Lattice QCD and Flavour Physics

Chris Sachrajda

Contents

LNF Spring School, *Bruno Touschek*, May 2004

3. Lattice QCD for the Unitarity Triangle

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In particular the non-perturbative quantities

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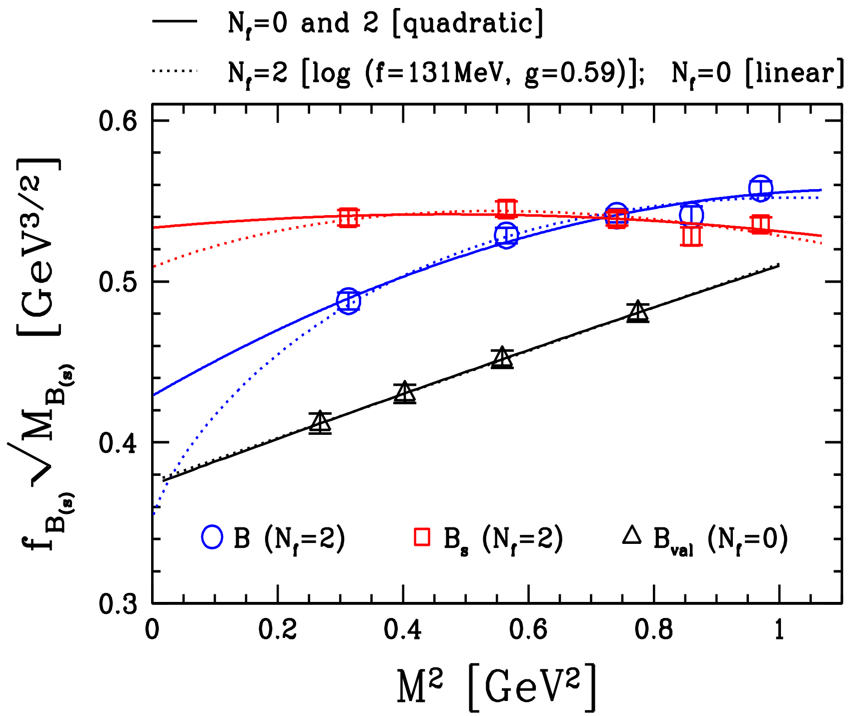
• In addition, lattice QCD is used to obtain the form factors for $B \to D^*(D)\ell\nu$ and $B \to \rho(\pi)\ell\nu$ decays.
Knowledge of these form-factors allow for the determination of V_{cb} and V_{ub} CKM Matrix elements from experimental measurements of the differential decay rates.

S.Hashimoto et al., JLQCD – hep-lat/0209091



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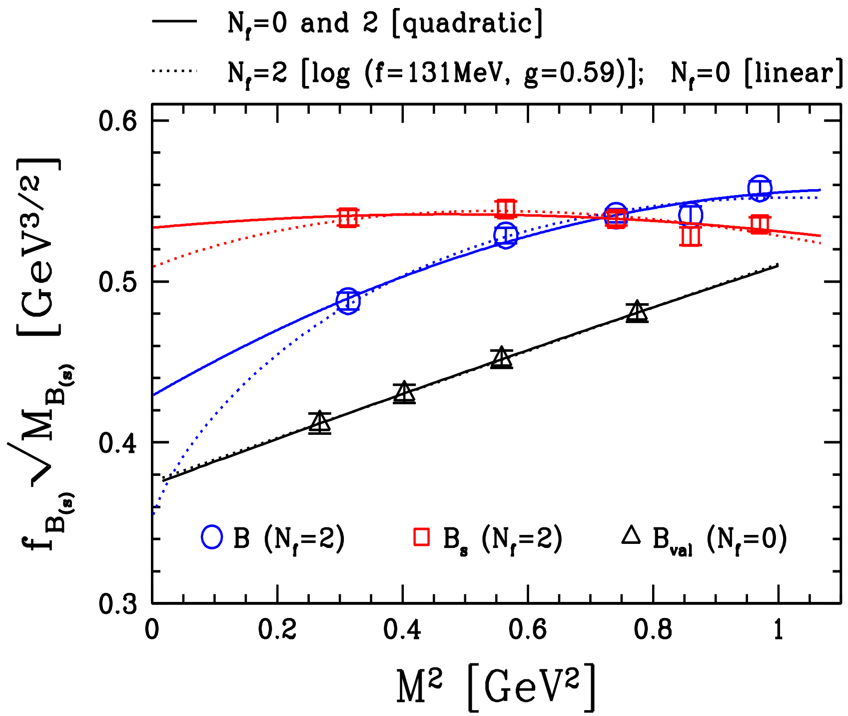

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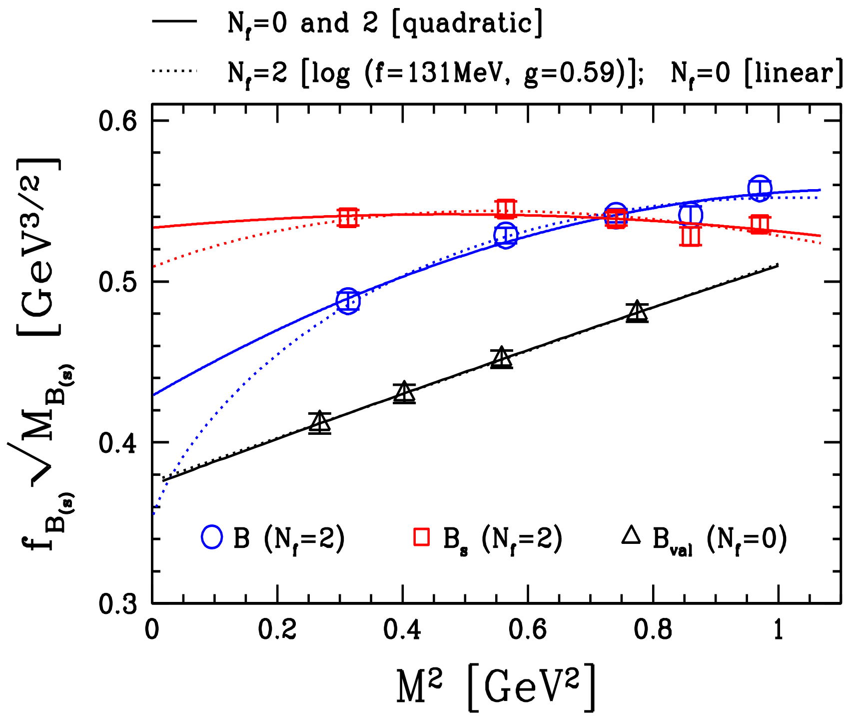
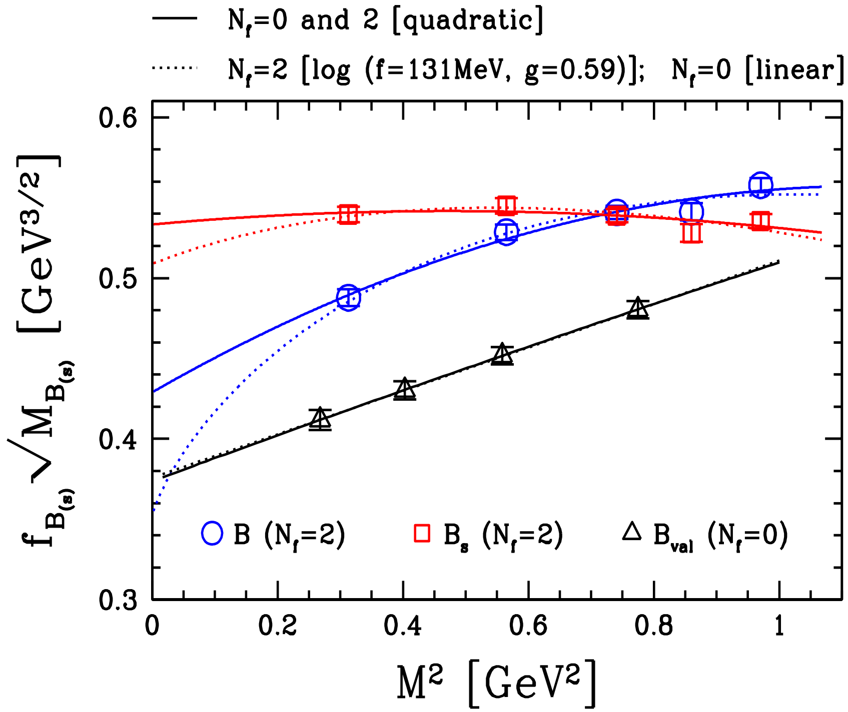
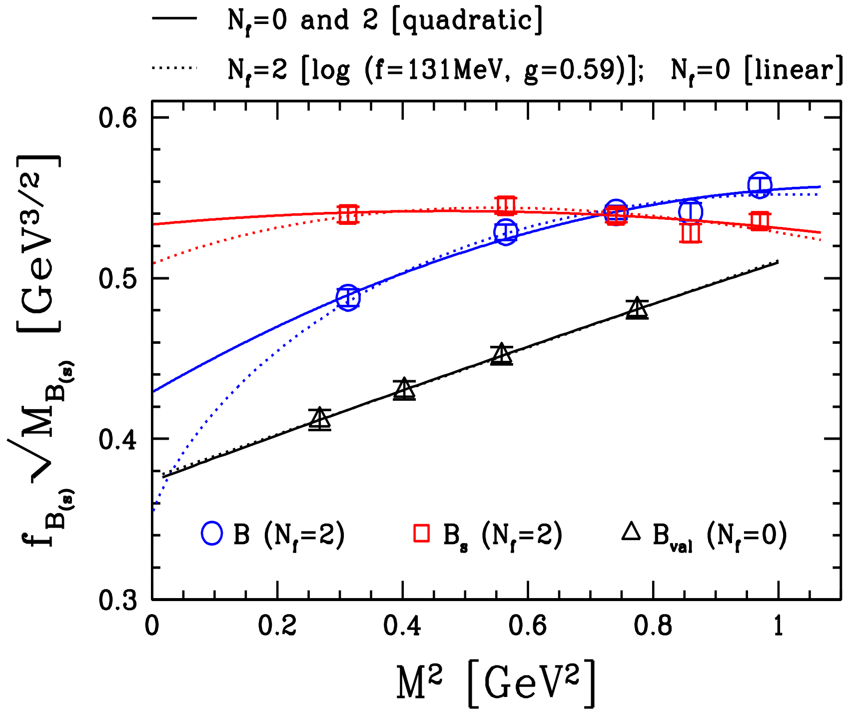


- See also S.Aoki et al., JLQCD – hep-ph/0307039.
- It seems doubtful that χPT is applicable the larger masses (for other quantities fits are poor in this region).
- The results are sensitive to the chiral behaviour in the region where there is no data!



• f_{B_s}/f_{B_d} is significantly larger if chiral perturbation theory is used to perform the extrapolation.



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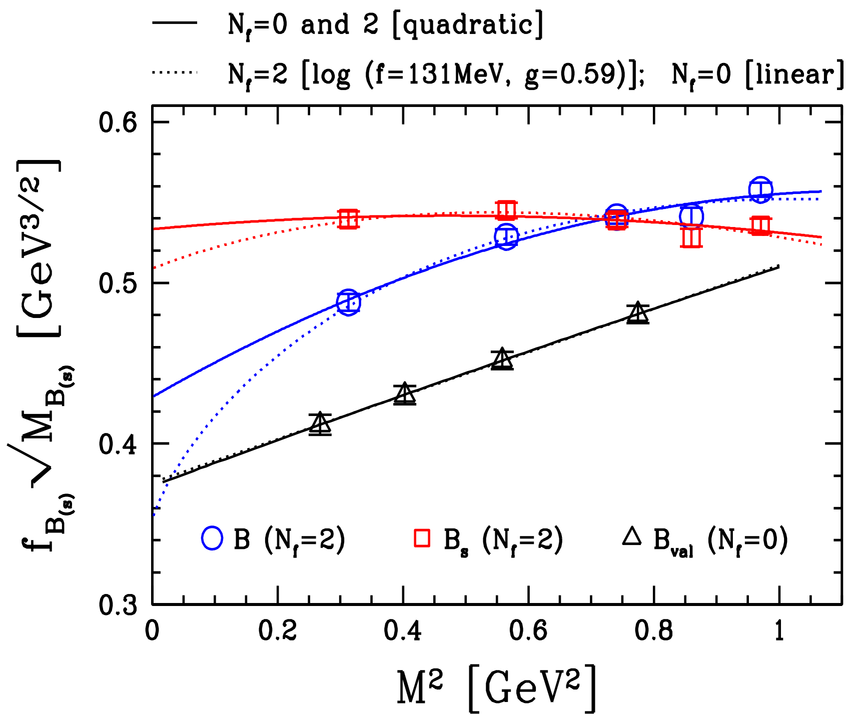

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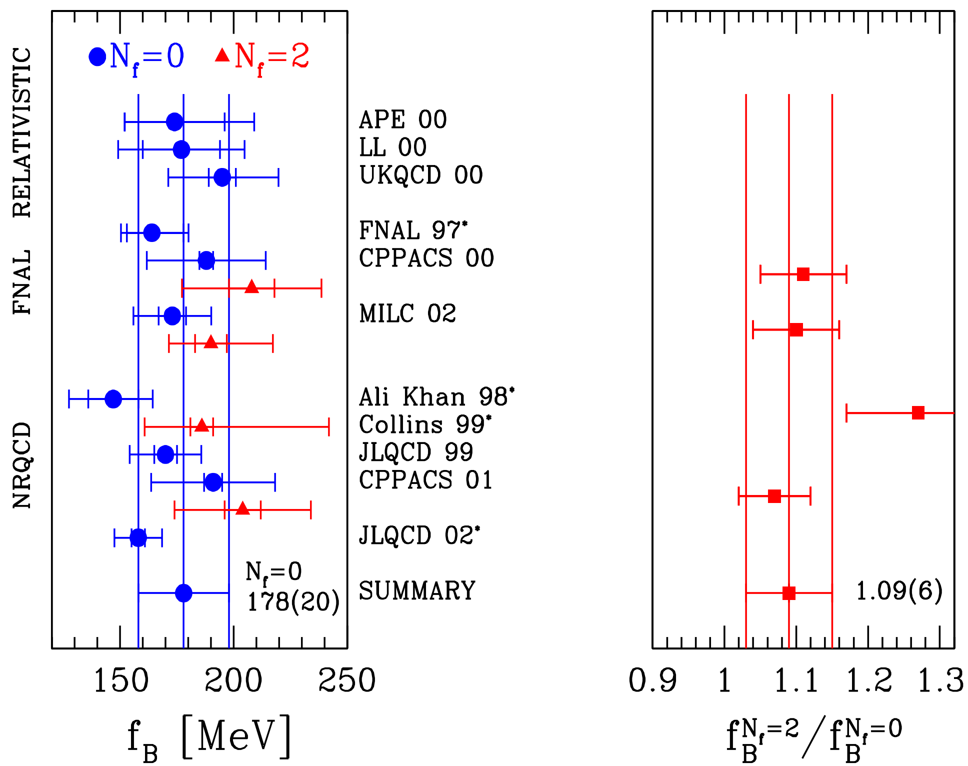
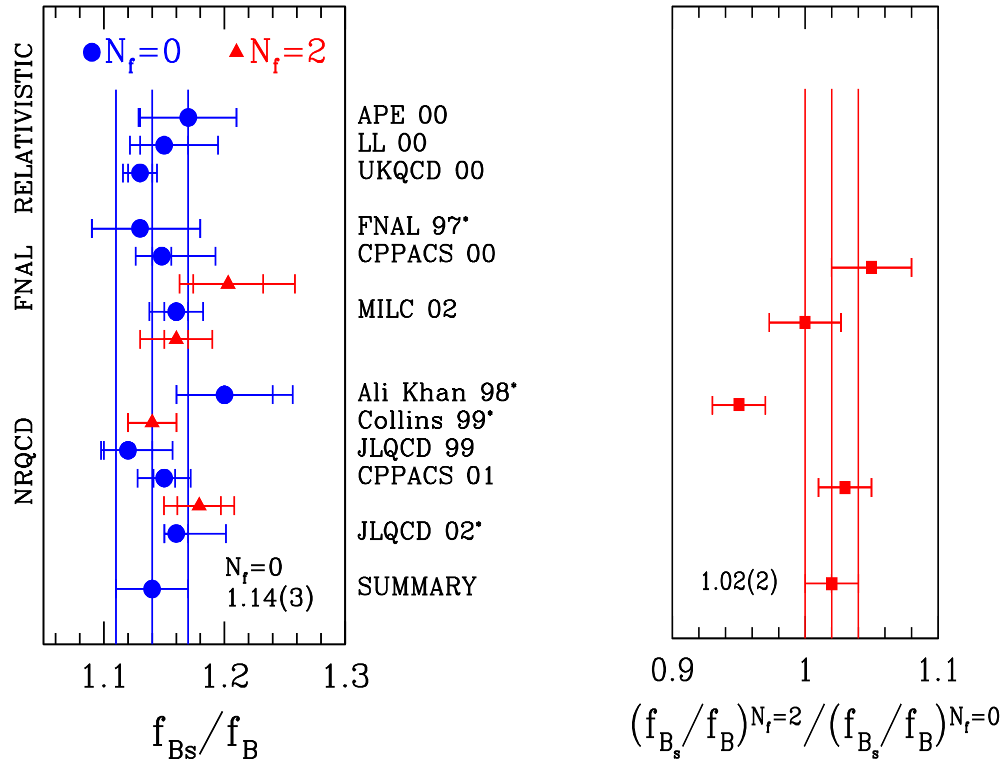
• The following compilations are taken from L.Lellouch's lecture at ICHEP 2002, hep-ph/0211359.



RELATIVISTIC
FNAL
NRQCD
N_f=0
N_f=2
APE 00
LL 00
UKQCD 00
FNAL 97*
CPPACS 00
MILC 02
Ali Khan 98*
Collins 99*
JLQCD 99
CPPACS 01
JLQCD 02*
SUMMARY
N_f=0
178(20)
150
200
250
f_B [MeV]
1.09(6)
0.9
1
1.1
1.2
1.3
f_B^{N_f=2} / f_B^{N_f=0}



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$N_f=0$
1.14(3)
1.1
1.2
1.3
f_{Bs}/f_B
1.02(2)
0.9
1
1.1
$(f_{B_s}/f_B)^{N_f=2}/(f_{B_s}/f_B)^{N_f=0}$

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Even if the quenched low-energy constants can be determined accurately, what does this imply for physical quantities at the chiral limit.

The goal is therefore to determine the low-energy constants in full QCD.

• A key point will be to demonstrate the validity of χPT in the region in which we have data.

• Partially quenched QCD?

Chiral Extrapolation of ξ

Kronfeld& Ryan, Bećirević, Fajfer, Prelovsek and Zupan, JLQCD, $\cdots$

- The quantity

$$\xi \equiv \frac{f_{B_s}\sqrt{B_s}}{f_{B_d}\sqrt{B_d}}\,,$$

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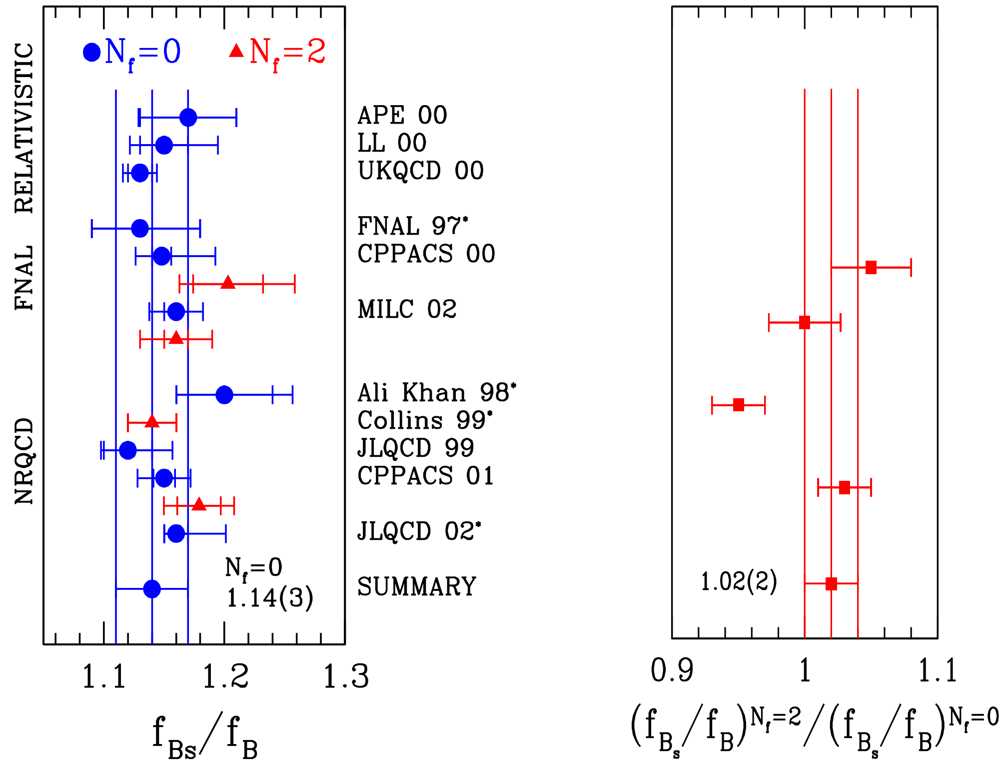
• In the $SU(3)_{\mathrm{flavour}}$ limit $\xi = 1$, and lattice calculations, assuming a linear (polynomial) extrapolation in m_q have been very stable, typically giving:

$$\xi = 1.16(5)\,.$$

This result is a key input into the unitarity triangle analysis.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I start by following the arguments of Kronfeld and Ryan.

• HQET + χPT $\Rightarrow \sqrt{M_{B_q}} f_{B_q} = \Phi\left[1 + \Delta f_q\right]$ and the contribution to ξ is

$$\xi_f - 1 \quad = \quad \Delta f_s - \Delta f_d$$
$$= (m_K^2 - m_\pi^2) f_2(\mu) - \frac{1+3g^2}{(4\pi f)^2}\left(\frac{1}{2}m_K^2 \ln\left[\frac{m_K^2}{\mu^2}\right] + \frac{1}{4}m_\eta^2 \ln\left[\frac{m_\eta^2}{\mu^2}\right] - \frac{3}{4}m_\pi^2 \ln\left[\frac{m_\pi^2}{\mu^2}\right]\right)$$

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• $f_2(\mu)$ is a low-energy constant which has to be determined.

• Similarly, HQET + χPT $\Rightarrow B_{B_q} = B[1 + \Delta B_q]$ and the contribution to ξ is

$$\begin{aligned}\xi_B^2 - 1 &= \Delta B_s - \Delta B_d \\ &= (m_K^2 - m_\pi^2) B_2(\mu) - \frac{1-3g^2}{(4\pi f)^2}\left(\frac{1}{2} m_\eta^2 \ln\left[\frac{m_\eta^2}{\mu^2}\right] - \frac{1}{2} m_\pi^2 \ln\left[\frac{m_\pi^2}{\mu^2}\right]\right)\end{aligned}$$

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$$\xi_f(r) - 1 = m_{ss}^2(1-r)\left\{\frac{1}{2}f_2(\mu) - \frac{1+3g^2}{(4\pi f)^2}\left[\frac{5}{12}\ln\left(\frac{m_{ss}^2}{\mu^2}\right) + l(r)\right]\right\},$$

where $r = m_{qq}^2/m_{ss}^2$ and the chiral logs are contained in

$$l(r) = \frac{1}{1-r}\left[\frac{1+r}{4}\ln\left(\frac{1+r}{2}\right) + \frac{2+r}{12}\ln\left(\frac{2+r}{3}\right) - \frac{3r}{4}\ln(r)\right].$$

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• The "physical" value of ξ_f is then determined from the χPT formula at $r \simeq 1/25$. The variation of the result with r_0 is included in the error.

- Kronfeld & Ryan, $\xi = 1.32(10)$.

- Becirevic, Fajfer, Prelovsek and Zupan, exploit the fact that the chiral logs in Φ_{B_s}/Φ_{B_d} and f_K/f_π are almost the same. They then study the ratio of these two ratios, finding $\xi = 1.22(7)$.

Part of the difference is due to the different choice of g (-0.02), part to the neglect of $\sqrt{m_{B_S}/m_{B_d}}$ (-0.01) and part to the determination of the low-energy constant f_2 (-0.06).

The difference in the errors is largely due to the analysis procedure.

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• Two recent reviews give:

N.Yamada (Lattice 2002)	$\xi = 1.16(6)^{+24}_{-0}$
L.Lellouch (ICHEP 2002)	$\xi = 1.18(4)^{+12}_{-0}$

These results are based largely on the JLQCD $N_f = 2$ data. In both cases the central value is obtained from a polynomial fit, and the second, asymmetric error is the estimated uncertainty due to the chiral logarithms.

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• Yamada quotes the 17% difference between a polynomial fit and that using χPT. Note however, that this requires χPT to hold up to about 1 GeV. (The data for f_π vs m_π^2 is inconsistent with χPT.)

• If we assume that χPT is valid only up to lower scales, the difference between polynomial and χPT fits is smaller.
Lellouch: "Given $[\cdots]$ the exploratory nature of the investigations of chiral-log effects, it seems reasonable $[\cdots]$ to add a -10% systematic error to f_B to account for the uncertainty in the chiral extrapolation and none to f_{B_s}.

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- Kronfeld in his Lattice 2003 review quotes

$$f_{B_s} = 240 \pm 35\,\text{MeV} \qquad \text{and} \qquad \xi = 1.25 \pm 0.10\,.$$

Comments

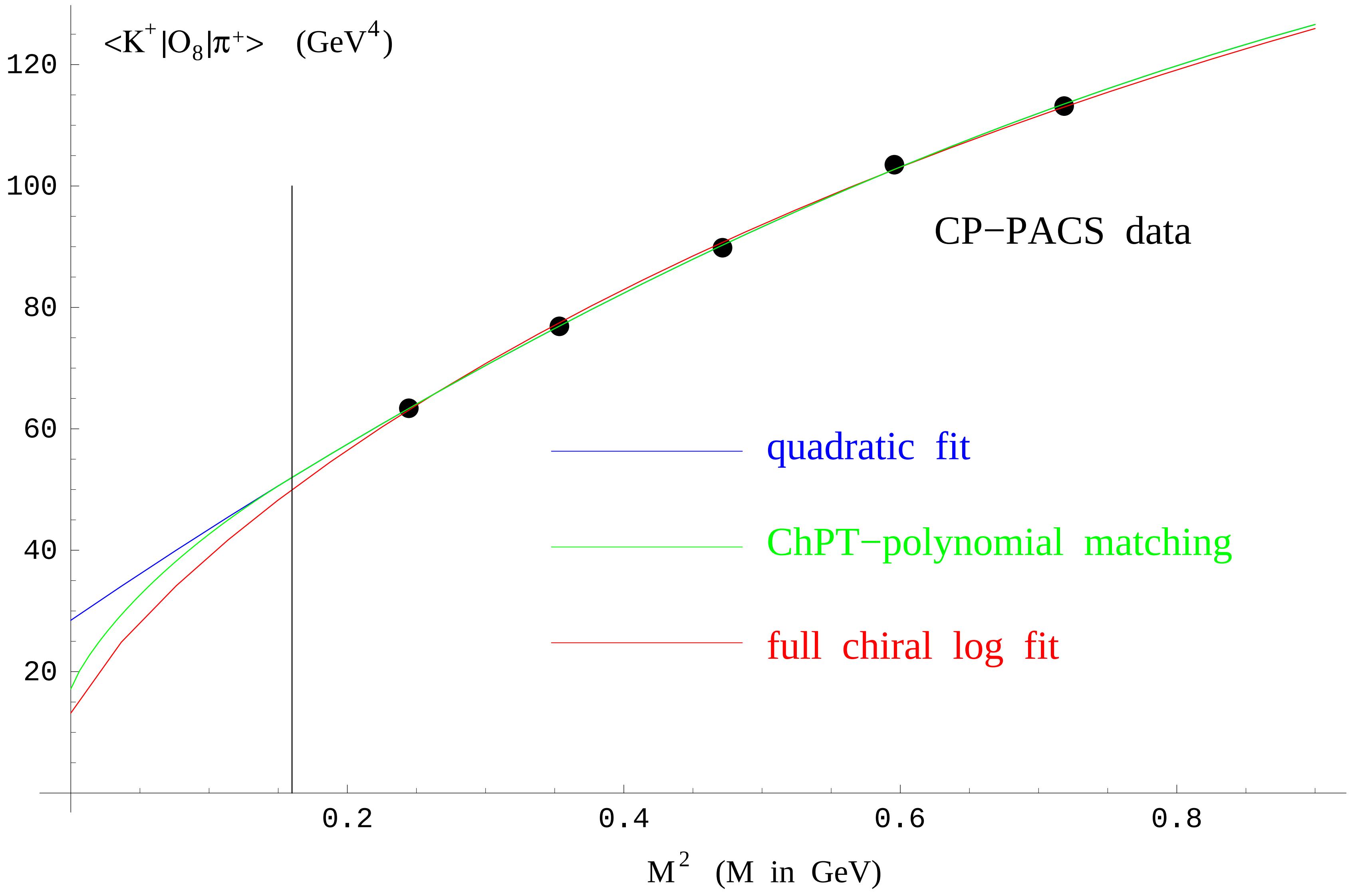
• This is just one example which highlights the importance and the difficulty of performing the chiral extrapolation.

In particular we would like to extend the lattice results to smaller masses to match unambiguously onto NLO chiral behaviour.

• In the absence of this, it is a statement of faith that there is an overlap region, in which both χPT and polynomial behaviour are valid.

My reservations also apply to ratios in which chiral logs almost cancel numerically. (Here I seem to disagree with some reviewers.)

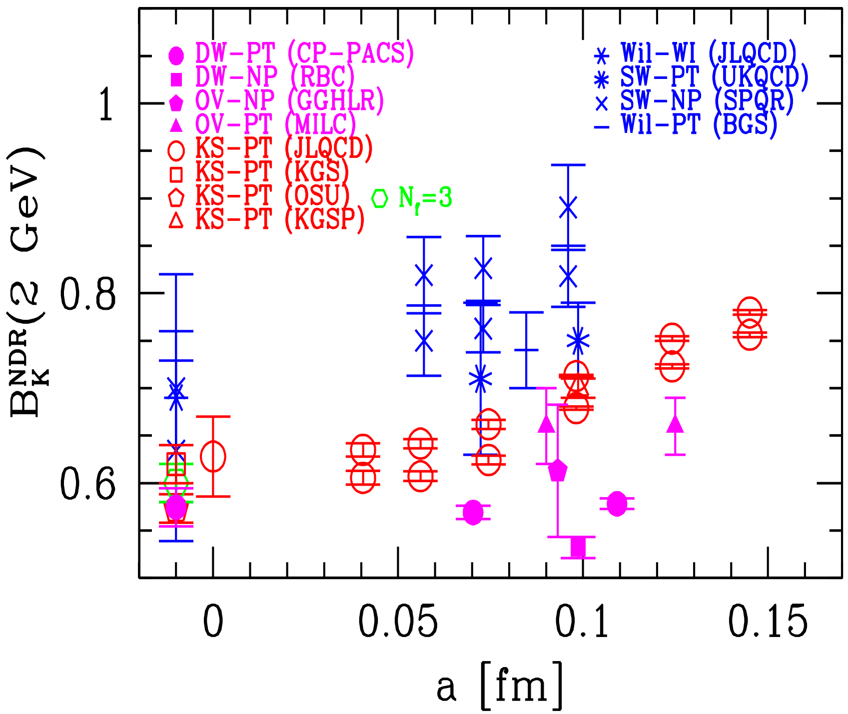
• Caveat - For the quantities discussed in this talk, such a detailed discussion has not been performed.



SPQR Collaboration - Preliminary

B_K

L.Lellouch – hep-ph/0211359



• Note that all but one of the simulations in this figure have been performed in the quenched approximation and kaons are composed of mass degenerate quarks (neglecting $(m_s - m_d)^2$ effects). At the lattice conference results will be presented using dynamical fermions.

• The reference result is the one obtained in the 1997 quenched staggered calculation of JLQCD.

• In the absence of chiral symmetry (e.g. when using Wilson-like quarks), the $\Delta S = 2$ operator $\bar{s}\gamma^\mu(1-\gamma^5)d\,\bar{s}\gamma_\mu(1-\gamma^5)d$ mixes with other dimension 6 operators:

$$
\begin{aligned}
O_1 &= V\otimes V + A\otimes A, \quad O_2 = V\otimes V - A\otimes A, \quad O_3 = S\otimes S + P\otimes P, \\
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\end{aligned}
$$

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- The reference result is the one obtained in the 1997 quenched staggered calculation of JLQCD.

- In the absence of chiral symmetry (e.g. when using Wilson-like quarks), the $\Delta S = 2$ operator $\bar{s}\gamma^{\mu}(1-\gamma^{5})d\,\bar{s}\gamma_{\mu}(1-\gamma^{5})d$ mixes with other dimension 6 operators:

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By making a change of fermionic field variables and exploiting the $d \leftrightarrow s$ symmetry it is possible to avoid these subtractions. This is the Wilson fermion manifestation of a similar feature noticed when using Twisted Mass version of lattice QCD.

Some Recent Studies

- RBC have performed a quenched study on a fine lattice, $a^{-1} = 3\,\mathrm{GeV}$, (with the primary motivation of including the charm quark) with the Domain-Wall Action and an improved gauge action.

$$B_K^{\mathrm{NDR}}(2\,\mathrm{GeV}) = 0.549(11); \qquad \hat{B}_K = 0.764(15)\,.$$

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T.Izubuchi – hep-lat/0310058

• T de Grand and the Milc collaborations have recently presented results obtained using quenched overlap fermions at two lattice spacings (0.13 and 0.09 fm):

$$B_K^{\mathrm{NDR}}(2\,\mathrm{GeV}) = 0.55(7); \qquad \hat{B}_K = 0.79(9)\,.$$

T.de Grand – hep-lat/0309026

- Alpha Collaboration have used quenched Twisted Mass QCD at $\beta \simeq 2.0\,\mathrm{GeV}$ and present 9 results, typically

$$B_K^{\mathrm{NDR}}(2\,\mathrm{GeV}) = 0.64(5); \qquad \hat{B}_K = 0.88(7)\,.$$

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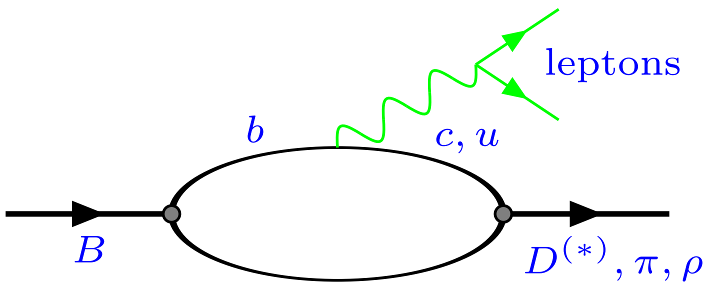
P.Dimopoulos et al. – hep-lat/0309134

Two recent reviews:

L.Lellouch (2002)	$B_K^{\mathrm{NDR}}(2\,\mathrm{GeV}) = 0.628(42)(99); \quad \hat{B}_K = 0.88(7)\,;$
D.Becirevic (2003)	$B_K^{\mathrm{NDR}}(2\,\mathrm{GeV}) = 0.63(4)(\pm 15\%); \quad \hat{B}_K = 0.87(6)(13)\,,$

where the second error is Steve Sharpe's estimate of the quenching effects.

Exclusive Semi-Leptonic B-Decays



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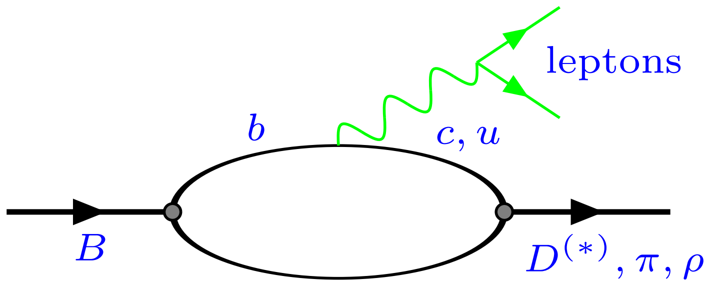

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Lorentz + Parity Invariance $\Rightarrow$ it is convenient to express the amplitudes in terms of invariant form-factors:

$$\langle\, P(p_P)\,|V_\mu(0)\,|\,B(p_B)\,\rangle = f^0(q^2)\,\frac{M_B^2 - M_P^2}{q^2}q_\mu + f^+(q^2)\,\left[(p_B + p_P)_\mu - \frac{M_B^2 - M_P^2}{q^2}q_\mu\right]$$

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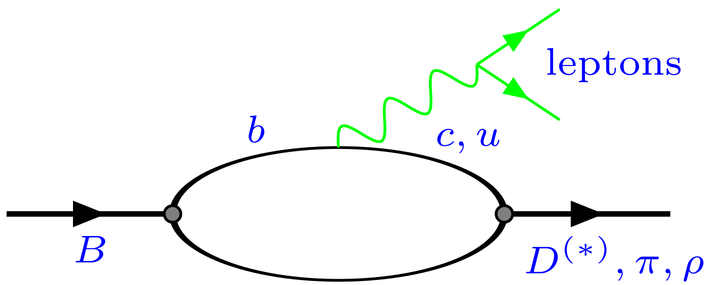

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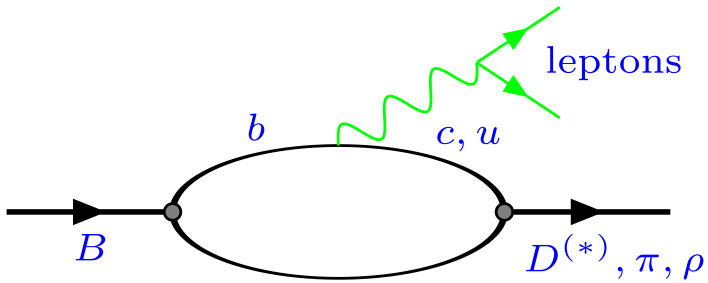

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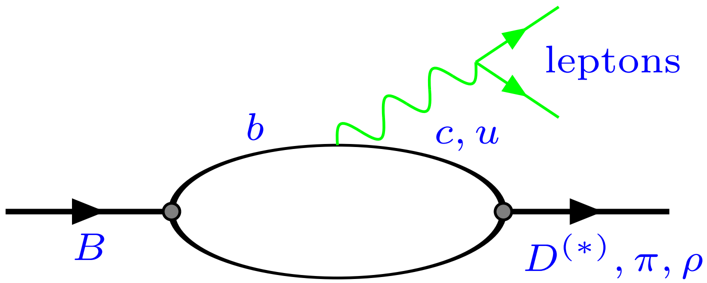

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- $q \equiv p_B - P_P$.
- Parity invariance $\Rightarrow$ only V (from V-A) contributes when the final-state hadron is a pseudoscalar.
- We require P_P to be small to avoid discretization errors. This implies that we can only compute the form-factors at directly at large q^2.

Exclusive Semi-Leptonic B-Decays – Cont.

$$\langle\, V(p_V)\,|\,A_\mu\,|\,B(p_B)\,\rangle \quad = \quad i(M_B + M_V)\, A_1(q^2)\varepsilon^*_\mu$$
$$-i\frac{A_2(q^2)}{M_B + M_V}\,\varepsilon^* \cdot p_B\,(p_B + p_V)_\mu + i\,\frac{A(q^2)}{q^2}\,2M_V\,\varepsilon^* \cdot p_B\,q_\mu$$

- ε is the polarization vector of the vector meson.
- $A_0 = A + A_3$, where

$$A_3 = \frac{M_B + M_V}{2M_V}\,A_1 - \frac{M_B - M_V}{2M_V}\,A_2.$$

- There are three independent vectors so that the vector current can mediate these decays.

$$\langle\, V(p_V)\,|\,V_\mu\,|\,B(p_B)\,\rangle = \frac{2\,V(q^2)}{M_B + M_V}\,\varepsilon^{\mu\gamma\delta\beta}\,\varepsilon^*_\beta\;p_{B\,\gamma}\;p_{V\,\delta}.$$

$\mathbf{B \to D^*}$ Semileptonic Decays

- For $B \to D^*$ decays

$$\frac{d\Gamma}{d\omega} = \frac{G_F^2}{48\pi^3}(m_B - m_{D^*})^2 m_{D^*}^3 \sqrt{\omega^2-1}\,(\omega+1)^2 \times$$
$$\left[1 + \frac{4\omega}{\omega+1}\frac{m_B^2 - 2\omega m_B m_{D^*} + m_{D^*}^2}{(m_B - m_{D^*})^2}\right] |V_{cb}|^2\, \mathcal{F}^2(\omega)\ ,$$

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where $\mathcal{F}(\omega)$ is the IW-function combined with perturbative and power corrections. $(\omega = v_B \cdot v_{D^*})$

- $\mathcal{F}(1) = 1$ up to power corrections and calculable power corrections.
- It is therefore very convenient to consider the distribution near the end-point $\omega = 1$:

$$\mathcal{F}(1) = \eta_{QED}\eta_A \left(1 + 0\,\frac{\Lambda_{QCD}}{m_Q} + c_2\frac{\Lambda_{QCD}^2}{m_Q^2} + \cdots\right) \ ,$$

where $\eta_{QED} = 1.007$ and η_A represents the QED and QCD perturbative corrections

$$\eta_A = 0.960 \pm 0.007 \, .$$

- The power corrections are much more difficult to estimate reliably. PDG take

$$\mathcal{F}(1) = 0.91 \pm 0.04$$

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"The dominant error is theoretical, but there are good prospects that lattice gauge calculations will improve significantly the accuracy of their estimate."

PDG 2002

• In 2002, the Edinburgh node (UKQCD) presented a study of $B \to D$ and $B \to D^*$ form-factors from quenched simulations at $\beta = 6.0$ and 6.2 using an $O(a)$ improved action. They publish a value for the slope ρ^2:

$$\xi(\omega) = 1 - \rho^2(\omega - 1) + O((\omega - 1)^2)\,,$$

where $\xi(\omega)$ is the Isgur-Wise.

Bowler, Douglas, Kenway, Lacagnina & Maynard

• Knowledge of the slope can be helpful in extrapolating the experimental data to $\omega = 1$.

• They find

$$\rho^2 = 0.83^{+15+24}_{-11-1}\,.$$

• This is consistent with previous quenched lattice calculations but lower than the experimentally determined slopes:

$$\begin{array}{rclr} \rho^2 & = & 1.67(11)(22) & \text{CLEO} \\ \rho^2 & = & 1.35(17)(19) & \text{Belle} \end{array}$$

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where the ℓ's are matrix elements of higher dimensional operators.

• The proposal is to determine the ℓ's from calculations of ratios of form-factors.

- For example

$$\mathcal{R}_+ = \frac{\langle D|\bar{c}\gamma^4 b|\bar{B}\rangle\,\langle\bar{B}|\bar{b}\gamma^4 c|D\rangle}{\langle D|\bar{c}\gamma^4 c|D\rangle\,\langle\bar{B}|\bar{b}\gamma^4 b|\bar{B}\rangle} = |h_+(1)|^2$$

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- Hashimoto et al., using the Fermilab formulation of heavy quarks in the quenched approximation, find:

$$\mathcal{F}_{B\to D^*}(1) = 0.913^{+0.024}_{-0.017} \pm 0.016^{+0.003+0.000+0.006}_{-0.014-0.016-0.014}.$$

$B \to \pi\,(\rho)\ell\nu$ decays and $|V_{ub}|$.

- In such decays the π or ρ mesons can have a momentum of $O(M_B/2)$. On present-day lattices this would lead to uncomfortably large discretization errors which are proportional to powers of $a\vec{p}$. Therefore, at present, only a limited kinematic range below the zero-recoil point can be reached without extrapolation.

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$$\Gamma(12.7\,\mathrm{GeV}^2 < q^2 < 18.2\,\mathrm{GeV}^2) = 4.9^{+12\,+0}_{-10\,-14}\,10^{12}\,\mathrm{s}^{-1}\ |V_{ub}|^2\ .$$

K.C.Bowler et al., hep-lat/0402023

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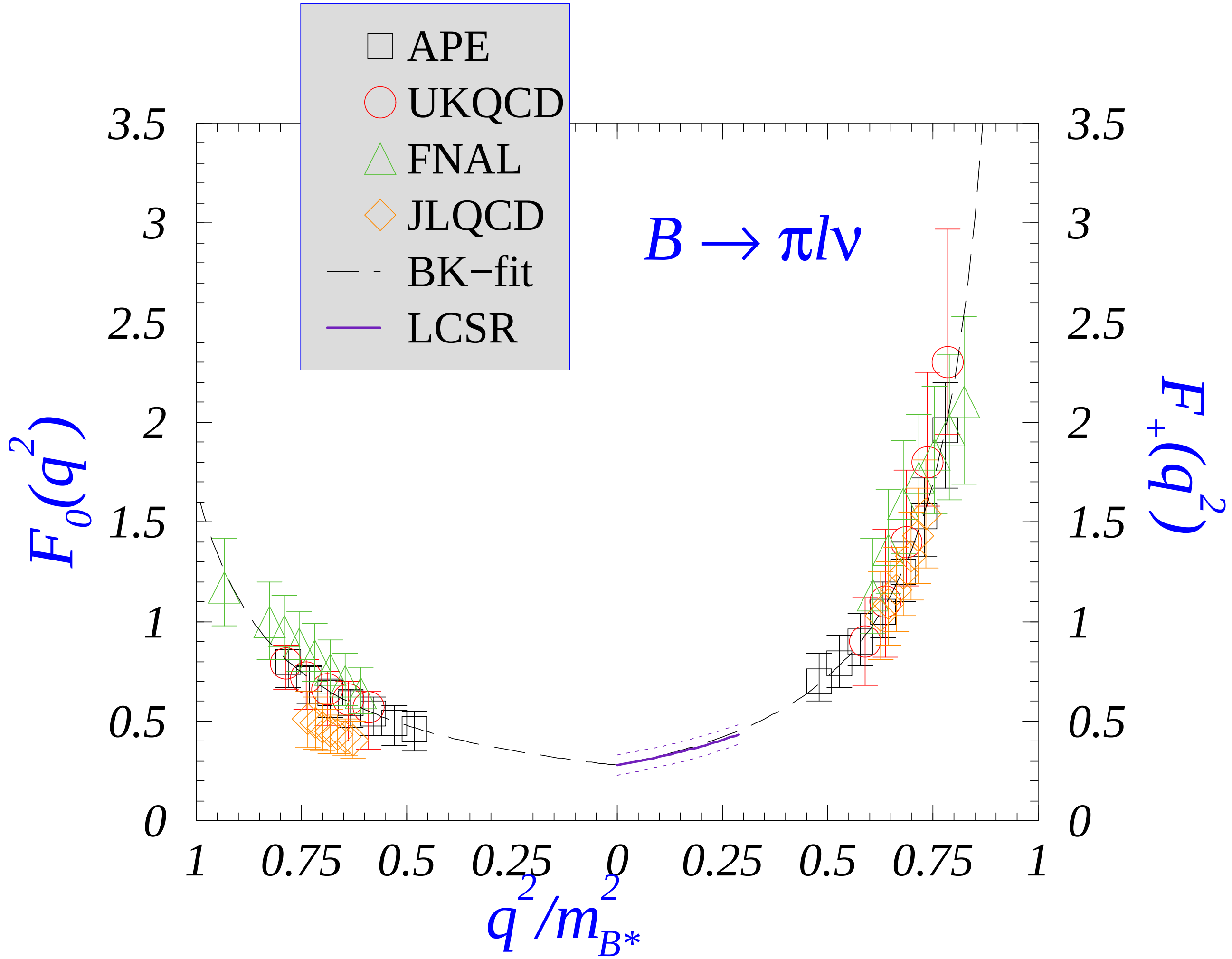
K.C.Bowler et al., hep-lat/0402023

• Extrapolations to lower values of q^2 can be performed using known constraints on the form-factors and give excellent agreement with light-cone sum rules. However, such extrapolations unavoidable have some model dependence.

D.Becirevic - ICHEP 2002



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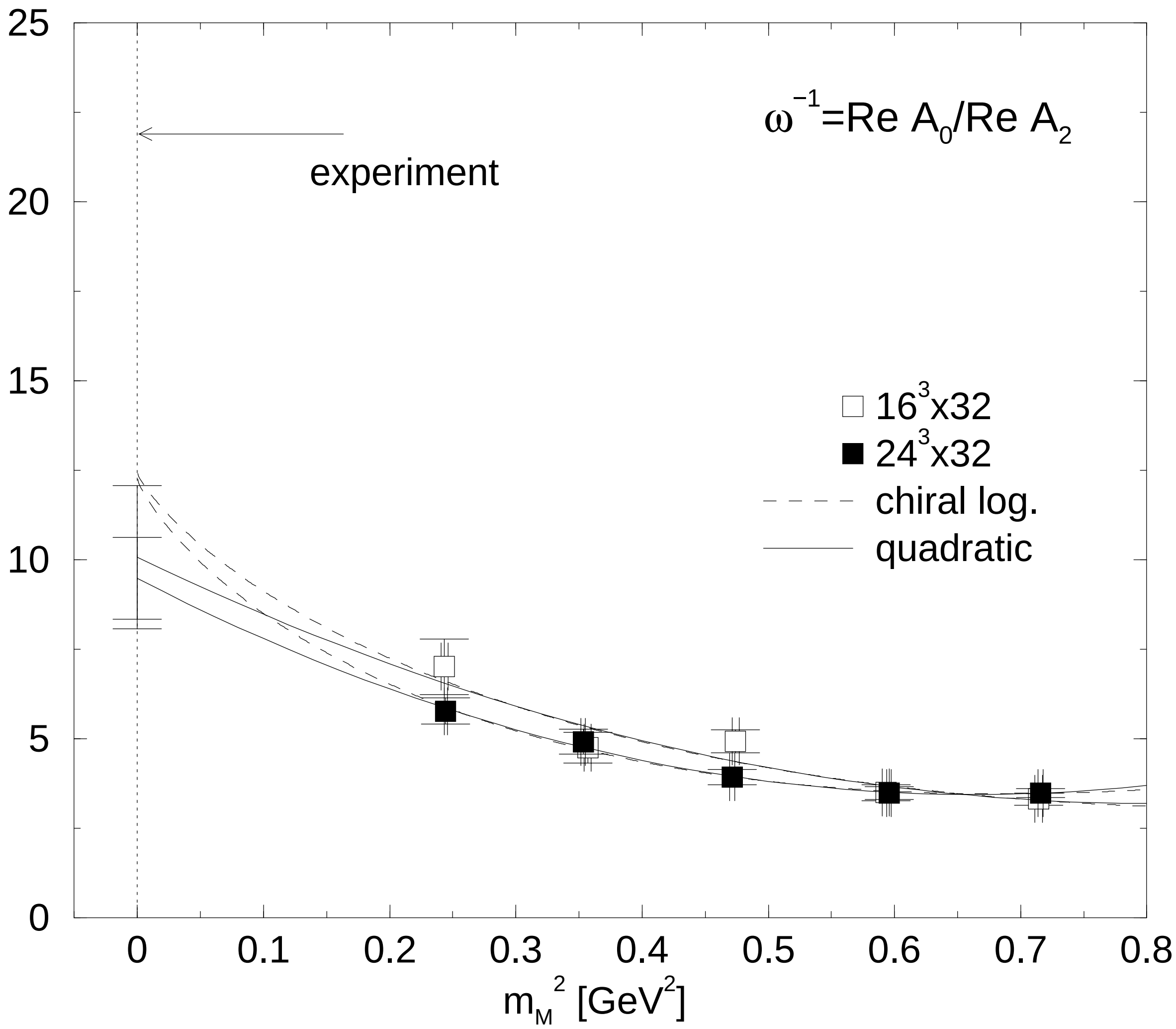
In 2001, two collaborations published some very interesting results on these quantities:

Collaboration(s)	$\mathtt{Re}\ A_0/\mathtt{Re}\ A_2$	ε'/ε
RBC*	25.3 ± 1.8	$-(4.0 \pm 2.3) \times 10^{-4}$
CP-PACS	9÷12	(-7÷-2)$\times 10^{-4}$
Experiments	22.2	$(17.2 \pm 1.8) \times 10^{-4}$

*updated results from July 2002 version of the paper.

$\mathrm{Re}A_0/\mathrm{Re}A_2$ as a function of the meson mass.

CP-PACS



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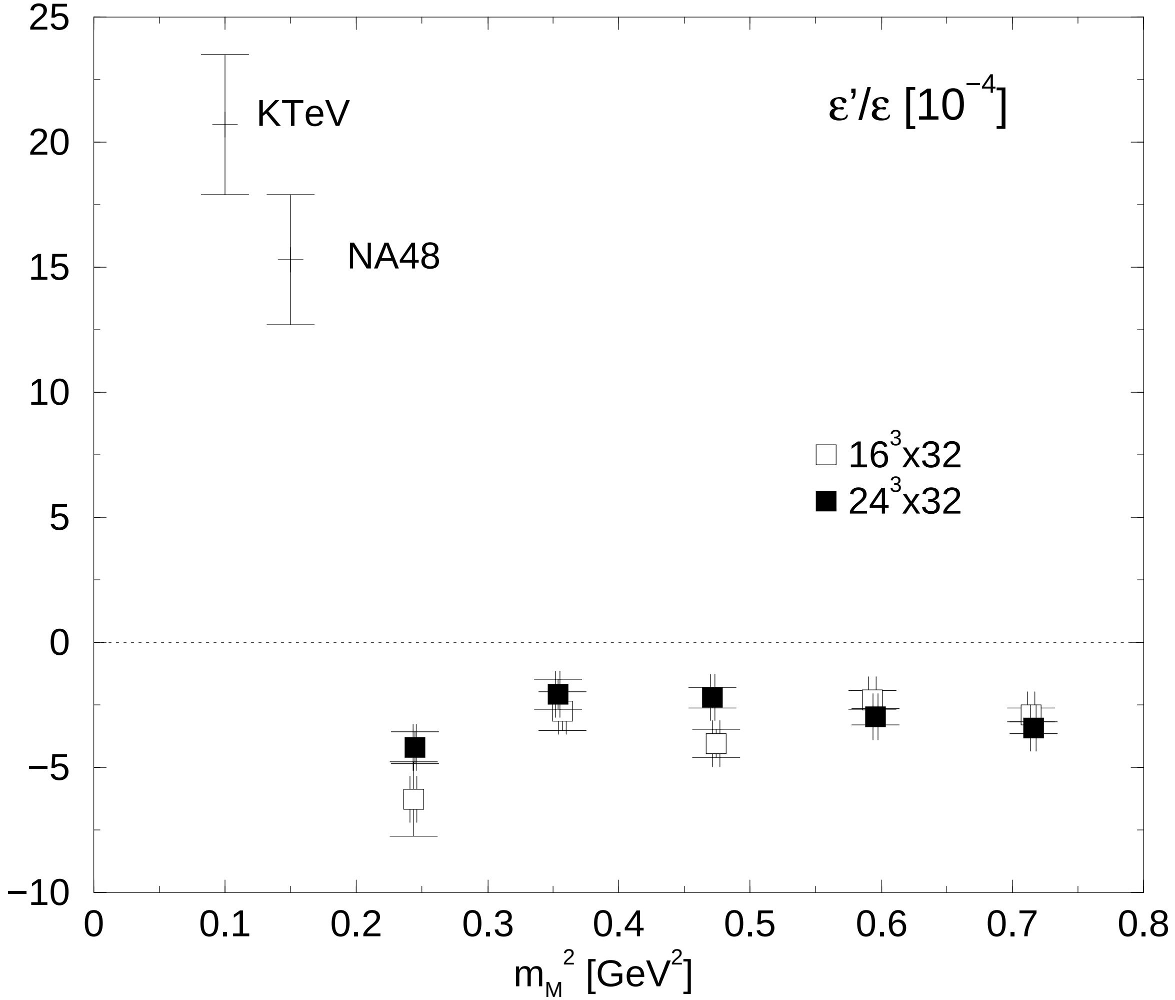
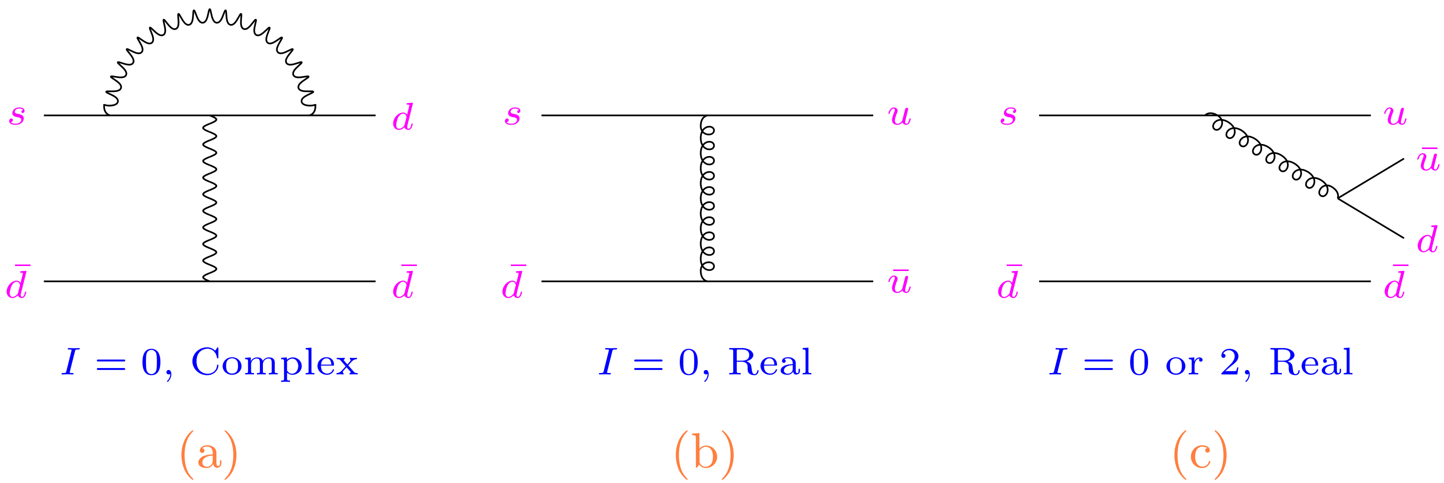


Physics of ε'/ε

- Consider the following contributions to $K \to \pi\pi$ decays:



  – Thus direct CP-violation in kaon decays manifests itself as a non-zero relative phase between the $I = 0$ and $I = 2$ amplitudes.

- We also have *strong phases*, δ_0 and δ_2 which are independent of the form of the weak Hamiltonian.

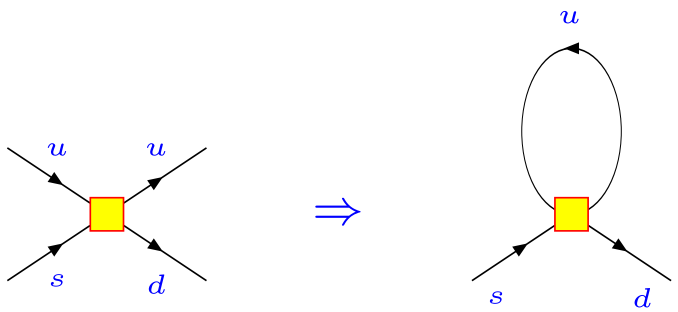
The $\Delta S = 1$ Weak Hamiltonian

$$\mathcal{H}_{eff}(\Delta S = 1) = \frac{G_F}{\sqrt{2}} V_{ud} V_{us}^* \sum_{i=1}^{10} C_i(\mu)\, O_i(\mu)$$

- Non-perturbative QCD effects are contained in the matrix elements of the operators $O_i(\mu)$.
- O_1, O_2 – Current-Current Operators
  e.g. $O_2 = (\bar{s}_L \gamma^\mu u_L)\,(\bar{u}_L \gamma_\mu d_L)$ - charm
- $O_3 - O_6$ – QCD Penguin Operators
  e.g. $O_6 = (\bar{s}_L^i \gamma^\mu d_L^j)\, \sum_q (\bar{q}_R^j \gamma_\mu q_R^i)$
- $O_7 - O_{10}$ – Electroweak Penguin Operators
  e.g. $O_8 = \frac{3}{2}\,(\bar{s}_L^i \gamma^\mu d_L^j)\, \sum_q\, e_q\,(\bar{q}_R^j \gamma_\mu q_R^i)$
- We would like to know the $K \to \pi\,\pi$ matrix elements of these operators in one of the standard continuum renormalization schemes.

Ultra-Violet Issues

- We need to obtain finite matrix elements of renormalized operators from those of the bare lattice operators. The lattice spacing provides a hard UV cut-off.
- Mixing of operators for $\Delta I = 1/2$ decays $\Rightarrow$ power divergences.



Dimension 6 operator $\Rightarrow$ $(\bar{d} \cdots s)$.

- The degree of divergence and which bilinear operators contribute depends on the lattice formulation of QCD being used and its symmetries (in particular its chiral structure).
- The problem of the subtraction of power divergences is simplified very significantly, but not eliminated, in formulations of lattice QCD with an explicit chiral symmetry.

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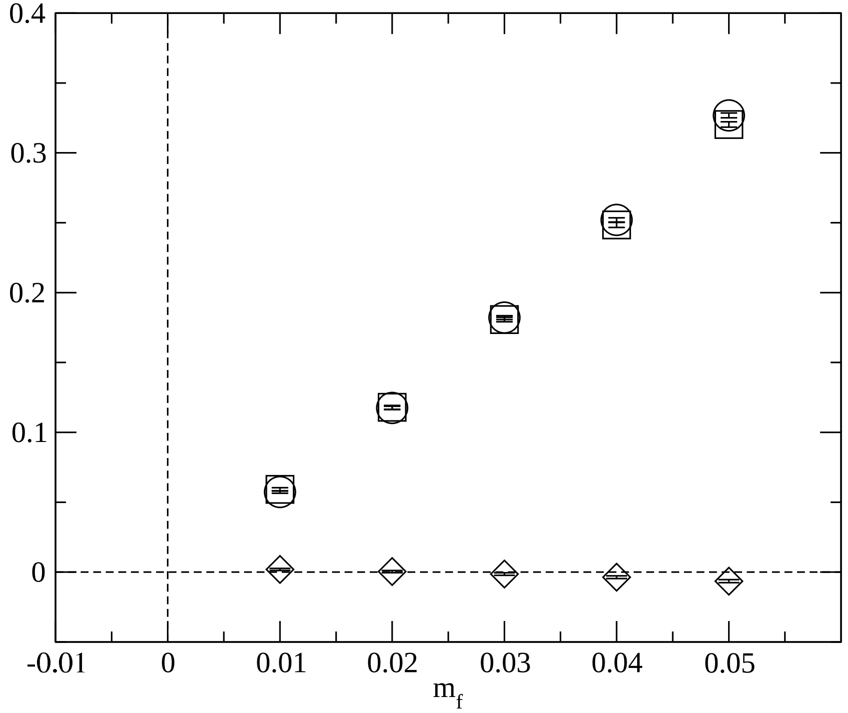
- RBC & CP-PACS use *domain wall fermions*, in which chiral symmetry is approached exponentially as $N_5 \to \infty$.
  They compute $K \to \pi$ and $K \to 0$ matrix elements and determine the corresponding $K \to \pi\pi$ matrix elements using lowest order χPT: for example there are two operators in the $\Delta S = 1$ weak chiral Lagrangian which transform as (8,1),

$$O^{(8,1)} = \alpha_1^{(8,1)} O_1^{(8,1)} + \alpha_2^{(8,1)} O_2^{(8,1)} \qquad \text{with} \qquad \langle 0|O^{(8,1)}|K^0\rangle = c_0\, \alpha_2^{(8,1)}\,;$$

$$\langle \pi^+|O^{(8,1)}|K^+\rangle = c_1 (\alpha_1^{(8,1)} - \alpha_2^{(8,1)})\,; \qquad \langle \pi^+\pi^-|O^{(8,1)}|K^0\rangle = c_2\, \alpha_1^{(8,1)}\,,$$

  where the c_i's are known kinematical constants.
- Thus from the evaluation of $K \to 0$ and $K \to \pi$ matrix elements $\alpha_1^{(8,1)}$, and hence $\langle \pi^+\pi^-|O^{(8,1)}|K^0\rangle$ can be determined. $\alpha_2^{(8,1)}$ contains the power divergences ($1/a^2$ in this case).

Example of the Subtraction of Power Divergences

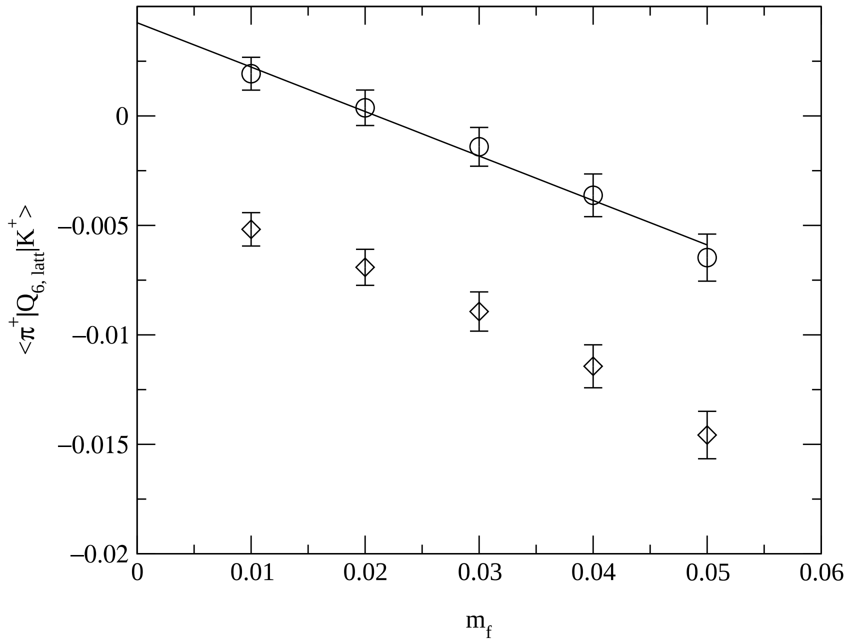


Operator O_6 from RBC:

- Squares - Lattice $K \to \pi$ Matrix Element
- Circles - Term to be Subtracted
- Diamonds - Difference

Highly correlated data $\Rightarrow$ make the subtractions possible.

Example of the Subtraction of Power Divergences - cont.



- Circles - Matrix element of O_6 after subtraction.
- Line should pass through zero if chiral symmetry were exact.
- $K \to \pi\pi$ matrix element obtained from the slope.
- Diamonds - Attempt to subtract residual chiral symmetry breaking effects.

Such studies highlight the necessity of having control of chiral symmetry.

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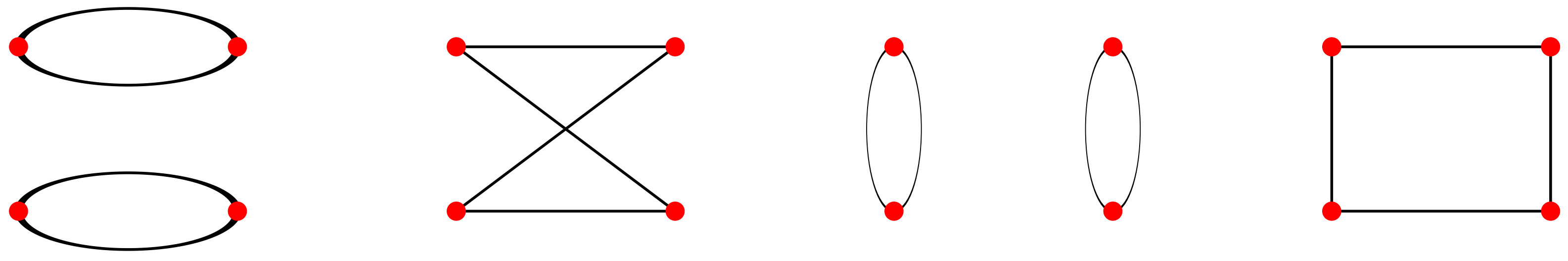
  Can Chiral Perturbation theory be used meaningfully to extrapolate quenched results?
  For ε'/ε there is a significant partial cancellation from the $\Delta I = 1/2$ and $\Delta I = 3/2$ contributions. Does this amplify the relative errors in general, and from the use of LO χPT in particular?

Diagrams Contributing to $\pi\pi$ Scattering



(a) (b) (c) (d)

● = Pion Source or Sink

- Only diagrams (a) and (b) contribute to $I = 2$ $\pi\pi$-scattering. For $I = 0$ scattering all four diagrams contribute.

- Diagrams (a) and (b) are relatively straightforward and cheap to evaluate. The most efficient way of evaluating diagram (c) in particular, requires some more investigation.

Infrared Issues

- Finite-Volume Effects
- χPT at NLO.

For the remainder of the talk I envisage evaluating $K \to \pi\pi$ decays directly.

Quantization Condition for Two-Pion States in a Finite Volume

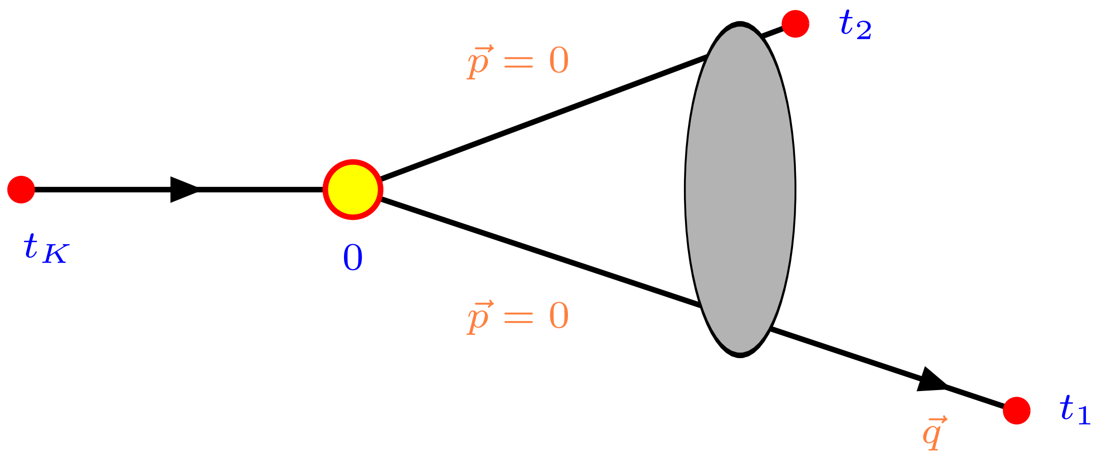
M.Lüscher (1986-91)

Finite-Volume Corrections to $K \to \pi\pi$ Matrix Elements

L.Lellouch & M.Lüscher (2000)

C.-J.D.Lin, G.Martinelli, CTS, M.Testa (2002)

Finite-Volume Effects



L.Maiani & M.Testa (1990) made the following two points about the computation of $K\to\pi\pi$ decays in Euclidean Space:

- At large times the correlator is dominated by the unphysical matrix element with the two-pions at threshold;
- In Euclidean space one obtains real quantities, such as

$$\frac{1}{2}\left\{{}_{\text{out}}\langle\,\pi\pi\,|\mathcal{H}_W\,|\,K\,\rangle + {}_{\text{in}}\langle\,\pi\pi\,|\mathcal{H}_W\,|\,K\,\rangle\right\}.$$

Following the Maiani-Testa paper there was a halt in the calculation of matrix elements between multi-hadron states.

Renewed interest was stimulated by L.Lellouch and M.Lüscher (2000) who:

- argued that by tuning the volume, one is in principle able to extract the matrix element corresponding to the physical kinematics for $K \to \pi\pi$ decays.

  – The correlation function will still be dominated by the matrix element with the two pions in the ground state (unphysical kinematics), so one has to determine the coefficient of a non-leading exponential.

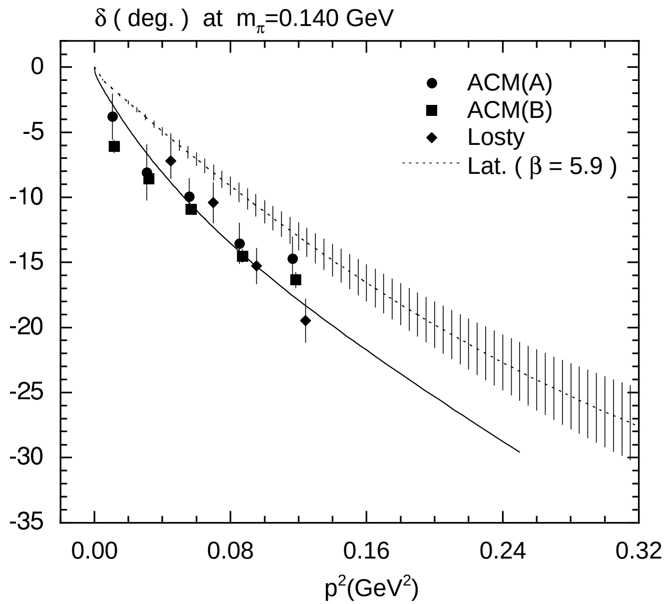
  – For a physical $K \to \pi\pi$ decay with the kaon at rest and the energy of the two-pions corresponding to $n = 1$, the first excited state, one needs a lattice of about 6 fm.

  – (Christ & Kim (2002) propose to use Wiese Boundary conditions so that the lowest energy state can correspond to m_K.)

- derived a formula relating the matrix elements in a finite volume to the modulus of the physical decay amplitudes, up to exponential corrections in the volume.

Two-Pion States in a Finite Cubic Volume - Status

- Finite Volume effects in the spectrum and matrix elements are well understood in the center of mass frame. They depend on the strong interaction phase shift.



Comparison of the lattice results for the I=2 Scattering Phase Shift $\delta(p)$ with experiments.

CP-PACS Collaboration, S.Aoki et al. `hep-lat/0209124`

Two-Pion States in a Finite Cubic Volume - Status (Cont.)

• LMST present a different derivation of the Lellouch-Lüscher formula which makes clear that it is also valid:

- for states with $n > 8$ (so that the infinite-volume limit can be taken at fixed physics);
- for non-zero momentum transfers at the weak operator. This is useful for studies of $K \to \pi\pi$ decays using chiral perturbation theory.

• At present we do not have a generalization of these results to a moving frame ($\vec{p}_K \neq 0$).

see however, Gottlieb & Rummukainen, `hep-lat/9503028`.

K $\rightarrow \pi\pi$ Decays at NLO in the Chiral Expansion

- The evaluation of $K \rightarrow \pi\pi$ decay amplitudes at physical kinematics will not be possible for some years yet. We will therefore continue to rely on χPT to estimate physical decay amplitudes from simulations at unphysical kinematics for some time.

- We have embarked on a major project to exploit χPT at NLO. The generic structure is of the form:

$$\langle\pi\pi|\mathcal{O}_W|K\rangle = \text{LO} * (1 + \text{Logs}) + \text{NLO counterterms.}$$

  The Logs are calculable in one-loop χPT. The idea is to use lattice computations of $K \rightarrow \pi\pi$ matrix elements, for a range of masses and momenta, in order to
  - determine the LO and NLO low-energy constants;
  - use these to determine the physical decay amplitudes.

- It appears that it is possible to determine all the required NLO low-energy constants with a simple set of masses and momenta.

Example – $\langle \boldsymbol{\pi}^+\boldsymbol{\pi}^0 \,|\mathcal{O}_4\,|\boldsymbol{K}^+\rangle$

1) For "physical" kinematics, i.e. for $p_{K^+} = p_{\pi^+} + p_{\pi^0}$,

$$\langle \pi^+\pi^0 \,|\mathcal{O}_4\,|\, K^+ \rangle_{\text{phys}} \;=\; -\frac{6\sqrt{2}}{f_K\, f_\pi^2}\Bigg\{ \alpha^{(27,1)}\Big(m_K^2 - m_\pi^2 + \text{chiral logs}\Big)$$

$$+(\beta_4 - \beta_5 + 4\beta_7 + 2\beta_{22})m_K^4 + (4\beta_2 - 4\beta_4 - 2\beta_7 - 16\beta_{24})m_\pi^4$$

$$+(-4\beta_2 + 3\beta_4 + \beta_5 - 2\beta_7 - 2\beta_{22} + 16\beta_{24})m_K^2 m_\pi^2 \Bigg\}\;.$$

Subscripts correspond to classification by Kambor, Missimer & Wyler.

2) For "SPQR" kinematics, i.e. with the kaon and one of the pions at rest and the other with energy E_π,

$$\langle\, \pi^+ \pi^0\, | \mathcal{O}_4 \,|\, K^+ \,\rangle_{\rm SPQR} \quad = \quad -\frac{6\sqrt{2}}{f_K\, f_\pi^2} \Bigg\{ \alpha^{(27,1)} \Big(E_\pi m_\pi + \frac{1}{2} m_K \left(E_\pi + m_\pi\right) + \text{chiral logs} \Big)$$

$$+4\beta_2 m_\pi^4 + (4\beta_4 + 2\beta_7) E_\pi m_\pi^3 + (\beta_4 - \beta_5 + \beta_7) m_\pi^3 m_K + (\beta_4 - \beta_5 + \beta_7 + 2\beta_{22}) E_\pi m_\pi^2 m_K +$$

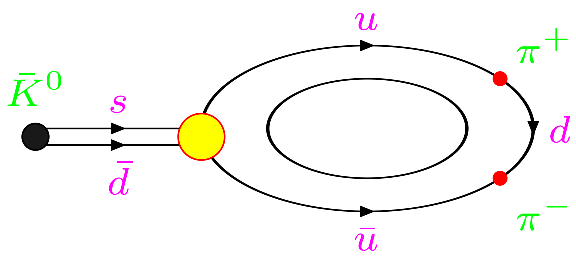
$$(-4\beta_2 + 8\beta_{24}) m_\pi^2 m_K^2 + (\beta_4 + 2\beta_7) E_\pi m_K^3 + (-2\beta_5 + 4\beta_7 + 4\beta_{22}) E_\pi m_\pi m_K^2 + (\beta_4 + 2\beta_7) m_\pi m_K^3$$

$$+(-16\beta_{24}) E_\pi^2 m_\pi^2 + 2\beta_{22} E_\pi^2 m_\pi m_K + 8\beta_{24} E_\pi^2 m_K^2 \Bigg\}.$$

By fitting the computed values of the matrix elements with the SPQR kinematics all the necessary low energy constants can be determined.

Lack of Unitarity in Quenched QCD

In full QCD we have the following contribution to the $\Delta I = 1/2$ decay $\bar{K}^0 \to \pi^+\pi^-$:



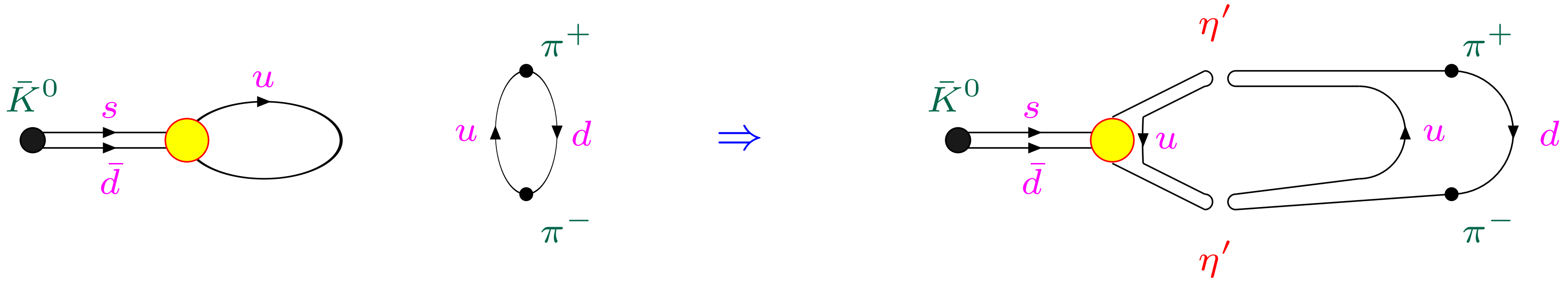
• In the quenched theory this contribution is absent. This is achieved, e.g. by introducing ghost-quarks (with the opposite statistics) to cancel the effect.
Internal particles are not the same as the external ones $\Rightarrow$ FSI depend on the operator.
Is there some meaningful way of overcoming this?

• At one-loop χPT this effect is not present for ΔI=3/2 decays.

• This effect is also present for partially quenched QCD, when $m_K > 2m_\pi$.

Hairpin Diagrams and Double Poles

As an example consider the following $\Delta I = 1/2$ contribution to decay $\bar{K}^0 \to \pi^+\pi^-$ in quenched QCD:



- Qualitatively the η' propagator is rewritten as the first two terms of the pion propagator.
Double Pole $\Rightarrow$ more singular long-distance behaviour.

- At one-loop order in the chiral expansion there are no such contributions to $\Delta I = 3/2$ transitions.

$K \longrightarrow \pi\pi$ Decays –Summary and Conclusions

• There has been a lot of theoretical progress in understanding the ingredients necessary to compute $K \to \pi\pi$ decay amplitudes in lattice computations.

- The non-perturbative subtraction of power divergences.
- Finite-Volume Effects. (But not for $\vec{p} \neq 0$?)
- Chiral Perturbation Theory at NLO $\Rightarrow$ determine all the LEC.
  – Calculation for $\Delta I = 3/2$ is complete for general kinematics is complete, and for $\Delta I = 1/2$ is in progress.
  – Significant difficulties for $\Delta I = 1/2$ decays in quenched and partially quenched QCD.

• **We now need to reap the harvest of this theoretical investment in numerical simulations of $K \to \pi\pi$ decays.**