2-m^2+2\lambda v\cdot(q+p)\right]^2}.$$

Let $q \to q-\lambda v$

$$-\left(\frac{8}{3}\right) g^2\mu^\epsilon \int_0^\infty d\lambda \int \frac{d^nq}{(2\pi)^n} \frac{1}{\left[q^2-m^2-\lambda^2+2\lambda v\cdot p\right]^2}.$$

Use the standard dim reg formula:

$$\int \frac{d^nq}{(2\pi)^n} \frac{(q^2)^\alpha}{(q^2-M^2)^\beta} =$$

$$\frac{i}{2^n\pi^{n/2}}(-1)^{\alpha+\beta}(M^2)^{\alpha-\beta+n/2}\frac{\Gamma(\alpha+n/2)\Gamma(\beta-\alpha-n/2)}{\Gamma(n/2)\Gamma(\beta)},$$

$$-\frac{i}{(4\pi)^{2-\epsilon/2}}\left(\frac{8}{3}\right) g^2\mu^\epsilon \Gamma\left(\epsilon/2\right) \int_0^\infty d\lambda \left[\lambda^2-2\lambda v\cdot p+m^2\right]^{-\epsilon/2}.$$

Recursion Relation

Evaluate λ integral using the recursion relation

$$I\left(a,b,c\right) \equiv \int_0^\infty d\lambda\left[\lambda^2+2b\lambda+c\right]^a$$

$$= \frac{1}{1+2a}\left[\left(\lambda^2+2b\lambda+c\right)^a\left(\lambda+b\right)\Big|_0^\infty+2a\left(c-b^2\right)I\left(a-1,b,c\right)\right],$$

to convert it to one that is convergent when $\epsilon=0$,

$$\int_0^\infty d\lambda\left[\lambda^2-2\lambda v\cdot p+m^2\right]^{-\epsilon/2}$$

$$= \frac{1}{1-\epsilon}\left[\left(\lambda^2-2\lambda v\cdot p+m^2\right)^{-\epsilon/2}\left(\lambda-v\cdot p\right)\Big|_0^\infty\right.$$

$$\left.-\epsilon\left(m^2-(v\cdot p)^2\right)\int_0^\infty d\lambda\left[\lambda^2-2\lambda v\cdot p+m^2\right]^{-1-\epsilon/2}\right].$$

Z_h

Can set $\epsilon = 0$ in the last term. Also use

$$\lim_{\lambda \to \infty} \lambda^z = 0,$$

to get

$$-i \frac{g^2}{3\pi^2 \epsilon} v \cdot p + \text{finite}$$

This gives

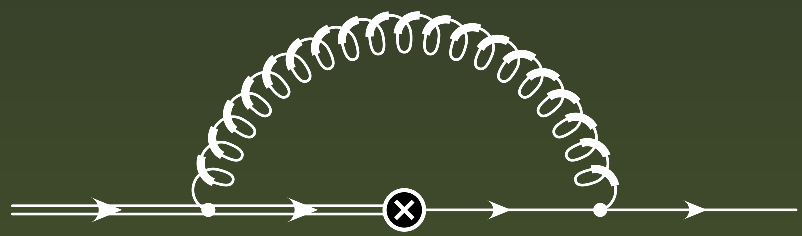
$$Z_h = 1 + \frac{g^2}{3\pi^2 \epsilon}, \qquad \gamma_h = \frac{1}{2} \frac{\mu}{Z_h} \frac{dZ_h}{d\mu} = -\frac{g^2}{6\pi^2}.$$

Note that $Z_q = 1 - \frac{g^2}{6\pi^2 \epsilon}$

Heavy-Light Current

Compute the anomalous dimension for

$$O_\Gamma = \bar{q}\Gamma Q_v$$



The light quark vertex is γ^μ and the heavy quark vertex is v^μ. Find that

$$\gamma_O = -\frac{g^2}{4\pi^2}.$$

independent of Γ.

Operator Mixing

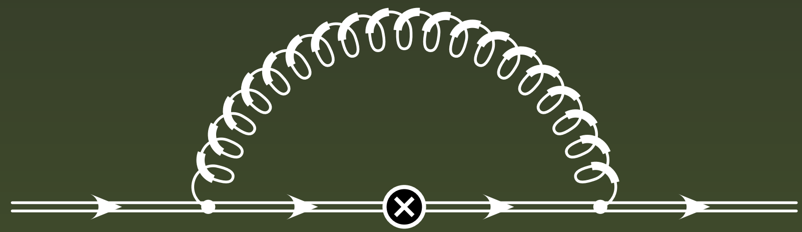
Use the convention that

$$\begin{aligned} \mu \frac{\mathrm{d}}{\mathrm{d}\mu} O_j &= -\gamma_{ji} O_i \\ \mu \frac{\mathrm{d}}{\mathrm{d}\mu} C_i &= \gamma_{ji} C_j \end{aligned}$$

with

$$L = C_i O_i$$

Heavy-Heavy Current



$$\gamma = \frac{g^2}{3\pi^2}\left[w\, r(w) - 1\right]$$

$$r(w) = \frac{1}{\sqrt{w^2-1}} \ln\left(w + \sqrt{w^2-1}\right)$$

$$\gamma = \frac{g^2}{\pi^2}\left[\frac{2}{9}(w-1) - \frac{1}{15}(w-1)^2 + \ldots\right],$$

that vanishes at $w = 1$.

Matching heavy-light currents

$$\bar{q}\gamma^\lambda Q \quad = \quad C_1^{(V)}\ \bar{q}\gamma^\lambda Q_v \quad + \quad C_2^{(V)}\ \bar{q}v^\lambda Q_v$$

where to lowest order $C_1 = 1$, $C_2 = 0$.

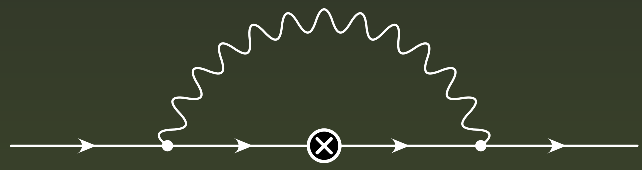
To compute the α_s corrections, compute on-shell matrix elements of both sides at one loop.

Computation of a matching coefficient. Need the difference between the full and effective theory results.

Full and EFT results can have IR divergences, but the difference is IR finite. [UV divergences taken care of by renormalization]

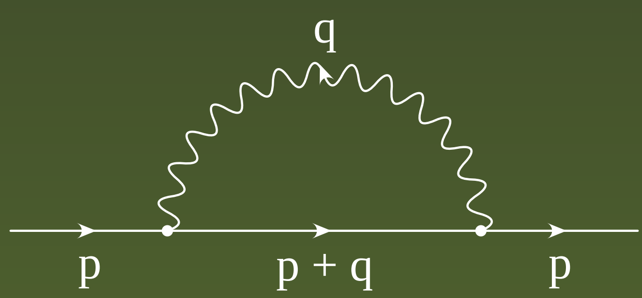
Full Theory

Full theory: evaluate on-shell with $p = m_Q v$



$$C_1 = -\frac{2\alpha_s}{3\pi}\left[\frac{1}{\epsilon_{IR}} + 1\right], \qquad C_2 = \frac{2\alpha_s}{3\pi}.$$

Wavefunction renormalization:



$$\Sigma(p) = A(p^2)m + B(p^2)\not{p}$$

$$\delta Z \equiv B + 2m^2 \frac{\mathrm{d}(A+B)}{\mathrm{d}p^2}\bigg|_{p^2=m^2}$$

Gives $C_1 = \delta Z_Q/2$,

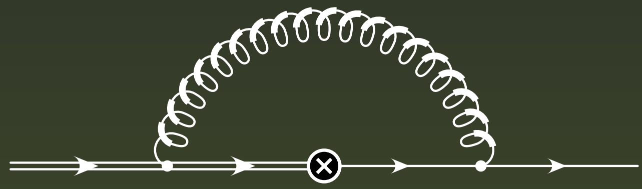
$$C_1 = -\frac{\alpha_s}{3\pi}\left[\frac{2}{\epsilon_{IR}} + 2 + 3\ln\frac{\mu}{m_Q}\right], \qquad C_2 = 0$$

(including UV counterterm)

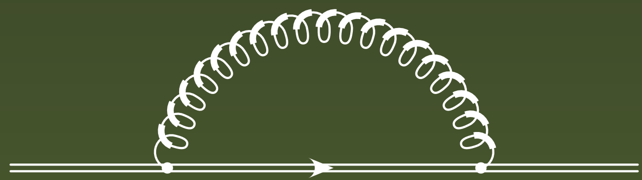
Neglect Z_q as it will cancel out.

Effective Theory

Effective theory:



$$C_1 = -\frac{2\alpha_s}{3\pi\epsilon_{IR}} \qquad C_2 = 0.$$



$$C_1 = -\frac{2\alpha_s}{3\pi\epsilon_{IR}} \qquad C_2 = 0$$

Light quark wavefunction cancels between full and EFT

Scaleless Integrals

EFT integrals: have no scale in them when evaluated on-shell. e.g. wavefunction graph

$$\int \frac{d^n q}{(2\pi)^n} \frac{1}{q^2 \;\; v\cdot(q+p)} \to \int \frac{d^n q}{(2\pi)^n} \frac{1}{q^2 \;\; (v\cdot q)^2}$$

and is zero in dim-reg. Adding the UV counterterm gives the $1/\epsilon$ terms.

$$\int \frac{d^n q}{(2\pi)^n} \frac{1}{q^4} = \frac{i}{8\pi^2 \epsilon_{UV}} - \frac{i}{8\pi^2 \epsilon_{IR}},$$

$$\int \frac{d^n q}{(2\pi)^n} \frac{1}{q^4} + \text{counterterm} = -\frac{i}{8\pi^2 \epsilon_{IR}},$$

Matching UV and IR

Full theory integrals can have finite parts, since they depend on m_Q.

IR divergence match between full and effective theory. Since the EFT integrals are scaleless, UV=IR in the EFT. Thus one finds

$$\left.\frac{1}{\epsilon_{IR}}\right|_{\mathrm{FULL}} = -\left.\frac{1}{\epsilon_{UV}}\right|_{\mathrm{EFT}}$$

The anomalous dimensions in the EFT are related to the IR behavior in the full theory.

Matching Correction

Full theory:

$$C_1 = -\frac{\alpha_s}{3\pi}\left[\frac{4}{\epsilon} + 4 + 3\ln\frac{\mu}{m_Q}\right], \qquad C_2 = \frac{2\alpha_s}{3\pi}.$$

EFT:

$$C_1 = -\frac{\alpha_s}{3\pi}\left[\frac{4}{\epsilon}\right] \qquad C_2 = 0$$

Difference

$$C_1 = -\frac{\alpha_s}{3\pi}\left[4 + 3\ln\frac{\mu}{m_Q}\right], \qquad C_2 = \frac{2\alpha_s}{3\pi}.$$

Or compute in full theory and drop $1/\epsilon$ terms.

Log in matching related to difference in anomalous dimensions in Full and EFT.

Usually choose matching scale $\mu = m_Q$, so one finds:

$$C_1^{(V)}(m) = 1 - \frac{4\alpha_s}{3\pi} \qquad C_2^{(V)}(m) = \frac{2\alpha_s}{3\pi}.$$

One can show that the matching for the axial current is $C_1 \to C_1$, $C_2 \to -C_2$

$$C_1^{(A)}(m) = 1 - \frac{4\alpha_s}{3\pi} \qquad C_2^{(A)}(m) = -\frac{2\alpha_s}{3\pi}.$$

Meson Decay Constants

Compute the radiative corrections to the meson decay constants:

Match at the scale m to the EFT:

$$\bar{q}\gamma^\mu\gamma_5 Q \to C_1^{(A)}(m)\ \bar{q}\gamma^\mu\gamma_5 Q_v + C_2^{(A)}(m)\ \bar{q}v^\mu\gamma_5 Q_v$$

Run in the EFT to μ:

$$\bar{q}\gamma^\mu\gamma_5 Q \to C_1^{(A)}(\mu)\ \bar{q}\gamma^\mu\gamma_5 Q_v + C_2^{(A)}(\mu)\ \bar{q}v^\mu\gamma_5 Q_v$$

μ dependence given by the anomalous dimension in the EFT

RGE

$$\mu\frac{\mathrm{d}C}{\mathrm{d}\mu} = \gamma_O C = \frac{\alpha_s}{\pi}C$$

Can integrate a one-loop anomalous dimension:

$$\begin{aligned}\mu\frac{\mathrm{d}C}{\mathrm{d}\mu} &= \gamma C, \qquad \gamma = \gamma_0\frac{g^2}{16\pi^2}\\ \mu\frac{\mathrm{d}g}{\mathrm{d}\mu} &= -\frac{g^3}{16\pi^2}b_0\end{aligned}$$

Then

$$\frac{\mathrm{d}C}{\mathrm{d}g} = -\frac{\gamma_0 C}{b_0 g},$$

Solution to one-loop RGE

$$\frac{C(\mu_1)}{C(\mu_2)} = \left[\frac{\alpha_s(\mu_1)}{\alpha_s(\mu_2)}\right]^{-\gamma_0/(2b_0)}$$

$$b_0 = 11 - \frac{2}{3}n_f$$

$$\gamma_O = -\frac{g^2}{4\pi^2}$$

so $\gamma_0 = -4$, $b_0 = 25/3$ below m_b, and

$$\frac{C(\mu)}{C(m_b)} = \left[\frac{\alpha_s(\mu)}{\alpha_s(m_b)}\right]^{6/25}$$

Compute matrix elements at μ:

$$a(\mu) \times \begin{cases} -iv^{\mu} P_v^{(Q)} & \text{if } \Gamma^{\mu} = \gamma^{\mu}\gamma_5, \\ iv^{\mu} P_v^{(Q)} & \text{if } \Gamma^{\mu} = v^{\mu}\gamma_5, \\ P_v^{*(Q)\mu} & \text{if } \Gamma^{\mu} = \gamma^{\mu}, \\ 0 & \text{if } \Gamma^{\mu} = v^{\mu}. \end{cases}$$

so that

$$\begin{aligned} f_{P^*} &= \sqrt{m_{P*}}\, a(\mu)\, C_1^{(V)}(\mu), \\ f_P &= \frac{1}{\sqrt{m_P}}\, a(\mu) \left(C_1^{(A)}(\mu) - C_2^{(A)}(\mu) \right). \end{aligned}$$

$a(\mu)C(\mu)$ is μ independent.

$$\frac{f_{P^*}}{f_P} = \sqrt{m_{P^*} m_P}\left\{\frac{C_1^{(V)}}{C_1^{(A)} - C_2^{(A)}}\right\} = \sqrt{m_{P^*} m_P}\left\{1 - \frac{2}{3}\frac{\alpha_s(m_Q)}{\pi}\right\}.$$

$$f_B\sqrt{m_B} = a(\mu)\left[\frac{\alpha_s(\mu)}{\alpha_s(m_b)}\right]^{6/25}\left\{1 - \frac{2}{3}\frac{\alpha_s(m_b)}{\pi}\right\}$$

$$f_D\sqrt{m_D} = a(m_c)\left\{1 - \frac{2}{3}\frac{\alpha_s(m_c)}{\pi}\right\}$$

$$\frac{f_B\sqrt{m_B}}{f_D\sqrt{m_D}} = \left[\frac{\alpha_s(m_c)}{\alpha_s(m_b)}\right]^{6/25}$$

One-loop running and tree-level matching

two-loop running and one-loop matching

Matching and running must be computed in the same scheme.

$1/m$ Corrections

$$\mathcal{L} = \bar{h}_v \left(iv \cdot D\right) h_v + \frac{1}{2m_Q} \bar{h}_v \left(iD_\perp\right)^2 h_v - c_F \frac{g}{4m_Q} \bar{h}_v \sigma_{\alpha\beta} G^{\alpha\beta} h_v$$

$$c_F(\mu) = \left[\frac{\alpha_s(m_Q)}{\alpha_s(\mu)}\right]^{9/(33-2n_f)}$$

Reparameterization Invariance

$$\begin{aligned} p_Q &= m_Q v + k, \\ v &\to v + \varepsilon/m_Q, \\ k &\to k - \varepsilon. \end{aligned}$$

Since $v^2 = 1$, $v \cdot \varepsilon = 0$.
Also $\not{v} Q_v = Q_v$ so the change in the field:

$$Q_v \to Q_v + \delta Q_v,$$

δQ_v satisfies

$$\left(\not{v} + \frac{\not{\varepsilon}}{m_Q} \right) (Q_v + \delta Q_v) = Q_v + \delta Q_v.$$

so that

$$(1-\not v)\delta Q_v = \frac{\not \varepsilon}{m_Q} Q_v.$$

One can choose:

$$\delta Q_v = \frac{\not \varepsilon}{2m_Q} Q_v.$$

[Not unique, one can always make field redefinitions]
L invariant under

$$\begin{aligned} v &\longrightarrow v + \varepsilon/m_Q, \\ Q_v &\longrightarrow e^{i\varepsilon\cdot x}\left(1 + \frac{\not \varepsilon}{2m_Q}\right) Q_v, \end{aligned}$$

$$\mathcal{L}_0 \quad \rightarrow \quad \mathcal{L}_0 + \frac{1}{m_Q}\bar{Q}_v\,(i\varepsilon\cdot D)\,Q_v,$$
$$\mathcal{L}_1 \quad \rightarrow \quad \mathcal{L}_1 - \frac{1}{m_Q}\bar{Q}_v\,(i\varepsilon\cdot D)\,Q_v.$$

so that the kinetic energy is not renormalized.

Other connections that follow form reparameterization invariance:

$$c_S = 2c_F - 1, \qquad\qquad \boldsymbol{\sigma}\cdot\nabla\times E$$

E.g. relates matching coefficients of leading order and $1/m$ operators, and their anomalous dimensions.

Luke's Theorem

Can compute $1/m$ corrections to meson form-factors. Two sources of $1/m$ corrections, those from the Lagrangian, and from the current. So one has

$$T\left(\mathcal{L}_1, J_0\right), \qquad J_1$$

where

$$\mathcal{L} = \mathcal{L}_0 + \frac{1}{m}\mathcal{L}_1 + \ldots, \qquad J = J_0 + \frac{1}{m}J_1 + \ldots$$

Can apply the same spurion analysis as before, and work out the form-factors. Complicated expressions involving more Isgur-Wise functions for the matrix elements that enter.

Luke's theorem: no $1/m$ corrections to the form-factor at zero recoil.

V_{cb}

Experimentally, measure $\bar{B} \to D^*$ which determines $|V_{cb}\mathcal{F}(1)|$.

$$\mathcal{F}(1) = \eta_A + 0 + \mathcal{O}\left(\frac{1}{m^2}\right)$$

$\eta_A = 0.96$, and $1/m^2 \approx -0.05$, so

$$\mathcal{F}(1) = 0.91 \pm 0.05$$

and from this one finds

$$|V_{cb}| = [\,38.6 \pm 1.5(\mathrm{exp}) \pm 2.0(\mathrm{th})\,] \times 10^{-3},$$