A numerical study of partially twisted boundary conditions

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UKQCD Collaboration

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Orientation - lattice phenomenology

- explore flavour sector of the SM

BR experiment	+	matrix elements theory	$\rightarrow$	$\lvert V_{CKM} \rvert$ CP-violation

- QCD contribution $\rightarrow$ non-perturbative techniques necessary
- lattice QCD
- often theoretical uncertainty dominant

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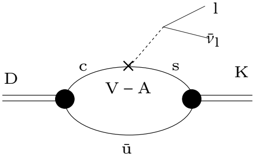
Focus - momentum dependence of observables

e.g. semi-leptonic meson decay



$$\Gamma(D \to K l \nu) = \frac{G_F^2 |V_{cs}|^2}{192\pi^3 m_D^3} \int_0^{(m_D - m_K)^2} dq^2 [\lambda(q^2)]^{3/2} |f_K^+(q^2)|^2$$

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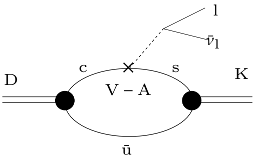

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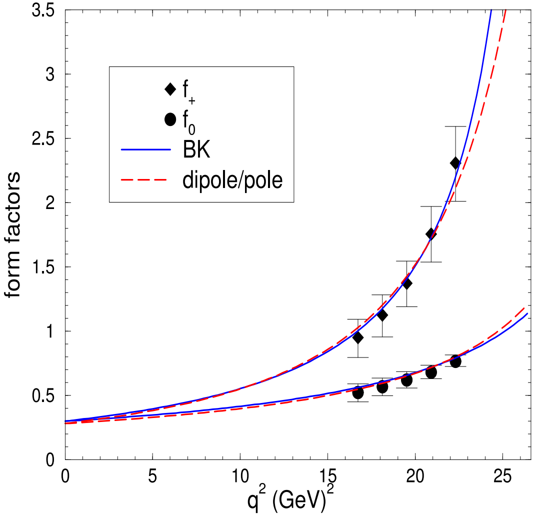
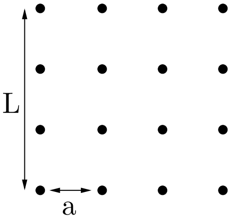
$$\underbrace{\langle K|O_\mu|D\rangle}_{\text{LQCD}} = f_K^+(q^2)\left(p_D + p_K - \frac{m_D^2 - m_K^2}{q^2}\right)_\mu + f^0(q^2) q_\mu \frac{m_D^2 - m_K^2}{q^2}$$

There is a problem



[UKQCD]

The lattice approach



$x_\mu = a n_\mu,\ n_\mu = 1, 2, \ldots, L/a$

Fields: $\overline{\psi}(x),\ \psi(x),\ U_\mu(x) \in SU(3)$

momentum cut-off: $\Lambda = \frac{\pi}{a}$

$$\langle \psi(x)\overline{\psi}(0)\rangle = \int \prod_x dU \frac{1}{\not{D}+m} e^{-S_{\rm eff}[U]}$$

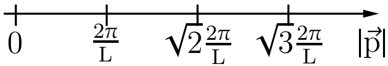
$$\overset{MC}{\approx} \frac{1}{N_{\rm meas}} \sum_{i=1}^{N_{\rm meas}} \{S(x,0)\}[\{U_i\}]$$

sea quarks

valence quarks

Momentum on the lattice

bc's	periodic	twisted
Dirac	$\bar{\tilde{\psi}}(x)(\not{D} + m)\tilde{\psi}(x)$	$\bar{\psi}(x)(\not{D} + m)\psi(x)$
fermions	$\tilde{\psi}(x + \hat{i}L) = \tilde{\psi}(x)$	$\psi(x + \hat{i}L) = e^{i\theta_i}\psi(x)$
latt-mom.	$\vec{p}_{lat} = \frac{2\pi}{L}\vec{n}$	$\vec{p}_{lat} = \frac{2\pi}{L}\vec{n} + \frac{\vec{\theta}}{L}$



[Bedaque], [de Divitiis, Petronzio, Tantalo], [Sachrajda, Villadoro]

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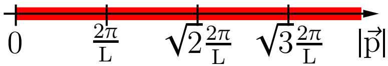

[Bedaque], [de Divitiis, Petronzio, Tantalo], [Sachrajda, Villadoro]

Chiral perturbation theory

[Sachrajda, Villadoro] - Twisted chiral perturbation theory

- $E_{\pi,\rho} = \sqrt{m_{\pi,\rho}^2 + (\vec{p}_{lat} - \frac{1}{L}(\vec{\theta}_1 - \vec{\theta}_2))^2}$
- partial twisting

  $\langle\bar{\psi}(x)\psi(0)\rangle \approx \frac{1}{N_{meas}} \sum_{i=1}^{N_{meas}} \{S(\vec{\theta}; x, 0)\}[\{U_i\}_{|\vec{\theta}=0}]$
- finite volume errors exponentially supressed
  - not for matrix elements with final state interactions
  - not above $\pi\pi$-threshold

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Our numerical study

- partial twisting of valence quarks $(\vec{\theta} = (0,0,0),(2,0,0),(0,\pi,0),(3,3,3))$
- combination with Fourier momenta $|\vec{p}L| = 0, 2\pi$
- Observables:

$$
\begin{aligned}
E^2_{\pi,\rho} &= m^2_{\pi,\rho} + \left(\vec{p}_{lat} - \frac{\vec{\theta}_1 - \vec{\theta}_2}{L}\right)^2 \\
E_\pi f_\pi &= \langle 0|A_0(0)|\pi(p)\rangle \\
Z_P &= \langle 0|P(0)|\pi(p)\rangle \\
\epsilon_\lambda(\lambda, p) m_\rho f_\rho &= \langle 0|V_i(0)|\rho(\lambda, p)\rangle
\end{aligned}
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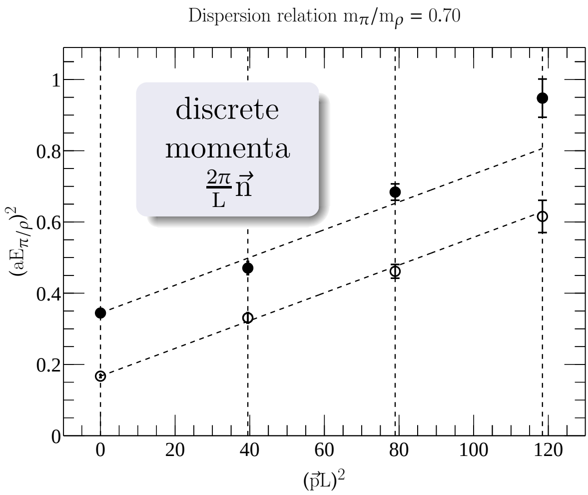
\epsilon_\lambda(\lambda,\mathrm{p})\mathrm{m}_\rho \mathrm{f}_\rho &= \langle 0|\mathrm{V}_\mathrm{i}(0)|\rho(\lambda,\mathrm{p})\rangle
\end{aligned}$$

Numerical results - Dispersion relation



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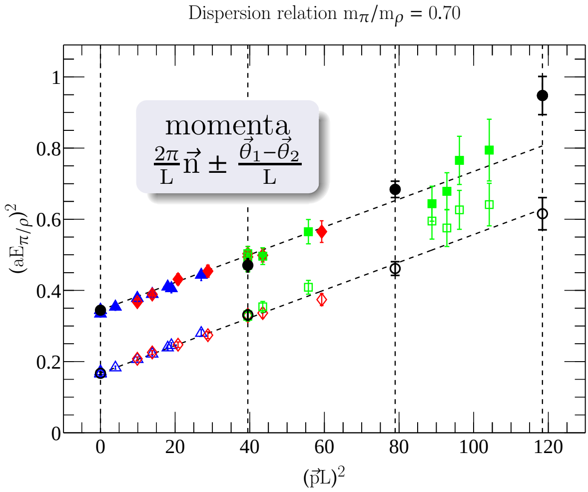

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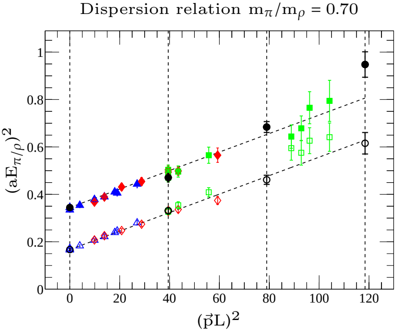
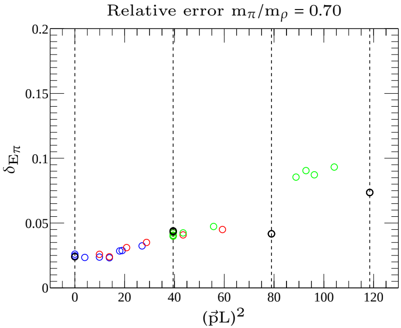
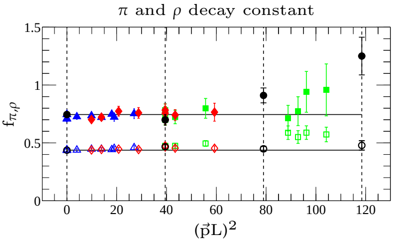
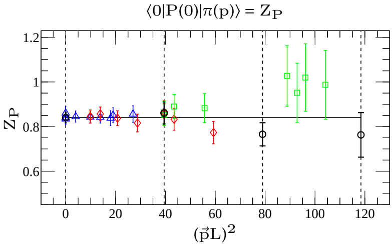




Numerical results - Matrix elements





Conclusions and outlook

Results:

- partial twisting → physical quantities at any momentum (cf. meson dispersion relation)
- matrix elements momentum-independent
- method does not introduce apreciable noise

Outlook:

- check for lighter quark masses (Finite volume effects $\propto \mathrm{e}^{-\mathrm{L}m_{\pi,\rho}}$)
- phenomenological application (semi-leptonic decays, ...)