Lattice QCD and Flavour Physics

Chris Sachrajda

University of Southampton

LNF Spring School, *Bruno Touschek*, May 2004

Preface

• At this school we have learned that frequently the largest uncertainty in extracting information about fundamental physics from experimental measurements is due to our inability to quantify non-perturbative QCD effects.

Preface

- At this school we have learned that frequently the largest uncertainty in extracting information about fundamental physics from experimental measurements is due to our inability to quantify non-perturbative QCD effects.

- *Lattice QCD* allows us to evaluate these effects for a large range of phenomenologically important quantities, from first principles (with no model assumptions or free parameters), by performing large scale numerical simulations of a discrete formulation of QCD.

This is why the organizers have included lectures on lattice QCD at this school.

Preface

- At this school we have learned that frequently the largest uncertainty in extracting information about fundamental physics from experimental measurements is due to our inability to quantify non-perturbative QCD effects.

- *Lattice QCD* allows us to evaluate these effects for a large range of phenomenologically important quantities, from first principles (with no model assumptions or free parameters), by performing large scale numerical simulations of a discrete formulation of QCD.

This is why the organizers have included lectures on lattice QCD at this school.

- In practice, systematic uncertainties reduce the precision which can be achieved.

With experience and improved computing facilities the systematic uncertainties are being reduced (or at least the errors on the errors are being reduced).

Aims of these Lectures

- In these lectures I will not try to:

Aims of these Lectures

- In these lectures I will not try to:

(i) give a course on the technology of lattice QCD;

Aims of these Lectures

- In these lectures I will not try to:

(i) give a course on the technology of lattice QCD;

(ii) try to review all physical quantities which have been (or could be) computed in lattice simulations;

Aims of these Lectures

• In these lectures I will not try to:

(i) give a course on the technology of lattice QCD;

(ii) try to review all physical quantities which have been (or could be) computed in lattice simulations;

(iii) present a catalogue of all the latest results.
These will be obsolete in any case by the end of Lattice 2004 in June.

Aims of these Lectures

• In these lectures I will not try to:

(i) give a course on the technology of lattice QCD;

(ii) try to review all physical quantities which have been (or could be) computed in lattice simulations;

(iii) present a catalogue of all the latest results.
These will be obsolete in any case by the end of Lattice 2004 in June.

• My aims instead are to give an introductory overview of the applications of lattice QCD to phenomenology, so that you will have some feel for:
(i) which quantities can be calculated on the lattice and which cannot;
(ii) the precision which might be reached.

Of course my presentation will necessarily include some theoretical background and many numerical results. (I will try to embed a discussion of some of the theoretical ideas into the discussion of the phenomenology.)

Contents

1. General Introduction to Lattice Phenomenology

• Lattice phenomenology starts with the evaluation of correlation functions of the form:

$$\langle 0|\, O(x_1, x_2, \cdots, x_n)\, |0\rangle = \frac{1}{Z} \int [dA_\mu]\, [d\psi]\, [d\bar{\psi}]\, e^{iS}\, O(x_1, x_2, \cdots, x_n) \ ,$$

where $O(x_1, x_2, \cdots, x_n)$ is a multilocal operator composed of quark and gluon fields and Z is the partition function:

$$Z = \int [dA_\mu]\, [d\psi]\, [d\bar{\psi}]\, e^{iS} \ .$$

1. General Introduction to Lattice Phenomenology

• Lattice phenomenology starts with the evaluation of correlation functions of the form:

$$\langle 0|\, O(x_1, x_2, \cdots, x_n)\, |0\rangle = \frac{1}{Z} \int [dA_\mu]\, [d\psi]\, [d\bar{\psi}]\, e^{iS}\, O(x_1, x_2, \cdots, x_n) \ ,$$

where $O(x_1, x_2, \cdots, x_n)$ is a multilocal operator composed of quark and gluon fields and Z is the partition function:

$$Z = \int [dA_\mu]\, [d\psi]\, [d\bar{\psi}]\, e^{iS} \ .$$

- These formulae are written in Minkowski space, whereas Lattice calculations are performed in Euclidean space ($\exp(iS) \rightarrow \exp(-S)$ etc.).

1. General Introduction to Lattice Phenomenology

• Lattice phenomenology starts with the evaluation of correlation functions of the form:

$$\langle 0|\, O(x_1, x_2, \cdots, x_n)\, |0\rangle = \frac{1}{Z} \int [dA_\mu]\, [d\psi]\, [d\bar{\psi}]\, e^{iS}\, O(x_1, x_2, \cdots, x_n)\ ,$$

where $O(x_1, x_2, \cdots, x_n)$ is a multilocal operator composed of quark and gluon fields and Z is the partition function:

$$Z = \int [dA_\mu]\, [d\psi]\, [d\bar{\psi}]\, e^{iS}\ .$$

- These formulae are written in Minkowski space, whereas Lattice calculations are performed in Euclidean space ($\exp(iS) \to \exp(-S)$ etc.).
- The physics which can be studied depends on the choice of the multilocal operator O.

1. General Introduction to Lattice Phenomenology

- Lattice phenomenology starts with the evaluation of correlation functions of the form:

$$\langle 0|\, O(x_1, x_2, \cdots, x_n)\, |0\rangle = \frac{1}{Z} \int [dA_\mu]\, [d\psi]\, [d\bar{\psi}]\, e^{iS}\, O(x_1, x_2, \cdots, x_n) \ ,$$

where $O(x_1, x_2, \cdots, x_n)$ is a multilocal operator composed of quark and gluon fields and Z is the partition function:

$$Z = \int [dA_\mu]\, [d\psi]\, [d\bar{\psi}]\, e^{iS} \ .$$

- These formulae are written in Minkowski space, whereas Lattice calculations are performed in Euclidean space ($\exp(iS) \to \exp(-S)$ etc.).
- The physics which can be studied depends on the choice of the multilocal operator O.
- The functional integral is performed by discretising space-time and using Monte-Carlo Integration.

Two-Point Correlation Functions

- Consider two-point correlation functions of the form:

$$C_2(t) = \int d^3x \; e^{i\vec{p}\cdot\vec{x}} \; \langle 0|\, J(\vec{x},t)\, J^\dagger(\vec{0},0)\,|0\rangle \ ,$$

where J and $J^\dagger$ are any interpolating operators for the hadron H which we wish to study and the time t is taken to be positive.

  - We assume that H is the lightest hadron which can be created by $J^\dagger$.
  - We take $t > 0$, but it should be remembered that lattice simulations are frequently performed on periodic lattices, so that both time-orderings contribute.

$$C_2(t) = \int d^3x \; e^{i\vec{p}\cdot\vec{x}} \; \langle 0|\, J(\vec{x},t)\, J^\dagger(\vec{0},0)\, |0\rangle \; ,$$

$$C_2(t) = \int d^3x \; e^{i\vec{p}\cdot\vec{x}} \; \langle 0| \, J(\vec{x},t) \, J^\dagger(\vec{0},0) \, |0\rangle \; ,$$

Inserting a complete set of states $\{|n\rangle\}$:

$$C_2(t) \quad = \quad \sum_n \int d^3x \; e^{i\vec{p}\cdot\vec{x}} \; \langle 0| \, J(\vec{x},t) \, |n\rangle \; \langle n| J^\dagger(\vec{0},0) \, |0\rangle$$

$$C_2(t) = \int d^3x \; e^{i\vec{p}\cdot\vec{x}} \; \langle 0| \, J(\vec{x}, t) \, J^\dagger(\vec{0}, 0) \, |0\rangle \ ,$$

Inserting a complete set of states $\{|n\rangle\}$:

$$\begin{aligned} C_2(t) &= \sum_n \int d^3x \; e^{i\vec{p}\cdot\vec{x}} \; \langle 0| \, J(\vec{x}, t) \, |n\rangle \; \langle n| J^\dagger(\vec{0}, 0) \, |0\rangle \\ &= \int d^3x \; e^{i\vec{p}\cdot\vec{x}} \; \langle 0| \, J(\vec{x}, t) \, |H\rangle \; \langle H| J^\dagger(\vec{0}, 0) \, |0\rangle + \cdots \end{aligned}$$

where the $\cdots$ represent contributions from heavier states with the same quantum numbers as H.

$$C_2(t) = \int d^3x \; e^{i\vec{p}\cdot\vec{x}} \; \langle 0|\, J(\vec{x},t)\, J^\dagger(\vec{0},0)\, |0\rangle \ ,$$

Inserting a complete set of states $\{|n\rangle\}$:

$$\begin{aligned} C_2(t) &= \sum_n \int d^3x \; e^{i\vec{p}\cdot\vec{x}} \; \langle 0|\, J(\vec{x},t)\, |n\rangle \; \langle n| J^\dagger(\vec{0},0)\, |0\rangle \\ &= \int d^3x \; e^{i\vec{p}\cdot\vec{x}} \; \langle 0|\, J(\vec{x},t)\, |H\rangle \; \langle H| J^\dagger(\vec{0},0)\, |0\rangle + \cdots \end{aligned}$$

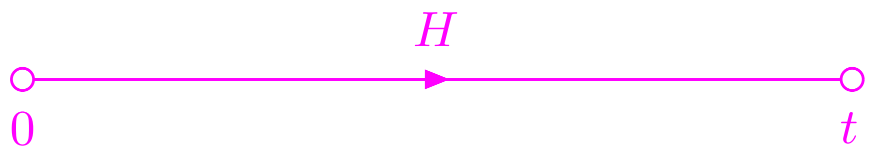
where the $\cdots$ represent contributions from heavier states with the same quantum numbers as H.

Finally using translational invariance:

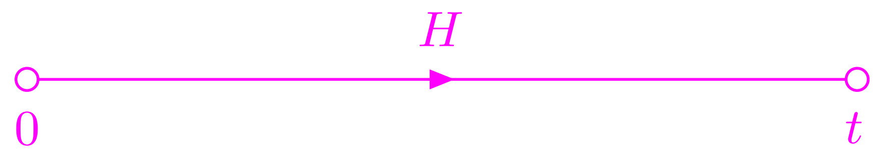
$$C_2(t) = \frac{1}{2E}\, e^{-iEt} \left| \langle 0| J(\vec{0},0) | H(p) \rangle \right|^2 + \cdots \ ,$$

where $E = \sqrt{m_H^2 + \vec{p}^{\,2}}$.

$$C_2(t) = \frac{1}{2E}\, e^{-iEt} \left| \langle 0 | J(\vec{0}, 0) | H(p) \rangle \right|^2 + \cdots \ .$$

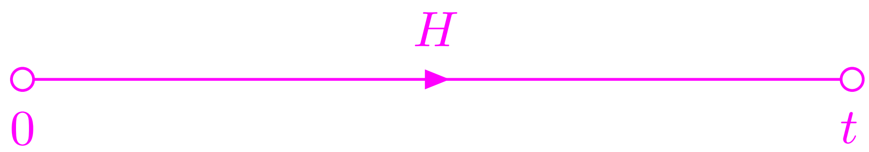


$$C_2(t) = \frac{1}{2E}\, e^{-iEt} \left| \langle 0 | J(\vec{0}, 0) | H(p) \rangle \right|^2 + \cdots \ .$$



- In Euclidean space $\exp(-iEt) \rightarrow \exp(-Et)$.

$$C_2(t) = \frac{1}{2E}\, e^{-iEt} \left| \langle 0 | J(\vec{0},0) | H(p) \rangle \right|^2 + \cdots \ .$$

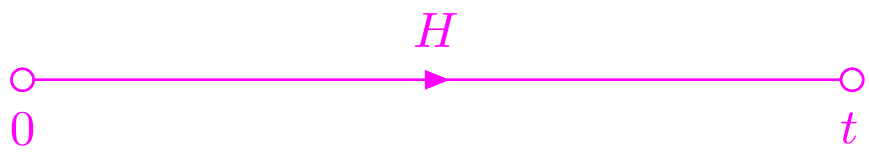


- In Euclidean space $\exp(-iEt) \to \exp(-Et)$.
- By fitting $C(t)$ to the form above, both the energy (or, if $\vec{p} = 0$, the mass) and the modulus of the matrix element

$$\left| \langle 0 | J(\vec{0},0) | H(p) \rangle \right|$$

can be evaluated.

$$C_2(t) = \frac{1}{2E}\, e^{-iEt} \left| \langle 0 | J(\vec{0},0) | H(p) \rangle \right|^2 + \cdots \quad .$$



- In Euclidean space $\exp(-iEt) \to \exp(-Et)$.
- By fitting $C(t)$ to the form above, both the energy (or, if $\vec{p}=0$, the mass) and the modulus of the matrix element

$$\left| \langle 0 | J(\vec{0},0) | H(p) \rangle \right|$$

  can be evaluated.
- Example: if $J = \bar{u}\gamma^\mu\gamma^5 d$ then the decay constant of the π-meson can be evaluated,

$$\left| \langle 0 |\, \bar{u}\gamma^\mu\gamma^5 d\, | \pi^+(p) \rangle \right| = f_\pi\, p^\mu \quad ,$$

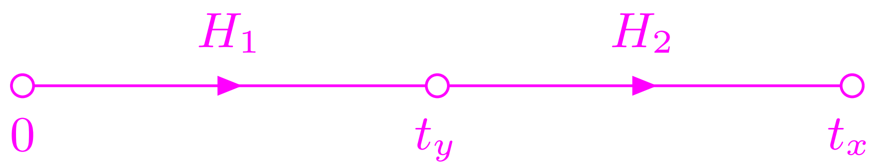
  and the physical value is $f_\pi \simeq 132\,\mathrm{MeV}$.

Three-Point Correlation Functions

- Consider now a three-point correlation function of the form:

$$C_3(t_x, t_y) = \int d^3x\, d^3y\; e^{i\vec{p}\cdot\vec{x}}\; e^{i\vec{q}\cdot\vec{y}}\; \langle 0|\, J_2(\vec{x}, t_x)\, O(\vec{y}, t_y)\, J_1^\dagger(\vec{0}, 0)\, |0\rangle\;,$$

where $J_{1,2}$ may be interpolating operators for different particles and we assume that $t_x > t_y > 0$.

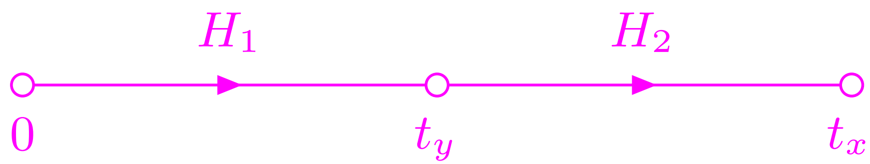


Three-Point Correlation Functions

- Consider now a three-point correlation function of the form:

$$C_3(t_x,t_y) = \int d^3x\, d^3y\; e^{i\vec{p}\cdot\vec{x}}\; e^{i\vec{q}\cdot\vec{y}}\; \langle 0|\, J_2(\vec{x},t_x)\, O(\vec{y},t_y)\, J_1^\dagger(\vec{0},0)\, |0\rangle\ ,$$

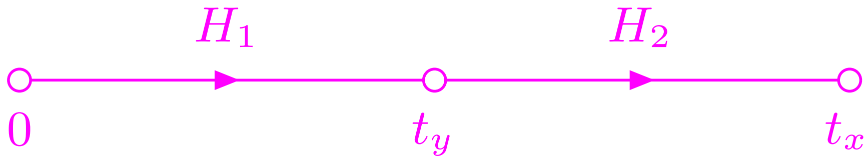
where $J_{1,2}$ may be interpolating operators for different particles and we assume that $t_x > t_y > 0$.



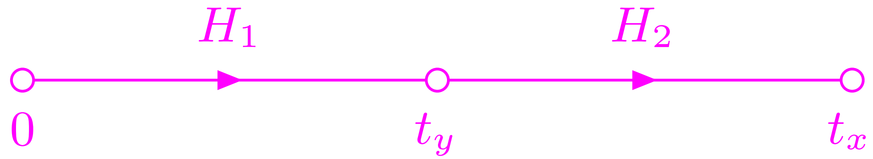
- For sufficiently large times t_y and $t_x - t_y$

$$C_3(t_x,t_y) \quad\simeq\quad \frac{e^{-E_1 t_y}}{2E_1}\, \frac{e^{-E_2(t_x-t_y)}}{2E_2}\, \langle 0|J_2(0)|H_2(\vec{p})\rangle$$
$$\times \langle H_2(\vec{p})|O(0)|H_1(\vec{p}+\vec{q})\rangle\ \langle H_1(\vec{p}+\vec{q})|J_1^\dagger(0)|0\rangle\ ,$$

where $E_1^2 = m_1^2 + (\vec{p}+\vec{q})^2$ and $E_2^2 = m_1^2 + \vec{p}^{\,2}$.



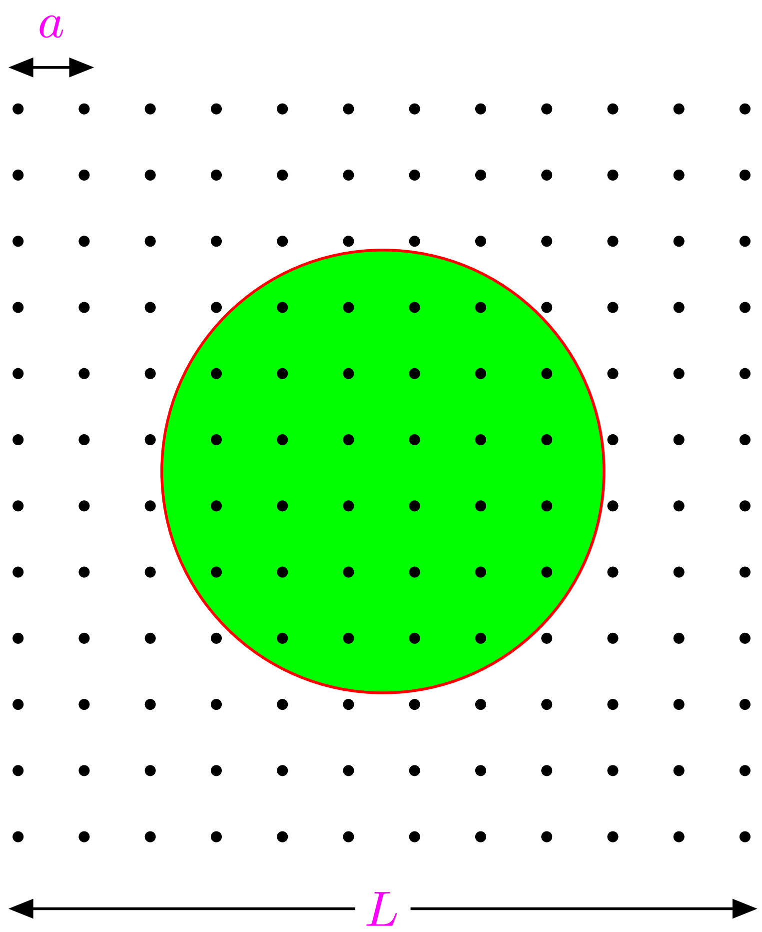
- From the evaluation of two-point functions we have the masses and the matrix elements of the form $|\langle 0|J|H(\vec{p})\rangle|$. Thus, from the evaluation of three-point functions we obtain matrix elements of the form $|\langle H_2|O|H_1\rangle|$.



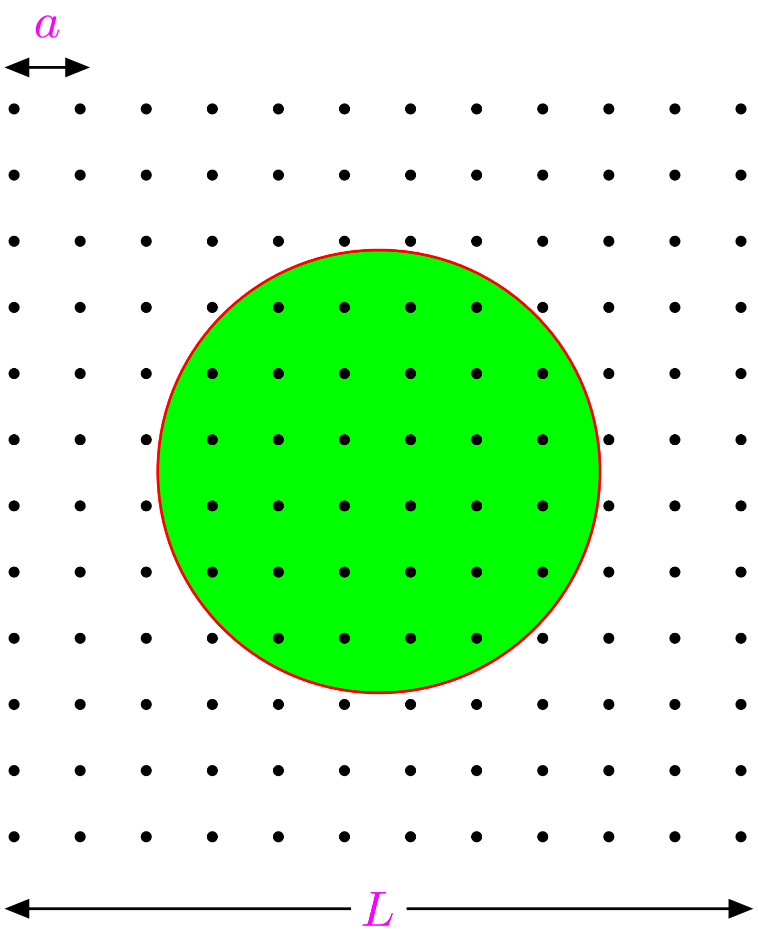
- From the evaluation of two-point functions we have the masses and the matrix elements of the form $|\langle 0|J|H(\vec{p})\rangle|$. Thus, from the evaluation of three-point functions we obtain matrix elements of the form $|\langle H_2|O|H_1\rangle|$.

- Examples: Important examples are $\bar{M}^0 - M^0$ mixing, the semileptonic and rare radiative decays of hadrons of the form $B \longrightarrow \pi, \rho +$ leptons or $B \rightarrow K^*\gamma$. The operators O in this case are quark bilinears.

Systematic Uncertainties



Systematic Uncertainties



- We want

$$L \gg 1\,\text{fm} \quad \text{and} \quad a^{-1} \gg \Lambda_{\text{QCD}}\ .$$

• Computing resources limit the number of lattice points which can be included, and hence the precision of the calculation. Typically in full QCD we can have about 24 points in each spatial direction ($O(50)$ points in quenched simulations) and so compromises have to be made.

• Computing resources limit the number of lattice points which can be included, and hence the precision of the calculation. Typically in full QCD we can have about 24 points in each spatial direction ($O(50)$ points in quenched simulations) and so compromises have to be made.

• **Statistical Errors:** The functional integral is evaluated by Monte-Carlo sampling. The statistical error is estimated from the fluctuations of computed quantities within different clusters of *configurations*. $\surd\surd\surd$

• Computing resources limit the number of lattice points which can be included, and hence the precision of the calculation. Typically in full QCD we can have about 24 points in each spatial direction ($O(50)$ points in quenched simulations) and so compromises have to be made.

• **Statistical Errors:** The functional integral is evaluated by Monte-Carlo sampling. The statistical error is estimated from the fluctuations of computed quantities within different clusters of *configurations*. $\surd\surd\surd$

• The different sources of systematic uncertainty are not independent of each other, so the following discussion is oversimplified.

- **Discretization Errors (Lattice Artefacts):** Current simulations are typically performed with

$$a \sim \left(\frac{1}{20} \text{ - } \frac{1}{10}\right)\ \mathrm{fm}$$

leading to errors of $O(a\Lambda_{\mathrm{QCD}})$ (with Wilson Fermions) or $O(a^2\Lambda^2_{\mathrm{QCD}})$ for *improved* fermion actions.

($0.1\,\mathrm{fm} \simeq 2\,\mathrm{GeV}$.)

• **Discretization Errors (Lattice Artefacts):** Current simulations are typically performed with

$$a \sim \left(\frac{1}{20}\text{ - }\frac{1}{10}\right)\ \text{fm}$$

leading to errors of $O(a\Lambda_{\rm QCD})$ (with Wilson Fermions) or $O(a^2\Lambda^2_{\rm QCD})$ for *improved* fermion actions.

($0.1\,\text{fm} \simeq 2\,\text{GeV}$.)

The errors can be reduced by:

• Performing simulations at several values of a and extrapolating to $a = 0$.

• **Discretization Errors (Lattice Artefacts):** Current simulations are typically performed with

$$a \sim \left(\frac{1}{20} \text{ - } \frac{1}{10}\right) \text{ fm}$$

leading to errors of $O(a\Lambda_{\rm QCD})$ (with Wilson Fermions) or $O(a^2\Lambda^2_{\rm QCD})$ for *improved* fermion actions.

$(0.1\,\text{fm} \simeq 2\,\text{GeV.})$

The errors can be reduced by:

• Performing simulations at several values of a and extrapolating to $a = 0$.

• *Improvement* (Symanzik), i.e. choosing a discretization of QCD so that the errors are formally smaller.

• **Discretization Errors (Lattice Artefacts):** Current simulations are typically performed with

$$a \sim \left(\frac{1}{20}\text{ - }\frac{1}{10}\right)\ \mathrm{fm}$$

leading to errors of $O(a\Lambda_{\mathrm{QCD}})$ (with Wilson Fermions) or $O(a^2\Lambda^2_{\mathrm{QCD}})$ for *improved* fermion actions.

$(0.1\,\mathrm{fm} \simeq 2\,\mathrm{GeV.})$

The errors can be reduced by:

• Performing simulations at several values of a and extrapolating to $a = 0$.

• *Improvement* (Symanzik), i.e. choosing a discretization of QCD so that the errors are formally smaller.

$$f'(x) = \frac{f(x+a) - f(x)}{a} \ +\ O(a) \quad \text{or} \quad f'(x) = \frac{f(x+a) - f(x-a)}{2a} \ +\ O(a^2)\ .$$

• **Discretization Errors (Lattice Artefacts):** Current simulations are typically performed with

$$a \sim \left(\frac{1}{20} \text{ - } \frac{1}{10}\right)\ \text{fm}$$

leading to errors of $O(a\Lambda_{\rm QCD})$ (with Wilson Fermions) or $O(a^2\Lambda^2_{\rm QCD})$ for *improved* fermion actions.

(0.1 fm $\simeq$ 2 GeV.)

The errors can be reduced by:

• Performing simulations at several values of a and extrapolating to $a = 0$.

• *Improvement* (Symanzik), i.e. choosing a discretization of QCD so that the errors are formally smaller.

$$f'(x) = \frac{f(x+a) - f(x)}{a} \ +\ O(a) \quad \text{or} \quad f'(x) = \frac{f(x+a) - f(x-a)}{2a} \ +\ O(a^2)\ .$$

For example, in this way it is possible to reduce the errors from $O(a)$ for Wilson fermions to ones of $O(a^2)$ by the addition of irrelevant operators.

$\surd\surd$

- **Finite Volume Effects and Chiral Extrapolations:**

The pion is light $\Rightarrow$ it can propagate over large distances.

Simulations are performed with heavier pions (typically with $m_{u,d} \geq m_s/2$) and the results are then extrapolated to the chiral limit.

Typically we impose that $m_\pi L > 4$.

The extrapolation to $m_{u,d} \simeq 0$ is a major uncertainty for many processes.

Use χPT to guide this extrapolation? Chiral logarithms suggest that there may be subtle effects at low masses.

This will be discussed in some detail later.

$\rho \to \pi\pi$ decays have not been achieved on the lattice up to now. $\surd$

- **Quenching:** Many large scale simulations were performed in the *quenched* approximation in which the effects of quark loops are neglected.

  (Repeated evaluation of fermion determinant extremely expensive.)

- **Quenching:** Many large scale simulations were performed in the *quenched* approximation in which the effects of quark loops are neglected.

(Repeated evaluation of fermion determinant extremely expensive.)

Increasingly this simulations with dynamical fermions are being performed, however the difficulty is to control the behaviour with the masses of the light quarks.

• **Quenching:** Many large scale simulations were performed in the *quenched* approximation in which the effects of quark loops are neglected.
(Repeated evaluation of fermion determinant extremely expensive.)

Increasingly this simulations with dynamical fermions are being performed, however the difficulty is to control the behaviour with the masses of the light quarks.

Since it is much cheaper to vary the mass of the valence quarks than of the sea quarks, it is not uncommon to perform computations with $m_S \neq m_V$ and maybe even $n_S \neq n_V$ (*partial quenching*).

It is claimed that in this case, since QCD is in the space of partially quenched theories, the low energy constants of chiral perturbation theory can be determined. $\surd$

• **Quenching:** Many large scale simulations were performed in the *quenched* approximation in which the effects of quark loops are neglected.
(Repeated evaluation of fermion determinant extremely expensive.)

Increasingly this simulations with dynamical fermions are being performed, however the difficulty is to control the behaviour with the masses of the light quarks.

Since it is much cheaper to vary the mass of the valence quarks than of the sea quarks, it is not uncommon to perform computations with $m_S \neq m_V$ and maybe even $n_S \neq n_V$ (*partial quenching*).

It is claimed that in this case, since QCD is in the space of partially quenched theories, the low energy constants of chiral perturbation theory can be determined. $\surd$

• **Renormalization of Lattice Operators:** Relating bare lattice operators to standard renormalized ones (e.g. $\overline{MS}$ ones) introduces uncertainties.

Non-perturbative renormalization is possible. $\surd\surd\surd$

Lattice QCD

- The quark fields are defined on the lattice sites, $\psi^i_\alpha(x_i)$.

Lattice QCD

- The quark fields are defined on the lattice sites, $\psi^i_\alpha(x_i)$.

- In order to ensure gauge invariance the gauge fields are introduced through *link variables*, defined on the links between two neighbouring points.

Lattice QCD

- The quark fields are defined on the lattice sites, $\psi^i_\alpha(x_i)$.

- In order to ensure gauge invariance the gauge fields are introduced through *link variables*, defined on the links between two neighbouring points.

$U_\mu(x_i)$ is the link variable between the points x_i and $x_i + \hat{\mu}$.

Lattice QCD

- The quark fields are defined on the lattice sites, $\psi^i_\alpha(x_i)$.
- In order to ensure gauge invariance the gauge fields are introduced through *link variables*, defined on the links between two neighbouring points.

$U_\mu(x_i)$ is the link variable between the points x_i and $x_i + \hat{\mu}$.

- Under a gauge transformation:

$$\psi(x_i) \to g(x)\psi(x_i) \qquad \text{and} \qquad U_\mu(x_i) \to g(x_i)U_\mu(x_i)g^\dagger(x_i+\hat{\mu})\,.$$

- Thus we can think of $U_\mu(x_i)$ as the path-ordered exponential of gauge fields between x_i and $x_i + \hat{\mu}$.

- Writing

$$U_\mu(x_i) = \exp\left\{ i g_0 A_\mu(x_i + \frac{\hat{\mu}}{2}) a \right\} \ ,$$

Wilson proposed the gauge action

$$S = \sum_{\mathcal{P}_{\mu\nu}} \mathcal{P}_{\mu\nu} \quad \text{where} \quad \mathcal{P}_{\mu\nu} = \beta \left\{ 1 - \frac{1}{3} \text{Re Tr} \left(U_\mu(x) U_\nu(x+\hat{\mu}) U_\mu^\dagger(x+\hat{\nu}) U_\nu^\dagger(x) \right) \right\}$$

where $\beta = 6/g_0^2$ and $\mathcal{P}$ is called the *plaquette*.

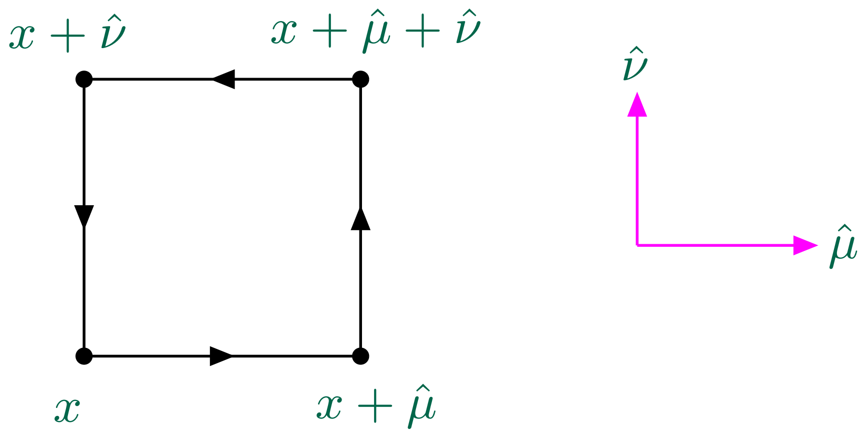
- Writing

$$U_\mu(x_i) = \exp\left\{ i g_0 A_\mu (x_i + \frac{\hat{\mu}}{2}) a \right\} ,$$

Wilson proposed the gauge action

$$S = \sum_{\mathcal{P}_{\mu\nu}} \mathcal{P}_{\mu\nu} \quad \text{where} \quad \mathcal{P}_{\mu\nu} = \beta \left\{ 1 - \frac{1}{3} \text{Re Tr} \left(U_\mu(x) U_\nu(x+\hat{\mu}) U^\dagger_\mu(x+\hat{\nu}) U^\dagger_\nu(x) \right) \right\}$$

where $\beta = 6/g_0^2$ and $\mathcal{P}$ is called the *plaquette*.



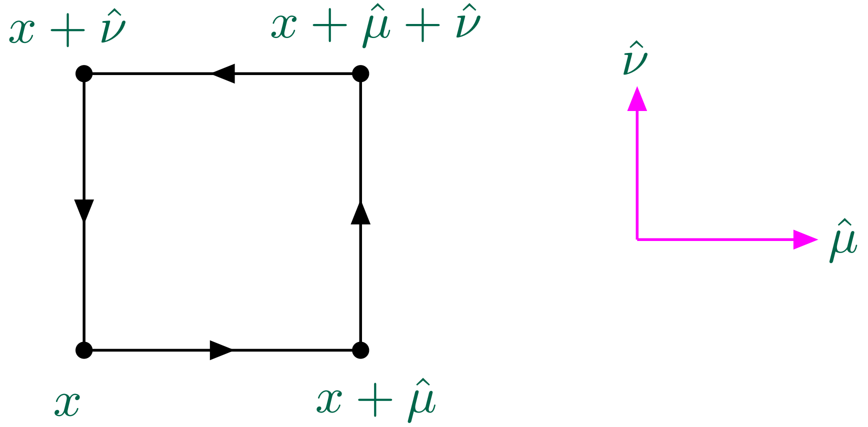
- Writing

$$U_\mu(x_i) = \exp\left\{ i g_0 A_\mu(x_i + \frac{\hat{\mu}}{2}) a \right\} ,$$

Wilson proposed the gauge action

$$S = \sum_{\mathcal{P}_{\mu\nu}} \mathcal{P}_{\mu\nu} \quad \text{where} \quad \mathcal{P}_{\mu\nu} = \beta \left\{ 1 - \frac{1}{3} \text{Re Tr} \left(U_\mu(x) U_\nu(x+\hat{\mu}) U_\mu^\dagger(x+\hat{\nu}) U_\nu^\dagger(x) \right) \right\}$$

where $\beta = 6/g_0^2$ and $\mathcal{P}$ is called the *plaquette.*



- "A little suppressed algebra" [Creutz] $\Rightarrow$

$$S = \frac{1}{2} \int d^4x \, \text{Tr}(F_{\mu\nu} F_{\mu\nu}) \quad + \text{terms suppressed by } a^2 .$$

Fermion Actions

• Naive Fermions ⇒ Fermion Doubling Problem. The inverse free propagator is

$$m + ia^{-1}\sum_{\mu}\gamma_{\mu}\sin(aq_{\mu})\,,$$

where q is the momentum.

At low momenta this is correct, but there are also similar contributions from $q_{\mu} \simeq \pi/a$ and we have $2^4 = 16$ independent Fermion species.

Fermion Actions

• Naive Fermions ⇒ Fermion Doubling Problem. The inverse free propagator is

$$m + ia^{-1} \sum_{\mu} \gamma_\mu \sin(aq_\mu) \,,$$

where q is the momentum.

At low momenta this is correct, but there are also similar contributions from $q_\mu \simeq \pi/a$ and we have $2^4 = 16$ independent Fermion species.

• As a result there is a plethora of Lattice Fermion Actions which overcome this problem:

1. Wilson Fermions (+ improved versions);
2. Staggered Fermions (and modified versions);
3. Twisted Mass QCD;
4. Actions satisfying the Ginsparg-Wilson relation (Domain Wall Fermions, Overlap Fermions, Perfect Actions)
5. $\cdots$

• Wilson Fermions: Add an irrelevant term to the action (proportional to $a\bar{\psi}D^2\psi$, adding a term proportional to $a^{-1}\sum_\mu\{1-\cos(aq_\mu)\}$ in the inverse propagator). This Wilson term breaks the chiral symmetry and induces artefacts of $O(a\Lambda_{\rm QCD})$.

The lattice aretefacts can be reduced to ones of $O(a^2{\Lambda_{\rm QCD}}^2)$ by adding further irrelevant operators to the action (and by calculating matrix elements of appropriate *improved* operators). Symanzik Improvement.

• Wilson Fermions: Add an irrelevant term to the action (proportional to $a\bar{\psi}D^2\psi$, adding a term proportional to $a^{-1}\sum_\mu\{1-\cos(aq_\mu)\}$ in the inverse propagator). This Wilson term breaks the chiral symmetry and induces artefacts of $O(a\Lambda_{\rm QCD})$.

The lattice aretefacts can be reduced to ones of $O(a^2{\Lambda_{\rm QCD}}^2)$ by adding further irrelevant operators to the action (and by calculating matrix elements of appropriate *improved* operators). Symanzik Improvement.

• Staggered Fermions: By a "spin diagonalization" of the γ-matrices the 16 fermion doublers can be reduced to 4. The chiral Ward Identities are still satisfied, the artefacts are of $O(a^2{\Lambda_{\rm QCD}}^2)$ but one has to worry about the different *tastes*.

Perturbative Coefficients tend to be large (reduced in recent versions).

• Wilson Fermions: Add an irrelevant term to the action (proportional to $a\bar{\psi}D^2\psi$, adding a term proportional to $a^{-1}\sum_\mu\{1-\cos(aq_\mu)\}$ in the inverse propagator). This Wilson term breaks the chiral symmetry and induces artefacts of $O(a\Lambda_{\rm QCD})$.

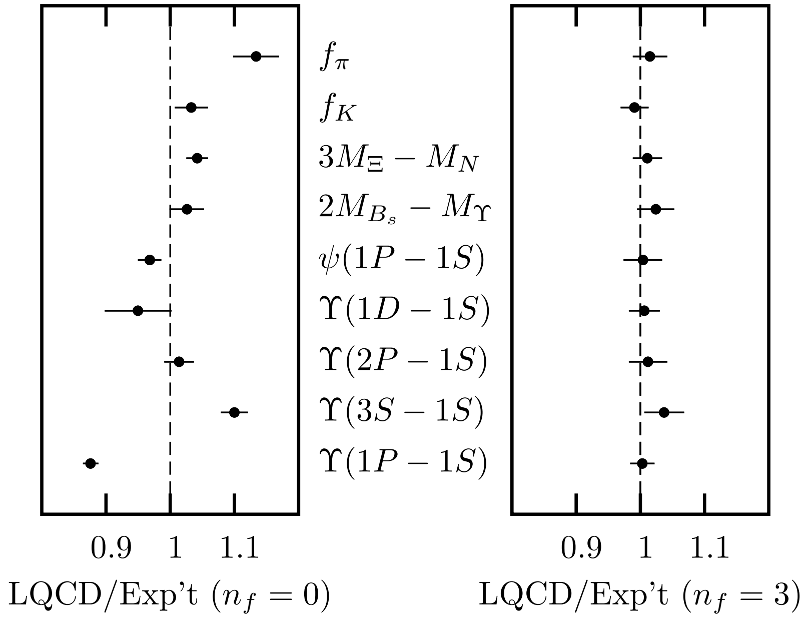
The lattice aretefacts can be reduced to ones of $O(a^2{\Lambda_{\rm QCD}}^2)$ by adding further irrelevant operators to the action (and by calculating matrix elements of appropriate *improved* operators). Symanzik Improvement.

• Staggered Fermions: By a "spin diagonalization" of the γ-matrices the 16 fermion doublers can be reduced to 4. The chiral Ward Identities are still satisfied, the artefacts are of $O(a^2{\Lambda_{\rm QCD}}^2)$ but one has to worry about the different *tastes*.

Perturbative Coefficients tend to be large (reduced in recent versions).

• Chiral Fermions: Much work is being devoted to developing algorithms for lattice fermions which have a continuum-like chiral symmetry even at finite lattice spacing.

This is ultimately likely to be the method of choice!



High-Precision Lattice QCD Confronts Experiment
C.T.H.Davies + 25 authors (HPQCD, UKQCD, MILC and Fermilab Lattice Collaborations),

`hep-lat/0304004`

• The authors claim that the use of *a new improved staggered-quark discretization* allows light quark masses to be reached down to about $m_s/8$ and represents "a breakthrough for phenomenology and, in particular, for heavy-quark physics".

Very Intriguing!!

2. Quark Masses

- Quark Masses and α_s are parameters of QCD.

We would like to know them at some standard renormalization scheme (e.g. $\overline{\mathrm{MS}}$) at a perturbative scale ($\mu \gg \Lambda_{\mathrm{QCD}}$).

2. Quark Masses

- Quark Masses and α_s are parameters of QCD.
We would like to know them at some standard renormalization scheme (e.g. $\overline{\text{MS}}$) at a perturbative scale ($\mu \gg \Lambda_{\text{QCD}}$).

- Lattice QCD is formulated in terms of bare fields and parameters and an ultra-violet cut-off, a^{-1}.

2. Quark Masses

• Quark Masses and α_s are parameters of QCD.
We would like to know them at some standard renormalization scheme (e.g. $\overline{\rm MS}$) at a perturbative scale ($\mu \gg \Lambda_{\rm QCD}$).

• Lattice QCD is formulated in terms of bare fields and parameters and an ultra-violet cut-off, a^{-1}.

• If a^{-1} and μ are sufficiently large (much larger than $\Lambda_{\rm QCD}$) then renormalized quantities can be obtained from bare ones using perturbation theory.

Apart from the fact that the Feynman rules in lattice QCD are more complicated than in continuum QCD and that therefore loop integrals have generally to be evaluated numerically, such a procedure is standard.

2. Quark Masses

• Quark Masses and α_s are parameters of QCD.
We would like to know them at some standard renormalization scheme (e.g. $\overline{\mathrm{MS}}$) at a perturbative scale ($\mu \gg \Lambda_{\mathrm{QCD}}$).

• Lattice QCD is formulated in terms of bare fields and parameters and an ultra-violet cut-off, a^{-1}.

• If a^{-1} and μ are sufficiently large (much larger than Λ_{QCD}) then renormalized quantities can be obtained from bare ones using perturbation theory.

Apart from the fact that the Feynman rules in lattice QCD are more complicated than in continuum QCD and that therefore loop integrals have generally to be evaluated numerically, such a procedure is standard.

• From our experience we know that perturbative coefficients in lattice QCD are frequently large, leading to significant uncertainties.

The uncertainties depend on the quantities being studied and on the lattice action. One loop effects of order 10-20% are typical, ones which are several times larger are not unusual.

2. Quark Masses

• Quark Masses and α_s are parameters of QCD.
We would like to know them at some standard renormalization scheme (e.g. $\overline{\text{MS}}$) at a perturbative scale ($\mu \gg \Lambda_{\text{QCD}}$).

• Lattice QCD is formulated in terms of bare fields and parameters and an ultra-violet cut-off, a^{-1}.

• If a^{-1} and μ are sufficiently large (much larger than Λ_{QCD}) then renormalized quantities can be obtained from bare ones using perturbation theory.

Apart from the fact that the Feynman rules in lattice QCD are more complicated than in continuum QCD and that therefore loop integrals have generally to be evaluated numerically, such a procedure is standard.

• From our experience we know that perturbative coefficients in lattice QCD are frequently large, leading to significant uncertainties.

The uncertainties depend on the quantities being studied and on the lattice action. One loop effects of order 10-20% are typical, ones which are several times larger are not unusual.

• It is for this reason that we have developed methods for *non-perturbative renormalization* (described later).

- Thus we can compare the lattice results for a number of physical quantities (e.g. some hadronic masses) with the experimental data to determine the bare quark masses and then use perturbation theory and/or non-perturbative renormalization to determine the renormalized values.

- Thus we can compare the lattice results for a number of physical quantities (e.g. some hadronic masses) with the experimental data to determine the bare quark masses and then use perturbation theory and/or non-perturbative renormalization to determine the renormalized values.

- For the coupling constant we exploit *dimensional transmutation* to use a fixed value of g_0 and take a^{-1} as the parameter.

• Thus we can compare the lattice results for a number of physical quantities (e.g. some hadronic masses) with the experimental data to determine the bare quark masses and then use perturbation theory and/or non-perturbative renormalization to determine the renormalized values.

• For the coupling constant we exploit *dimensional transmutation* to use a fixed value of g_0 and take a^{-1} as the parameter.

• There are a number of ways being used to determine the quark masses. These should give identical results if the systematic errors (in particular, lattice artefacts) are negligible.

• Thus we can compare the lattice results for a number of physical quantities (e.g. some hadronic masses) with the experimental data to determine the bare quark masses and then use perturbation theory and/or non-perturbative renormalization to determine the renormalized values.

• For the coupling constant we exploit *dimensional transmutation* to use a fixed value of g_0 and take a^{-1} as the parameter.

• There are a number of ways being used to determine the quark masses. These should give identical results if the systematic errors (in particular, lattice artefacts) are negligible.

• One standard method exploits the axial Ward identity

$$\partial_\mu A^\mu = (m_1 + m_2) P$$

where A and P are the axial current and pseudoscalar density corresponding to quarks with masses $m_{1,2}$:

$$A^\mu(x) = \bar{\psi}_1(x)\gamma^\mu\gamma^5\psi_2(x) \qquad \text{and} \qquad P(x) = \bar{\psi}_1(x)\gamma^5\psi_2(x)\,.$$

• Thus we can compare the lattice results for a number of physical quantities (e.g. some hadronic masses) with the experimental data to determine the bare quark masses and then use perturbation theory and/or non-perturbative renormalization to determine the renormalized values.

• For the coupling constant we exploit *dimensional transmutation* to use a fixed value of g_0 and take a^{-1} as the parameter.

• There are a number of ways being used to determine the quark masses. These should give identical results if the systematic errors (in particular, lattice artefacts) are negligible.

• One standard method exploits the axial Ward identity

$$\partial_\mu A^\mu = (m_1 + m_2) P$$

where A and P are the axial current and pseudoscalar density corresponding to quarks with masses $m_{1,2}$:

$$A^\mu(x) = \bar{\psi}_1(x)\gamma^\mu\gamma^5\psi_2(x) \qquad \text{and} \qquad P(x) = \bar{\psi}_1(x)\gamma^5\psi_2(x)\,.$$

For simplicity here let me take $m_1 = m_2$.

$$\partial_\mu A^\mu = (m_1 + m_2) P$$

- Calculate the two-point correlation function

$$\langle\, 0 \,|\, P^l(t) P^{s\,\dagger}(0) \,|\, 0 \,\rangle = \frac{Z_P^l Z_P^s}{2 m_P} \left\{ \exp(-m_P t) + \exp(-m_P (L_t - t)) \right\} \; .$$

- m_P is the mass of the pseudoscalar meson.
- The superscripts l and s stand for *local* and *smeared* respectively.

Optional

- The $Z_P^{l,s}$'s are the matrix elements $\langle 0 | P^{l,s}(0) | P \rangle$.

$$\partial_\mu A^\mu = 2m_q P$$

$$\langle\, 0 \,|\, P^l(t) P^{s\,\dagger}(0) \,|\, 0 \,\rangle = \frac{C_P^l C_P^s}{2m_P} \left\{ \exp(-m_P t) + \exp(-m_P(L_t - t)) \right\} \ .$$

- Calculate also

$$\langle\, 0 \,|\, A_4^l(t) P^{s\,\dagger}(0) \,|\, 0 \,\rangle = \frac{C_A^l C_P^s}{2m_P} \left\{ \exp(-m_P t) - \exp(-m_P(L_t - t)) \right\} \ .$$

$$\partial_\mu A^\mu = 2m_q P$$

$$\langle\, 0 \,|\, P^l(t) P^{s\,\dagger}(0) \,|\, 0 \,\rangle = \frac{C_P^l C_P^s}{2m_P} \left\{ \exp(-m_P t) + \exp(-m_P(L_t - t)) \right\} \,.$$

- Calculate also

$$\langle\, 0 \,|\, A_4^l(t) P^{s\,\dagger}(0) \,|\, 0 \,\rangle = \frac{C_A^l C_P^s}{2m_P} \left\{ \exp(-m_P t) - \exp(-m_P(L_t - t)) \right\} \,.$$

- Thus we obtain

$$m_q^{(0)\,\mathrm{AWI}} \equiv \frac{m_P C_A^l}{2C_P^l} \,.$$

$$\partial_\mu A^\mu = 2m_q P$$

$$\langle\, 0 \,|\, P^l(t) P^{s\,\dagger}(0) \,|\, 0 \,\rangle = \frac{C_P^l C_P^s}{2m_P} \left\{ \exp(-m_P t) + \exp(-m_P(L_t - t)) \right\} \,.$$

- Calculate also

$$\langle\, 0 \,|\, A_4^l(t) P^{s\,\dagger}(0) \,|\, 0 \,\rangle = \frac{C_A^l C_P^s}{2m_P} \left\{ \exp(-m_P t) - \exp(-m_P(L_t - t)) \right\} \,.$$

- Thus we obtain

$$m_q^{(0)\,\mathrm{AWI}} \equiv \frac{m_P C_A^l}{2C_P^l} \,.$$

- Now we would like the mass in some standard renormalization scheme, and the axial current and pseudoscalar density are both multiplicatively renormalizable. The renormalization constants can be fixed and we obtain the masses.

$$\partial_\mu A^\mu = 2m_q P$$

$$\langle 0 \,|\, P^l(t) P^{s\,\dagger}(0) \,|\, 0 \rangle = \frac{C_P^l C_P^s}{2m_P} \{\exp(-m_P t) + \exp(-m_P(L_t - t))\} \ .$$

- Calculate also

$$\langle 0 \,|\, A_4^l(t) P^{s\,\dagger}(0) \,|\, 0 \rangle = \frac{C_A^l C_P^s}{2m_P} \{\exp(-m_P t) - \exp(-m_P(L_t - t))\} \ .$$

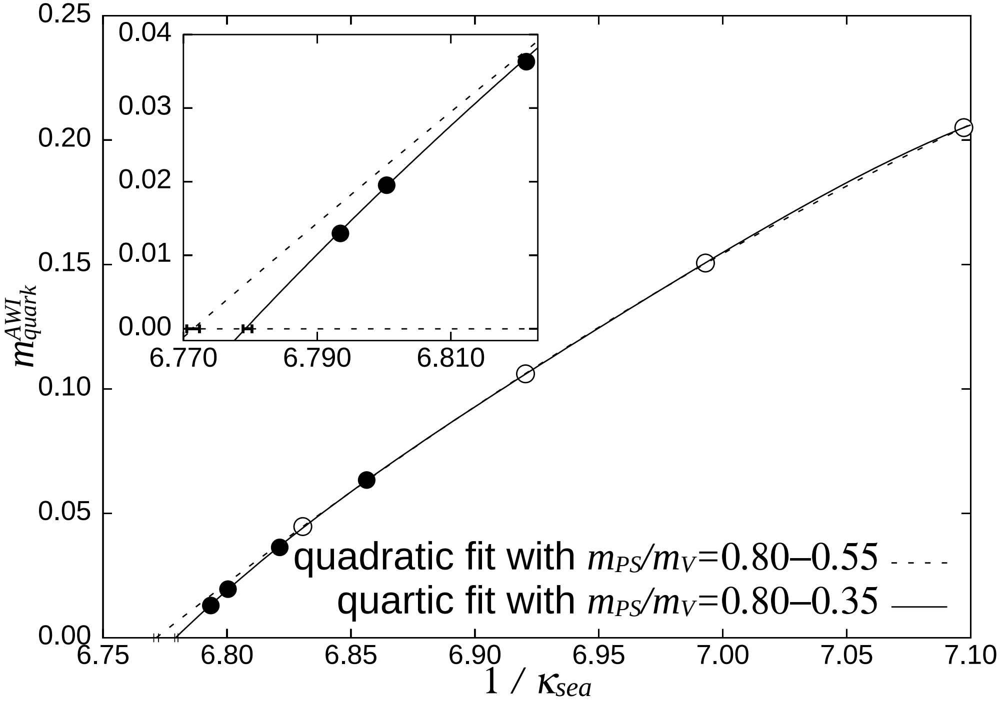
- Thus we obtain

$$m_q^{(0)\,\mathrm{AWI}} \equiv \frac{m_P C_A^l}{2C_P^l} \ .$$

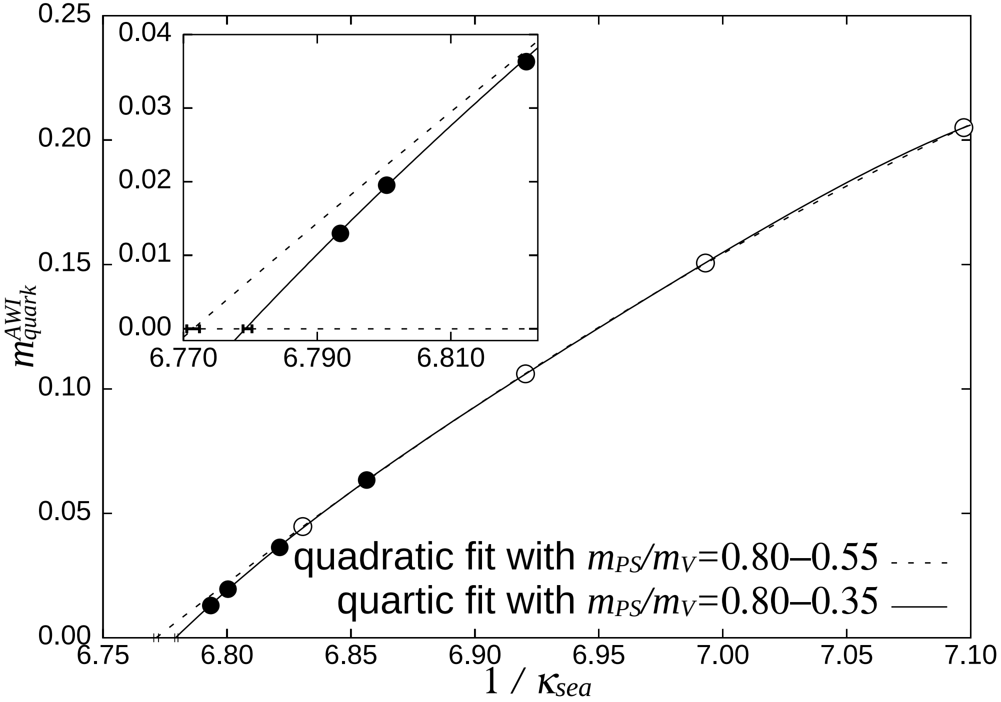
- Now we would like the mass in some standard renormalization scheme, and the axial current and pseudoscalar density are both multiplicatively renormalizable. The renormalization constants can be fixed and we obtain the masses.

Vector Ward Identities can also be used.

Y Namekawa et al., CP-PACS – hep-lat/0404014

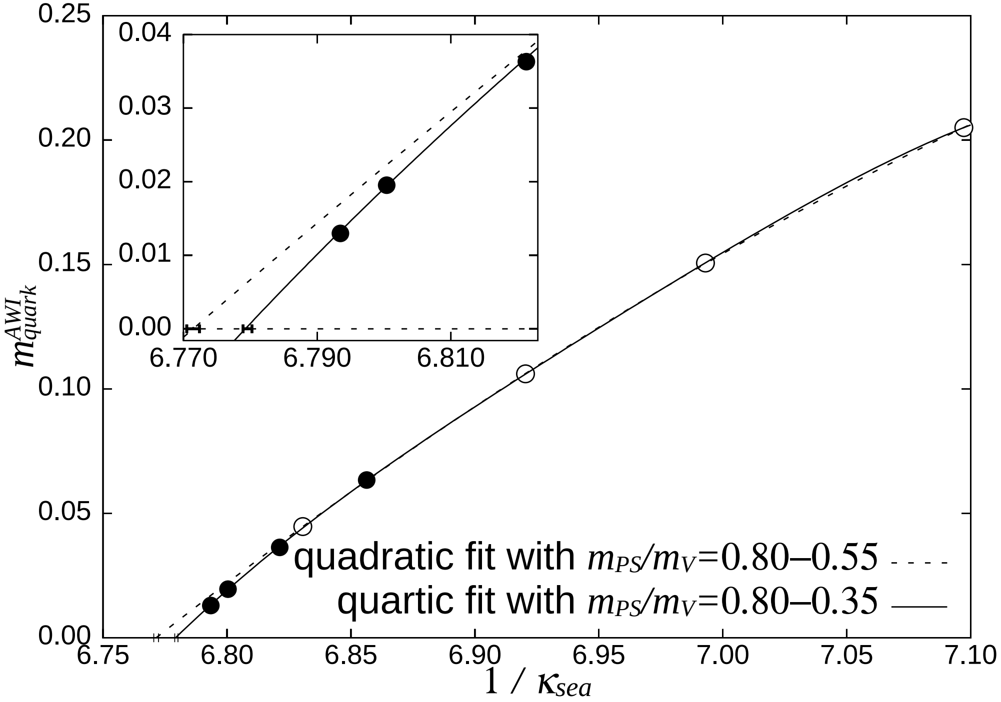


Y Namekawa et al., CP-PACS – hep-lat/0404014



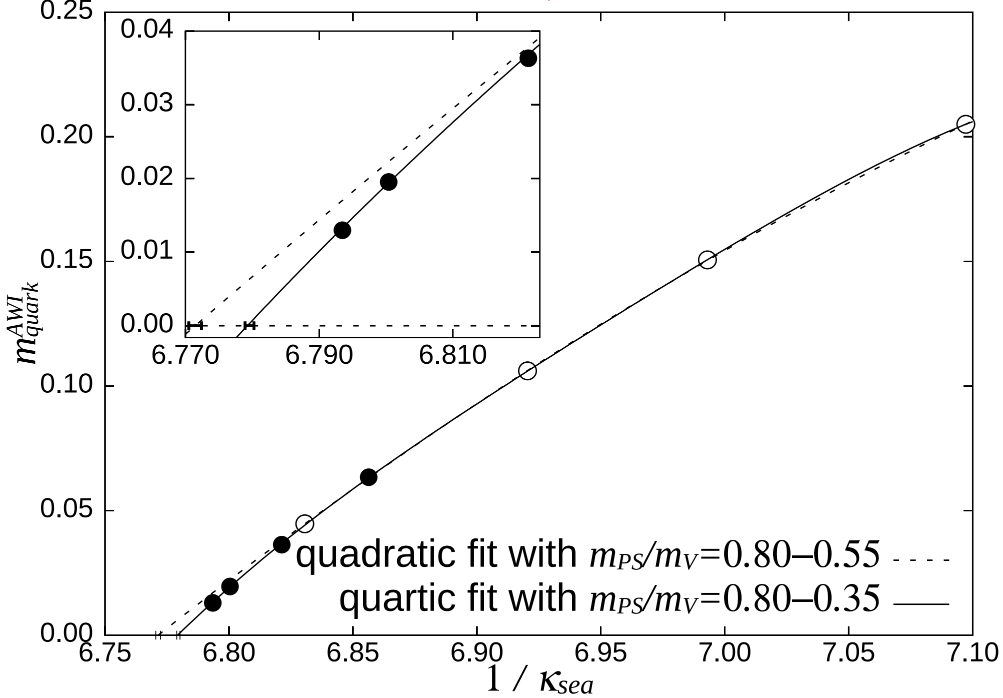
• For this paper the authors use an $(N_f = 2)$ action for which the renormalization constants are not known non-perturbatively ⇒ use perturbation theory.

Y Namekawa et al., CP-PACS – hep-lat/0404014



- For this paper the authors use an $(N_f = 2)$ action for which the renormalization constants are not known non-perturbatively $\Rightarrow$ use perturbation theory.
- They use the π and ρ masses to determine a ($a \simeq 0.2\,\text{fm}$) and the (single) quark mass.

Y Namekawa et al., CP-PACS – hep-lat/0404014

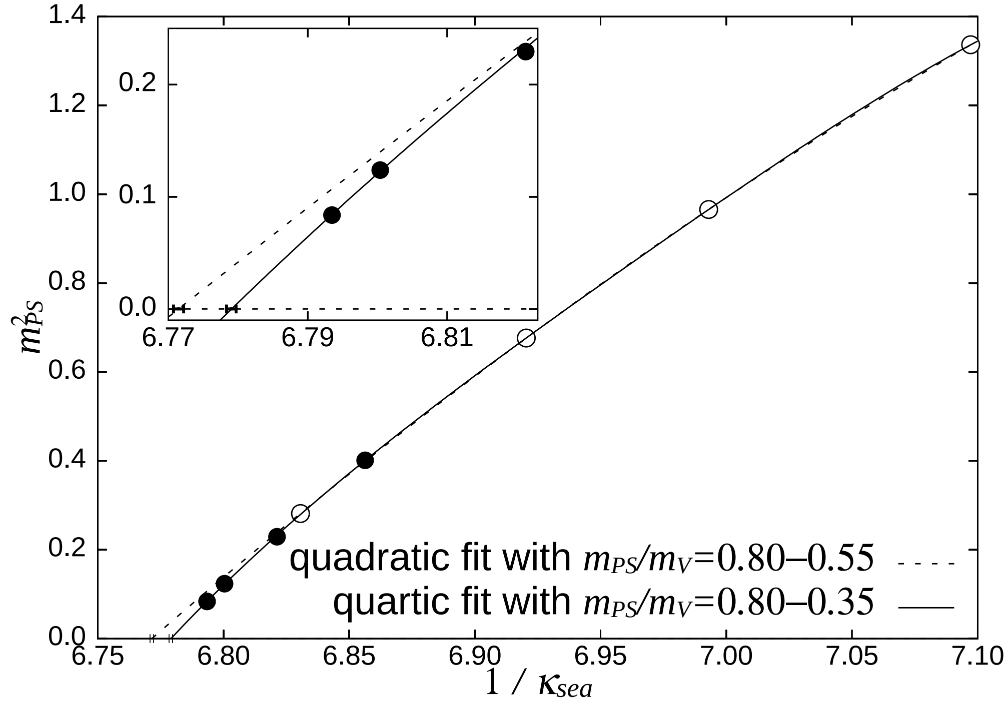


• For this paper the authors use an ($N_f = 2$) action for which the renormalization constants are not known non-perturbatively ⇒ use perturbation theory.

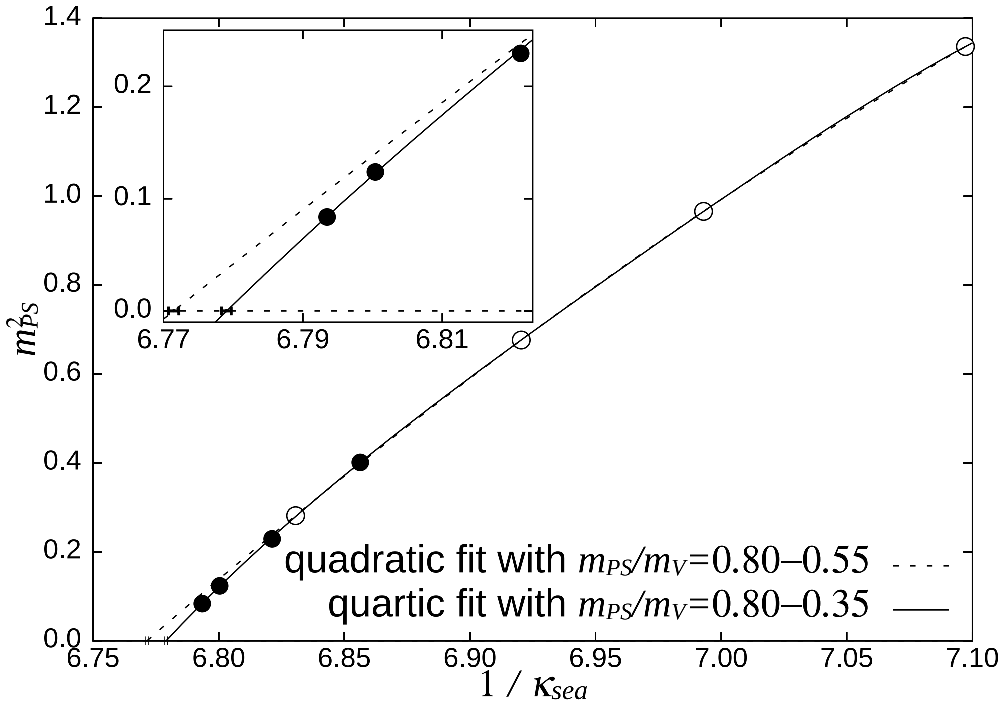
• They use the π and ρ masses to determine a ($a \simeq 0.2\,\mathrm{fm}$) and the (single) quark mass.

• The primary aim was to simulate with light masses and to study the chiral behaviour.

Y Namekawa et al., CP-PACS – hep-lat/0404014

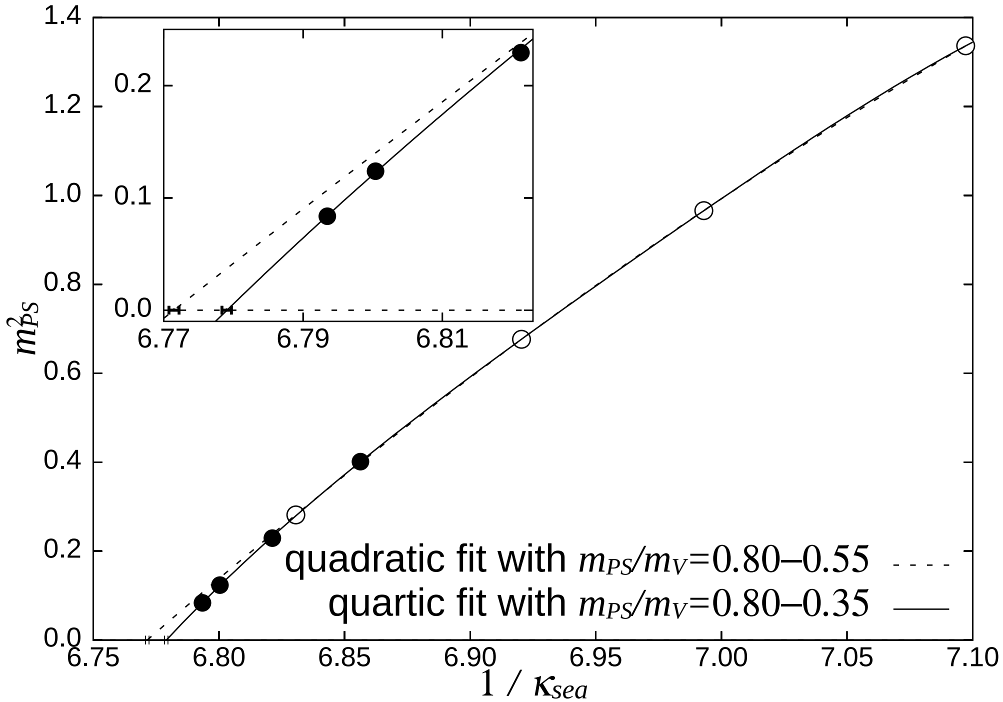


Y Namekawa et al., CP-PACS – hep-lat/0404014



- The authors claim that neither the quadratic fits that they had used earlier, nor continuum χPT fit the data well, but that WχPT, which includes a^2 effects associated with explicit chiral symmetry breaking fits the whole data.

Y Namekawa et al., CP-PACS – hep-lat/0404014



- The authors claim that neither the quadratic fits that they had used earlier, nor continuum χPT fit the data well, but that WχPT, which includes a^2 effects associated with explicit chiral symmetry breaking fits the whole data.

- Reanalyzing their previous data with WχPT lowers $m_{u,d}$ by about 10% ($m_{u,d}^{\overline{\rm MS}} = 3.11(17)\,{\rm MeV}$).

R.Gupta hep-ph/0311033

	Action Renorm.	$\bar{m}$ $= (m_u + m_d)/2$	$m_s(M_K)$	$m_s(M_\phi)$	scale $1/a$
JLQCD (1999)	Staggered RI/MOM	4.23(29)	106(7)	129(12)	M_ρ
CPPACS (1999)	Wilson 1-loop TI	4.57(18)	116(3)	144(6)	M_ρ
CP-PACS (2000)	Iwasaki+SW 1-loop TI	4.37^{+13}_{-16}	111^{+3}_{-4}	132^{+4}_{-5}	M_ρ
ALPHA-UKQCD (1999)	O(a) SW SF		97(4)		f_K
QCDSF (1999)	O(a) SW SF	4.4(2)	105(4)		r_0
QCDSF (1999)	Wilson RI/MOM	3.8(6)	87(15)		r_0
SPQcdR (2002)	O(a) SW RI/MOM	4.4(1)(4)	106(2)(8)		r_0

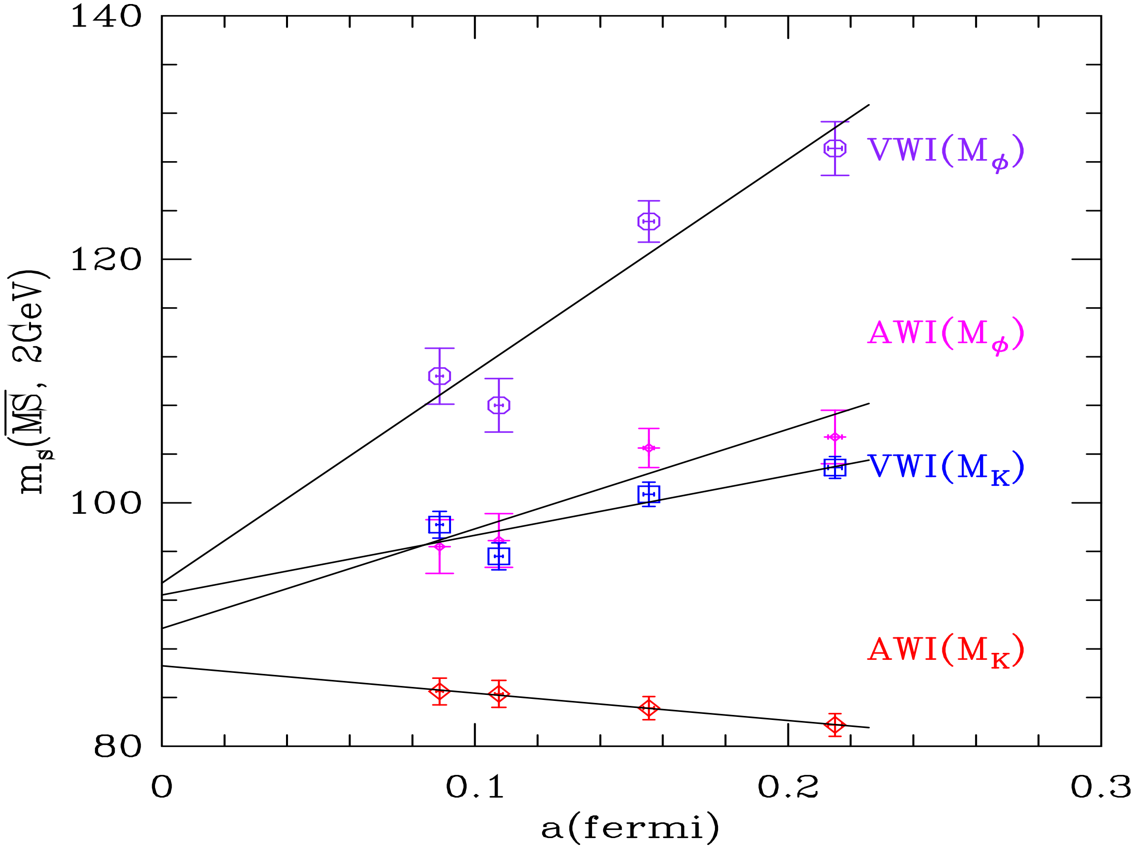
State of the art quenched results for quark masses.

R.Gupta hep-ph/0311033

	Action Renorm	$\bar{m}$	$m_s(M_K)$	$m_s(M_\phi)$	scale $1/a$ (GeV)
JLQCD (2002)	Wilson+SW 1-loop TI	3.22(4)	84.5(1.1)	96.4(2.2)	M_ρ (2.22)
CP-PACS (2000)	Iwasaki+SW 1-loop TI	$3.45^{+0.14}_{-0.20}$	89^{+3}_{-6}	90^{+5}_{-11}	M_ρ $(a \to 0)$
QCDSF-UKQCD (2003)	O(a) SW 1-loop TI	3.5(2)	90^{+5}_{-10}		r_0 $[1.9 - 2.2]$
QCDSF-UKQCD (2003)	O(a) SW RI-MOM		85(1)		r_0 $[1.9 - 2.2]$

Recent $N_f = 2$ results for quark masses.

R.Gupta hep-ph/0311033



Estimates of m_s from the $N_f = 2$ simulations by the CP-PACS and JLQCD (points on the finest lattice with $a \approx 0.09$ fermi) collaborations.

• Of course we would like to simulate with dynamical strange quarks ($N_f = 3$), and this still has major uncertainties.

	Action Renorm	$\bar{m}$	$m_s(M_K)$ AWI	scale $1/a$ (GeV)
JLQCD (2003)	Iwasaki+SW 1-loop TI	2.89(6)	75.6(3.4)	M_ρ (2.05(5))
MILC (2003)	AsqTad 1-loop TI	2.5(1)	66(1)	M_ρ (1.6)
MILC (2003)	AsqTad 1-loop TI	2.6(1)	68(1)	M_ρ (2.2)

Recent $N_f = 3$ results for quark masses from Gupta's review.
AsqTad is a perturbatively improved version of Staggered fermions which reduces "taste" symmetry breaking. The quoted errors in the MILC results include both statistical and those due to varying q^* in the 1-loop matching between $(1/a \to 2/a)$.

Non-Perturbative Renormalization

- Lattice computations are of bare quantities with a^{-1} as the ultraviolet cut-off.

Non-Perturbative Renormalization

• Lattice computations are of bare quantities with a^{-1} as the ultraviolet cut-off.

• As an example consider a local operator, such as the pseudoscalar density $\bar{\psi}_1\gamma^5\psi_2$.

In lattice simulations we compute

$$\langle f|O_B(a)|i\rangle\,,$$

whereas we would like to know

$$\langle f|O_R(\mu)|i\rangle\,,$$

in some standard renormalization scheme R.

The long distance physics is the same in both.

Non-Perturbative Renormalization

- Lattice computations are of bare quantities with a^{-1} as the ultraviolet cut-off.

- As an example consider a local operator, such as the pseudoscalar density $\bar{\psi}_1\gamma^5\psi_2$.

In lattice simulations we compute

$$\langle f|O_B(a)|i\rangle\,,$$

whereas we would like to know

$$\langle f|O_R(\mu)|i\rangle\,,$$

in some standard renormalization scheme R.

The long distance physics is the same in both.

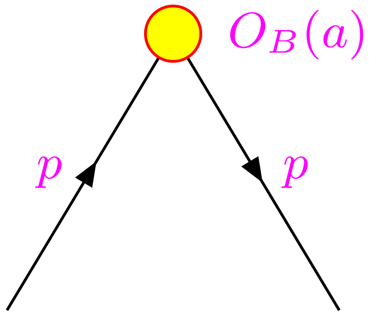
- For sufficiently large scale, a^{-1} and $\mu \gg \Lambda_{\rm QCD}$, the relation between these two matrix elements can be determined in perturbation theory, but the coefficients in Lattice PT are frequently large.

- It is possible to perform the renormalization non-pertubatively, eliminating the need for lattice perturbation theory.

• It is possible to perform the renormalization non-pertubatively, eliminating the need for lattice perturbation theory.

For example (there are more sophisticated schemes), let us define the renormalized operator O_R as being the one whose matrix element between quark states, at some scale $p^2 = \mu^2$ and in some gauge (the Landau gauge say) is the tree-level one. We compute

$$\langle\, p | O_B(a) | p\,\rangle =$$



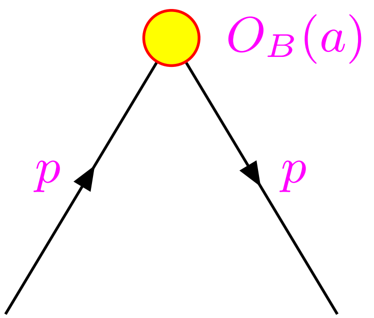
and determine the renormalization constant $Z_O(a\mu)$ by requiring that

$$Z_O(a\mu)\,\langle\, p | O_B(a) | p\,\rangle_{p^2=\mu^2} \;=\; \text{tree level value.}$$

• It is possible to perform the renormalization non-pertubatively, eliminating the need for lattice perturbation theory.

For example (there are more sophisticated schemes), let us define the renormalized operator O_R as being the one whose matrix element between quark states, at some scale $p^2 = \mu^2$ and in some gauge (the Landau gauge say) is the tree-level one. We compute

$\langle\, p|O_B(a)|p\,\rangle =$



and determine the renormalization constant $Z_O(a\mu)$ by requiring that

$$Z_O(a\mu)\,\langle\, p|O_B(a)|p\,\rangle_{p^2=\mu^2} \;=\; \text{tree level value.}$$

• The renormalized operator

$$O_R^{RI\;Mom}(\mu) \equiv Z_O(a\mu)\, O_B(a)$$

is independent of the regularization (RI) and can be used in hadronic matrix elements.

• To go from $RI\ Mom$ scheme to any other standard scheme (such as the $\overline{\mathrm{MS}}$ scheme) only requires continuum perturbation theory.

• To go from *RI Mom* scheme to any other standard scheme (such as the $\overline{\rm MS}$ scheme) only requires continuum perturbation theory.

• In lattice simulations we necessarily work on a finite volume ⇒ momentum is quantized:

$$\vec{p} = \frac{2\pi}{L}\,\vec{n}\,.$$

We require

$$\Lambda^2_{\rm QCD} \ll p^2 \ll a^{-2}$$

and this *window* is small, in practice.

• To go from *RI Mom* scheme to any other standard scheme (such as the $\overline{\mathrm{MS}}$ scheme) only requires continuum perturbation theory.

• In lattice simulations we necessarily work on a finite volume ⇒ momentum is quantized:

$$\vec{p} = \frac{2\pi}{L}\,\vec{n}\,.$$

We require

$$\Lambda^2_{\mathrm{QCD}} \ll p^2 \ll a^{-2}$$

and this *window* is small, in practice.

By calculating the matrix element between quark states on a sequence of lattices with decreasing a (and hence smaller volumes) and matching, it is possible to eliminate the constraint $p^2 \gg \Lambda^2_{\mathrm{QCD}}$. This procedure is called *step scaling.*

- For the mass of the charm quark, the most detailed study has been performed by Rolf and Sint (hep-ph/0209255) who study all systematic errors in detail, except for quenching.

- For the mass of the charm quark, the most detailed study has been performed by Rolf and Sint (hep-ph/0209255) who study all systematic errors in detail, except for quenching.

  - $O(a)$ improved action $\Rightarrow$ discretization errors of $O(a^2)$.

• For the mass of the charm quark, the most detailed study has been performed by Rolf and Sint (hep-ph/0209255) who study all systematic errors in detail, except for quenching.

- $O(a)$ improved action $\Rightarrow$ discretization errors of $O(a^2)$.
- Non-perturbative renormalization a la Alpha Collaboration $\Rightarrow$ only need to use continuum perturbation theory at large scales ($\sim 30\,\mathrm{Gev}$) to get m_c^{RGI} and N^3LO perturbation theory to get $\bar{m}_c$.

• For the mass of the charm quark, the most detailed study has been performed by Rolf and Sint (hep-ph/0209255) who study all systematic errors in detail, except for quenching.

- $O(a)$ improved action $\Rightarrow$ discretization errors of $O(a^2)$.
- Non-perturbative renormalization a la Alpha Collaboration $\Rightarrow$ only need to use continuum perturbation theory at large scales ($\sim 30\,$Gev) to get $m_c^{\rm RGI}$ and N^3LO perturbation theory to get $\bar{m}_c$.
- 5 different definitions of m_c based on Ward Identities.

• For the mass of the charm quark, the most detailed study has been performed by Rolf and Sint (hep-ph/0209255) who study all systematic errors in detail, except for quenching.

- $O(a)$ improved action ⇒ discretization errors of $O(a^2)$.
- Non-perturbative renormalization a la Alpha Collaboration ⇒ only need to use continuum perturbation theory at large scales ($\sim 30\,$Gev) to get $m_c^{\rm RGI}$ and N^3LO perturbation theory to get $\bar{m}_c$.
- 5 different definitions of m_c based on Ward Identities.

They find

$$\bar{m}_c = 1.301(34)\,\text{GeV}\,,$$

where the scale has been set from f_K.

- For the mass of the charm quark, the most detailed study has been performed by Rolf and Sint (hep-ph/0209255) who study all systematic errors in detail, except for quenching.

  - $O(a)$ improved action $\Rightarrow$ discretization errors of $O(a^2)$.
  - Non-perturbative renormalization a la Alpha Collaboration $\Rightarrow$ only need to use continuum perturbation theory at large scales ($\sim 30\,\text{Gev}$) to get m_c^{RGI} and N^3LO perturbation theory to get $\bar{m}_c$.
  - 5 different definitions of m_c based on Ward Identities.

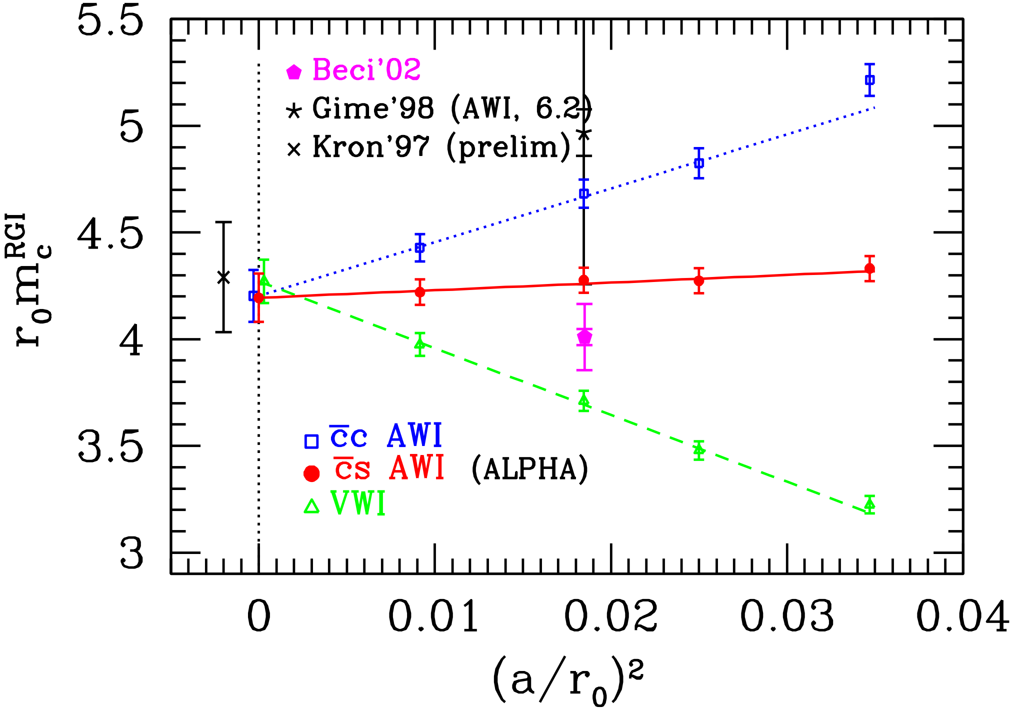
They find

$$\bar{m}_c = 1.301(34)\,\text{GeV}\,,$$

where the scale has been set from f_K.

- Setting the scale by m_p increases $\bar{m}_c$ by about 3%.

L.Lellouch – hep-ph/0211359



Continuum extrapolation of the RGI charm mass by ALPHA (Rolf and Sint).

The Mass of the b-quark

$m_b \gg a^{-1} \gg \Lambda_{\mathrm{QCD}} \quad \Rightarrow$ Effective Theories (HQET or NRQCD)

The Mass of the b-quark

$m_b \gg a^{-1} \gg \Lambda_{\rm QCD} \quad \Rightarrow$ Effective Theories (HQET or NRQCD)

Consider the correlation function

$$C(t) = \sum_{\vec{x}} \langle\, 0 |\, A_0(\vec{x}, t)\, A_0(\vec{0}, 0)\, |\, 0\, \rangle$$

in the HQET. A_μ is the axial current,

$$A_\mu = \bar{h}\, \gamma_\mu \gamma^5\, q\ .$$

h and q are heavy- and light-quark fields.

The Mass of the b-quark

$$m_b \gg a^{-1} \gg \Lambda_{\rm QCD} \quad \Rightarrow \text{Effective Theories} \quad \text{(HQET or NRQCD)}$$

Consider the correlation function

$$C(t) = \sum_{\vec{x}} \langle\, 0 |\, A_0(\vec{x}, t)\, A_0(\vec{0}, 0)\, |\, 0\, \rangle$$

in the HQET. A_μ is the axial current,

$$A_\mu = \bar{h}\, \gamma_\mu \gamma^5\, q\ .$$

h and q are heavy- and light-quark fields.

- At large t:

$$C(t) \simeq Z^2 \exp(-\xi t)$$

where from Z we obtain the value of the decay constant, f_B, in the static approximation.

(E.Eichten - 1987)

The Mass of the b-quark

$m_b \gg a^{-1} \gg \Lambda_{\rm QCD}$ $\Rightarrow$ Effective Theories (HQET or NRQCD)

Consider the correlation function

$$C(t) = \sum_{\vec{x}} \langle 0 |\, A_0(\vec{x}, t)\, A_0(\vec{0}, 0)\, | 0 \rangle$$

in the HQET. A_μ is the axial current,

$$A_\mu = \bar{h}\, \gamma_\mu \gamma^5\, q \ .$$

h and q are heavy- and light-quark fields.

- At large t:

$$C(t) \simeq Z^2 \exp(-\xi t)$$

where from Z we obtain the value of the decay constant, f_B, in the static approximation. (E.Eichten - 1987)

- From the measured value of ξ one can obtain m_b (up to $\Lambda^2_{\rm QCD}/m_b$ corrections.) (M.Crisafulli, V,Giménez, G.Martinelli, CTS - 1995)

$$M_B = m_b^{\mathrm{pole}} + \xi - \delta m$$

$$M_B = m_b^{\mathrm{pole}} + \xi - \delta m$$

- ξ is computed directly in lattice simulations. It diverges linearly as $a \longrightarrow 0$.

$$M_B = m_b^{\rm pole} + \xi - \delta m$$

- ξ is computed directly in lattice simulations. It diverges linearly as $a \to 0$.
- δm is the perturbation series for the residual mass

$$\delta m = \frac{1}{a} \sum_n X_n \, \alpha_s^n(m_b) \ .$$

Each term in the series for δm also diverges linearly as $a \to 0$. This (partially) cancels the divergence in ξ ($\Rightarrow$ a cannot be too small).

$$M_B = m_b^{\text{pole}} + \xi - \delta m$$

- ξ is computed directly in lattice simulations. It diverges linearly as $a \longrightarrow 0$.
- δm is the perturbation series for the residual mass

$$\delta m = \frac{1}{a} \sum_n X_n \, \alpha_s^n(m_b) \ .$$

Each term in the series for δm also diverges linearly as $a \longrightarrow 0$. This (partially) cancels the divergence in ξ ($\Rightarrow$ a cannot be too small).

The perturbative series for δm is not *Borel Summable*, leading to a *renormalon ambiguity*.

$$M_B = m_b^{\rm pole} + \xi - \delta m$$

- ξ is computed directly in lattice simulations. It diverges linearly as $a \longrightarrow 0$.

- δm is the perturbation series for the residual mass

$$\delta m = \frac{1}{a} \sum_n X_n \, \alpha_s^n(m_b) \ .$$

Each term in the series for δm also diverges linearly as $a \longrightarrow 0$. This (partially) cancels the divergence in ξ ($\Rightarrow$ a cannot be too small).

The perturbative series for δm is not *Borel Summable*, leading to a *renormalon ambiguity.*

- The pole mass is also not a physical quantity containing a renormalon ambiguity.

(M.Beneke and V.Braun (1994); I.Bigi,M.Shifman, N.Uraltsev and A.Vainshtein (1994)).

$$M_B = m_b^{\rm pole} + \xi - \delta m$$

- ξ is computed directly in lattice simulations. It diverges linearly as $a \longrightarrow 0$.
- δm is the perturbation series for the residual mass

$$\delta m = \frac{1}{a} \sum_n X_n \, \alpha_s^n(m_b) \ .$$

Each term in the series for δm also diverges linearly as $a \longrightarrow 0$. This (partially) cancels the divergence in ξ ($\Rightarrow$ a cannot be too small).

The perturbative series for δm is not *Borel Summable*, leading to a *renormalon ambiguity.*

- The pole mass is also not a physical quantity containing a renormalon ambiguity.

(M.Beneke and V.Braun (1994); I.Bigi,M.Shifman, N.Uraltsev and A.Vainshtein (1994)).

- The renormalons in the pole mass and δm cancel.

$$
\begin{aligned}
\overline{m}_b &= [1 + \sum_n D_n \, \alpha_s^n(m_b)] \, m_B^{\text{pole}} \\
&= [1 + \sum_n D_n \, \alpha_s^n(m_b)][M_B - \xi + \frac{1}{a} \sum_n X_n \, \alpha_s^n(m_b)] \; .
\end{aligned}
$$

Perturbative Matching

$$\overline{m}_b = [1 + \sum_n D_n \, \alpha_s^n(m_b)][M_B - \xi + \frac{1}{a} \sum_n X_n \, \alpha_s^n(m_b)] \ .$$

Perturbative Matching

$$\overline{m}_b = [1 + \sum_n D_n\, \alpha_s^n(m_b)][M_B - \xi + \frac{1}{a} \sum_n X_n\, \alpha_s^n(m_b)]\ .$$

- Relation between the Pole Mass and the $\overline{\mathrm{MS}}$ Mass:

$$\overline{m}_b = [1 + \sum_n D_n\, \alpha_s^n(m_b)]\, m_B^{\mathrm{pole}}.$$

  - D_2 – N.Gray, D.J.Broadhurst, W.Grafe and K.Schilcher (1990)
  - D_3 – K.Melnikov and T.van Ritbergen (1999)

Perturbative Matching

$$\overline{m}_b = [1 + \sum_n D_n\, \alpha_s^n(m_b)][M_B - \xi + \frac{1}{a} \sum_n X_n\, \alpha_s^n(m_b)]\ .$$

- Relation between the Pole Mass and the $\overline{\mathrm{MS}}$ Mass:

$$\overline{m}_b = [1 + \sum_n D_n\, \alpha_s^n(m_b)]\, m_B^{\mathrm{pole}}.$$

  - D_2 – N.Gray, D.J.Broadhurst, W.Grafe and K.Schilcher (1990)
  - D_3 – K.Melnikov and T.van Ritbergen (1999)

- Relation between the Bare Lattice Coupling and the $\overline{\mathrm{MS}}$ Coupling:

$$\alpha_s(\mu) = \left[1 + \sum_n d_n(a\mu)\alpha_0^n\right] \alpha_0.$$

  - d_2 – M.Lüscher and P.Weisz (1995)
    C.Christou, A.Feo, H.Panagopoulos and E.Vicari (1998)

Perturbative Matching - Continued

$$\overline{m}_b = [1 + \sum_n D_n\, \alpha_s^n(m_b)][M_B - \xi + \frac{1}{a} \sum_n X_n\, \alpha_s^n(m_b)]\ .$$

Perturbative Matching - Continued

$$\overline{m}_b = [1 + \sum_n D_n \, \alpha_s^n(m_b)][M_B - \xi + \frac{1}{a} \sum_n X_n \, \alpha_s^n(m_b)] \ .$$

- Perturbative expansion of the Residual Mass

$$a\delta m = C_1 \alpha_0 + C_2 \, \alpha_0^2 + C_3 \alpha_0^3 + \cdots$$

Perturbative Matching - Continued

$$\overline{m}_b = [1 + \sum_n D_n \, \alpha_s^n(m_b)][M_B - \xi + \frac{1}{a} \sum_n X_n \, \alpha_s^n(m_b)] \; .$$

- Perturbative expansion of the Residual Mass

$$a\delta m = C_1 \alpha_0 + C_2 \, \alpha_0^2 + C_3 \alpha_0^3 + \cdots$$

- Analytical Calculation of C_2 from

$$W(R,T) \sim \exp(-2\delta m(R+T))$$

$$C_1 = 2.1173 \text{ and } C_2 = 11.152 + n_f(-0.282 + 0.035 c_{SW} - 0.391 c_{SW}^2).$$

G.Martinelli and CTS (1998)

Perturbative Matching - Continued

$$\overline{m}_b = [1 + \sum_n D_n \, \alpha_s^n(m_b)][M_B - \xi + \frac{1}{a} \sum_n X_n \, \alpha_s^n(m_b)] \ .$$

- Perturbative expansion of the Residual Mass

$$a\delta m = C_1 \alpha_0 + C_2 \, \alpha_0^2 + C_3 \alpha_0^3 + \cdots$$

  - Analytical Calculation of C_2 from

$$W(R,T) \sim \exp(-2\delta m(R+T))$$

$C_1 = 2.1173$ and $C_2 = 11.152 + n_f(-0.282 + 0.035 c_{SW} - 0.391 c_{SW}^2)$.

G.Martinelli and CTS (1998)

  - Stochastic Perturbation Theory (Numerical):
    C_1=2.09(4), C_2=10.7(7) and C_3=86.2(5),
    R.Alfieri, F. DiRenzo, E.Onofri and L. Scorzato, (1999) and (2000)

Perturbative Matching - Continued

$$\overline{m}_b = [1 + \sum_n D_n\, \alpha_s^n(m_b)][M_B - \xi + \frac{1}{a}\sum_n X_n\, \alpha_s^n(m_b)]\ .$$

- Perturbative expansion of the Residual Mass

$$a\delta m = C_1\alpha_0 + C_2\,\alpha_0^2 + C_3\alpha_0^3 + \cdots$$

  - Analytical Calculation of C_2 from

$$W(R,T) \sim \exp(-2\delta m(R+T))$$

$C_1 = 2.1173$ and $C_2 = 11.152 + n_f(-0.282 + 0.035 c_{SW} - 0.391 c_{SW}^2)$.

G.Martinelli and CTS (1998)

  - Stochastic Perturbation Theory (Numerical):
    C_1=2.09(4), C_2=10.7(7) and C_3=86.2(5),
    R.Alfieri, F. DiRenzo, E.Onofri and L. Scorzato, (1999) and (2000)

  - Large β Monte-Carlo: C_3=80(2)+5.6(1.8)
    G.P.Lepage, P.B.Mackenzie, N.H.Shakespeare and H.D.Trottier (1999)

Perturbative Matching - Continued

$$\overline{m}_b = [1 + \sum_n D_n\, \alpha_s^n(m_b)][M_B - \xi + \frac{1}{a}\sum_n X_n\, \alpha_s^n(m_b)]\ .$$

- Perturbative expansion of the Residual Mass

$$a\delta m = C_1\alpha_0 + C_2\,\alpha_0^2 + C_3\alpha_0^3 + \cdots$$

- Analytical Calculation of C_2 from

$$W(R,T) \sim \exp(-2\delta m(R+T))$$

$C_1 = 2.1173$ and $C_2 = 11.152 + n_f(-0.282 + 0.035 c_{SW} - 0.391 c_{SW}^2)$.

G.Martinelli and CTS (1998)

- Stochastic Perturbation Theory (Numerical):
  C_1=2.09(4), C_2=10.7(7) and C_3=86.2(5),
  R.Alfieri, F. DiRenzo, E.Onofri and L. Scorzato, (1999) and (2000)

- Large β Monte-Carlo: C_3=80(2)+5.6(1.8)
  G.P.Lepage, P.B.Mackenzie, N.H.Shakespeare and H.D.Trottier (1999)

For NRQCD only the first coefficient (C_1) is known.

Quenched Results

- $\overline{m}_b = 4.15 \pm 0.05 \pm 0.20\,\mathrm{GeV}$
  Quenched NLO; Second error estimate of uncertainty due to higher order perturbation theory. V.Giménez, G.Martinelli and CTS (1996)
- $\overline{m}_b = 4.30 \pm 0.05 \pm 0.10\,\mathrm{GeV}$
  Quenched $\mathrm{N^2LO}$ G.Martinelli and CTS (1998)
- $\overline{m}_b = 4.30 \pm 0.05 \pm 0.05\,\mathrm{GeV}$
  Quenched $\mathrm{N^3LO}$ V.Giménez at al. (2000) private comm.
- $\overline{m}_b = 4.34 \pm 0.03 \pm 0.06\,\mathrm{GeV}$
  Quenched $\mathrm{N^3LO}$ from NRQCD in the static limit. C.Davies et al. (2000)

Unquenched Results

- $\overline{m}_b = 4.26 \pm 0.06 \pm 0.07\,\mathrm{GeV}$
  $n_f = 2$, $\mathrm{N^2LO}$. V.Giménez, L.Giusti, G.Martinelli and F.Rapuano (2000)

Results using NRQCD are in good agreement, but we need to understand the errors due to higher orders of perturbation theory in that case.

• A strategy to renormalize the HQET non-perturbatively is being successfully developed by the DESY-Zeuthen group.

J.Heitger & R.Sommer; M.Kurth & R.Sommer; Heitger, Kurth & Sommer, · · ·