# EFFECTIVE FIELD THEORY

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# Outline

1) **General Aspects of Effective Field Theory**

- Dimensional Analysis
- Relevant, Irrelevant and Marginal
- Quantum Loops
- Decoupling. Matching. Scaling

2) **Chiral Perturbation Theory**

- Goldstone Theorem
- Chiral Symmetry
- Effective Goldstone Theory
- Massive States

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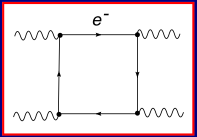


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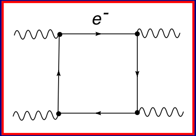


$$\Longrightarrow \qquad a = -\frac{\alpha^2}{36} \quad , \quad b = \frac{7\alpha^2}{90}$$

$$\sigma(\gamma\gamma \to \gamma\gamma) \propto \frac{\alpha^4 E^6}{m_e^8}$$

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Low-energy scattering of photons with neutral atoms

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**Blue light is scattered more strongly than red one**

# Dimensions

$$S \; = \; \int d^4x \; \mathcal{L}(x) \qquad \longrightarrow \qquad [\mathcal{L}] = E^4$$

$$\mathcal{L}_{\rm KG} \; = \; \partial_\mu\phi^\dagger\partial^\mu\phi - m^2\,\phi^\dagger\phi \qquad \longrightarrow \qquad [\phi] \; = \; [V^\mu] \; = \; [A^\mu] \; = \; E$$

$$\mathcal{L}_{\rm Dirac} \; = \; \bar\psi\,(i\gamma^\mu\partial_\mu - m)\,\psi \qquad \longrightarrow \qquad [\psi] \; = \; E^{3/2}$$

$$[\sigma] \; = \; E^{-2} \qquad , \qquad [\Gamma] \; = \; E$$

# Scalar Field Theory

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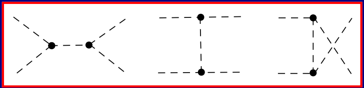

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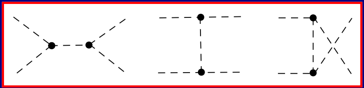

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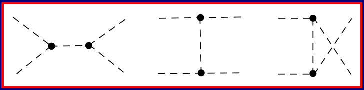

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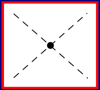

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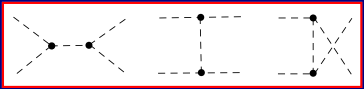

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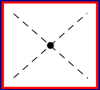

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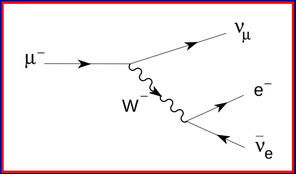
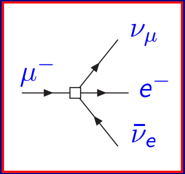



$$\frac{-g_{\mu\nu} + q_\mu q_\nu / M_W^2}{q^2 - M_W^2} \xrightarrow{q^2 \ll M_W^2} \frac{g_{\mu\nu}}{M_W^2}$$

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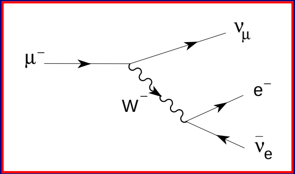


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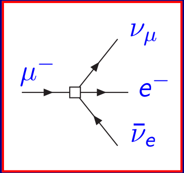


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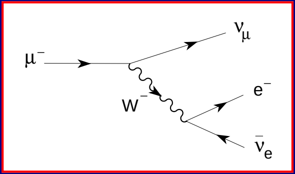


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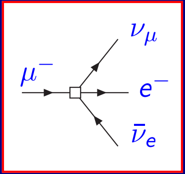


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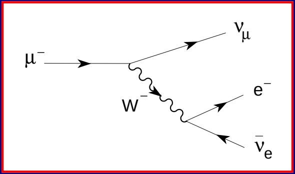


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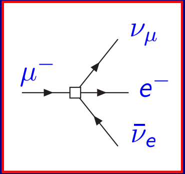


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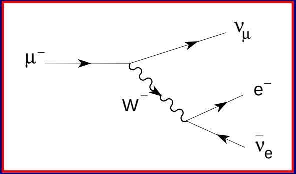


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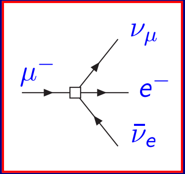


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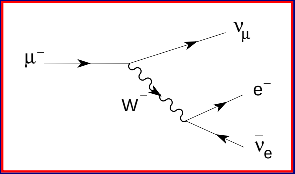


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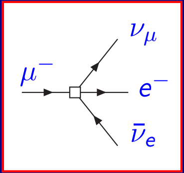


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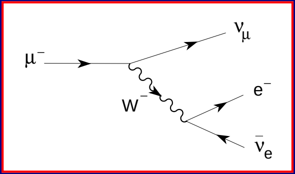


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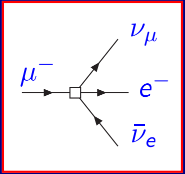


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- $\sigma(\nu_\mu e^- \to \mu^- \nu_e) \sim G_F^2 s$ $\qquad$ **Violates unitarity at high energies**

# Relevant, Irrelevant & Marginal

$$\mathcal{L} = \sum_i c_i \, O_i \qquad , \qquad [O_i] = d_i \quad \longrightarrow \quad c_i \sim \frac{1}{\Lambda^{d_i - 4}}$$

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**Low-energy behaviour:**

- **Relevant** $\mathbf{(d_i < 4)}$**:** Enhanced by $(\Lambda/E)^{4-d_i}$

  $I\, ,\ \phi^2\, ,\ \phi^3\, ,\ \bar\psi\psi$

- **Marginal** $\mathbf{(d_i = 4)}$

- **Irrelevant** $\mathbf{(d_i > 4)}$**:** Suppressed by $(E/\Lambda)^{d_i-4}$

$$\boxed{\alpha(Q^2) = \frac{\alpha(Q_0^2)}{1 - \beta_1 \frac{\alpha(Q_0^2)}{2\pi} \log\left(Q^2/Q_0^2\right)}}$$

# QED:

$$\beta_1^{\mathrm{QED}} = \frac{2}{3} \sum_f Q_f^2 N_f > 0 \quad \Longrightarrow \quad \lim_{Q^2 \to 0} \alpha(Q^2) = 0$$

Quantum corrections make **QED irrelevant** at low energies

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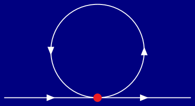
# QCD:

$$\beta_1^{\mathrm{QCD}} = \frac{2\,N_F \,-\, 11\,N_C}{6} \;<\; 0 \qquad \Longrightarrow \qquad \lim_{Q^2\to 0}\,\alpha_s(Q^2) \;=\; \infty$$

Quantum corrections make **QCD relevant** at low energies

# QUANTUM LOOPS

$$\mathcal{L} = \bar{\psi}\,(i\gamma^\mu\partial_\mu - m)\,\psi - \frac{a}{\Lambda^2}(\bar{\psi}\psi)^2 - \frac{b}{\Lambda^4}(\bar{\psi}\,\Box\psi)(\bar{\psi}\psi) + \cdots$$



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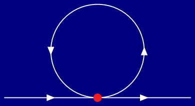

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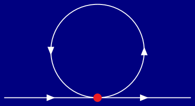

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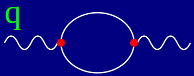
- **Cut-off regularization:** $\delta m \sim \dfrac{m}{\Lambda^2}\,\Lambda^2 \sim m$ **Not suppressed!**
- **Dimensional regularization:**

$$\delta m \sim 2a\,m\,\frac{m^2}{16\pi^2\Lambda^2}\,\mu^{2\epsilon}\left\{\frac{1}{\epsilon} + \gamma_E - \log\,(4\pi) + \log\left(\frac{m^2}{\mu^2}\right) - 1 + \mathcal{O}(\epsilon)\right\}$$

**Well-defined expansion**

# VACUUM POLARIZATION $(m_f \neq 0)$

$$\frac{1}{\hat\epsilon} \equiv \frac{2}{D-4} + \gamma_E - \ln 4\pi$$




$$i\,\Pi^{\mu\nu}(q) \;=\; i\,\left(-q^2 g^{\mu\nu} + q^\mu q^\nu\right)\,\Pi(q^2)$$

$$\Pi(q^2) \;=\; -\frac{\alpha Q_f^2}{3\pi}\left\{\frac{\mu^{2\epsilon}}{\hat\epsilon} + 6\int_0^1 dx\; x(1-x)\,\log\left(\frac{m_f^2 - q^2 x(1-x)}{\mu^2}\right)\right\}$$

$$\equiv\; \Delta\Pi_\epsilon(\mu^2) \;+\; \Pi_R(q^2/\mu^2)$$

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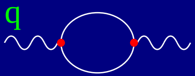


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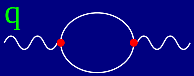


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$$\mu\,\frac{d\alpha}{d\mu} \;\equiv\; \alpha\,\beta(\alpha) \;=\; \alpha\left\{\beta_1\,\frac{\alpha}{\pi} + \beta_2\,\left(\frac{\alpha}{\pi}\right)^2 + \cdots\right\}$$

$$\longrightarrow \qquad \alpha(Q^2) \;\approx\; \frac{\alpha(Q_0^2)}{1 - \beta_1\,\frac{\alpha(Q_0^2)}{2\pi}\,\log\left(Q^2/Q_0^2\right)}$$

## Mass-Dependent Scheme: $\Delta\Pi_\epsilon(\mu^2) \equiv \Pi(-\mu^2)$

$$\Pi_R(q^2/\mu^2) \; = \; -Q_f^2 \, \frac{\alpha}{3\pi} \, 6 \int_0^1 dx \, x(1-x) \, \log\left[\frac{m_f^2 - q^2 x(1-x)}{m_f^2 + \mu^2 x(1-x)}\right]$$

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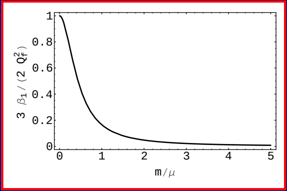

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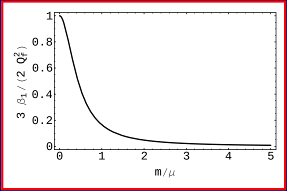


- $\mathbf{m_f^2 \ll \mu^2\,,\ q^2}$**:** $\quad \beta_1 = \frac{2}{3}\,Q_f^2 \quad , \quad \Pi_R(q^2/\mu^2) = -Q_f^2\,\frac{\alpha}{3\pi}\,\log\left(-q^2/\mu^2\right)$

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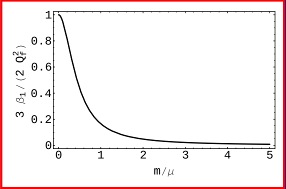


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- $\mathbf{m_f^2 \gg \mu^2, q^2}$: $\quad \beta_1 \sim \frac{2}{15} Q_f^2 \frac{\mu^2}{m_f^2} \quad , \quad \Pi_R(q^2/\mu^2) \sim Q_f^2 \frac{\alpha}{15\pi} \frac{q^2+\mu^2}{m_f^2}$

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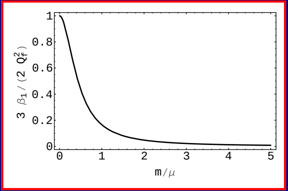


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➡ **DECOUPLING** (Appelquist-Carazzone Theorem)

$\overline{\mathrm{MS}}$ Scheme: $$\Delta\Pi_\epsilon(\mu^2) \;\equiv\; -Q_f^2\,\frac{\alpha_0\mu^{2\epsilon}}{3\pi}\,\frac{1}{\hat\epsilon}$$

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- $\beta_1 = \frac{2}{3} \, Q_f^2$ **Independent of $\mathbf{m_f}$**

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**Perturbation theory breaks down**

**SOLUTION: Integrate Out Heavy Particles**

# MATCHING

$\mu$

EFT with heavy fermion $\qquad \beta_1 = \frac{2}{3}\, Q_f^2 + \beta_1^{\mathrm{light}}$

$\mathbf{m_f}$

EFT without heavy fermion $\qquad \beta_1 = \beta_1^{\mathrm{light}}$

- **Two different EFTs** (with and without the heavy fermion $f$)
- **Same S-matrix elements for light-particle scattering at** $\mu = \mathbf{m_f}$

# EFFECTIVE FIELD THEORY

$$\mathcal{L} = \frac{1}{2}(\partial\phi)^2 + \frac{1}{2}(\partial\Phi)^2 - \frac{1}{2}m^2\phi^2 - \frac{1}{2}M^2\Phi^2 - \frac{\lambda}{2}\phi^2\Phi$$

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$$\sigma(\phi\phi \to \phi\phi) \sim \frac{1}{E^2} \times \begin{cases} (\lambda/E)^4 \;, & (m \ll M \ll E) \\ (\lambda/M)^4 \;, & (m, E \ll M) \end{cases}$$

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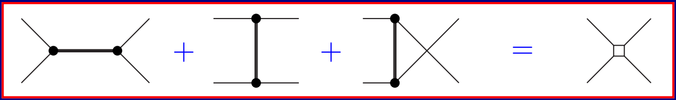

$\mathbf{E, m \ll M}$

$$\frac{\lambda^2}{s - M^2} = -\frac{\lambda^2}{M^2}\sum_{n=0}\frac{s}{M^2} \quad \Longrightarrow \quad \mathcal{L}_{\mathrm{eff}}(\phi) = \sum_i c_i\, O_i(\phi)$$

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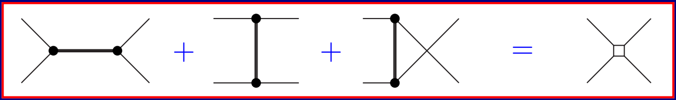

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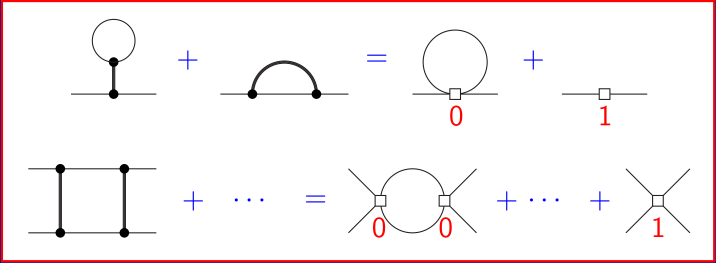
$$[O_i] = d_i \qquad ; \qquad c_i \sim \frac{\lambda^2}{M^2}\,\frac{1}{M^{d_i-4}}$$

One–Loop: $\qquad \mathcal{L}_{\mathrm{eff}} = \frac{1}{2}\, a\, (\partial\phi)^2 - \frac{1}{2}\, b\, \phi^2 + c\, \frac{\lambda^2}{8M^2}\, \phi^4 + \cdots$

**MATCHING**




$$a = 1 + a_1\, \frac{\lambda^2}{16\pi^2 M^2} + \cdots \qquad ; \qquad b = m^2 + b_1\, \frac{\lambda^2}{16\pi^2} + \cdots$$

$$c = 1 + c_1\, \frac{\lambda^2}{16\pi^2 M^2} + \cdots \qquad ; \qquad \cdots$$

# PRINCIPLES OF EFFECTIVE FIELD THEORY

- **Low-energy dynamics** independent of details at high energies
- Appropriate physics description at the analyzed scale **(degrees of freedom)**
- **Energy gaps:** $\quad 0 \leftarrow m \ll E \ll M \rightarrow \infty$
- Non-local heavy-particle exchanges replaced by a **tower of local interactions** among the light particles
- **Accuracy:** $\quad (E/M)^{(d_i-4)} \gtrsim \epsilon \quad \Longleftrightarrow \quad d_i \lesssim 4 + \dfrac{\log(1/\epsilon)}{\log(M/E)}$
- **Same infrared** (but different ultraviolet) **behaviour** than the underlying fundamental theory
- The only remnants of the high-energy dynamics are in the **low-energy couplings** and in the **symmetries** of the EFT

# Evolution from High to Low Scales

Large $\mu$

$\mathcal{L}(\phi_i) + \mathcal{L}(\phi_i, \Phi)$ $\quad\downarrow\quad$ $\phi_i, \Phi$

**Renormalization Group**

$\mu = M$ **Matching**

$\mathcal{L}(\phi_i) + \delta\mathcal{L}(\phi_i)$ $\quad\downarrow\quad$ $\phi_i$

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Small $\mu$

# QCD MATCHING

$$(\mu > M) \qquad \boxed{\mathcal{L}_{\rm QCD}^{(N_F)} \quad \Longleftrightarrow \quad \mathcal{L}_{\rm QCD}^{(N_F-1)} + \sum_{d_i>4} \frac{c_i}{M^{d_i-4}}\, O_i} \qquad (\mu < M)$$

# QCD MATCHING

$$(\mu > M) \qquad \boxed{\mathcal{L}_{\mathrm{QCD}}^{(N_F)} \longleftrightarrow \mathcal{L}_{\mathrm{QCD}}^{(N_F-1)} + \sum_{d_i>4} \frac{c_i}{M^{d_i-4}}\, O_i} \qquad (\mu < M)$$

$$\alpha_s^{(N_F)}(\mu^2) \; = \; \alpha_s^{(N_F-1)}(\mu^2) \, \left\{ 1 + \sum_{k=1}^{\infty} C_k(L) \, \left[ \frac{\alpha_s^{(N_F-1)}(\mu^2)}{\pi} \right]^k \right\}$$

$$L \equiv \ln\left(\mu^2/m_q^2\right)$$

$$m_q^{(N_F)}(\mu^2) \; = \; m_q^{(N_F-1)}(\mu^2) \, \left\{ 1 + \sum_{k=1}^{\infty} H_k(L) \, \left[ \frac{\alpha_s^{(N_F-1)}(\mu^2)}{\pi} \right]^k \right\}$$

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- Matching conditions known to 4 (3) loops: $C_{1,2,3,4}\;,\; H_{1,2,3}$
  (Chetyrkin et al, Larin et al)
- $L$ dependence known to 4 loops: $H_4(L)$
- $\alpha_s(\mu^2)$ is not continuous at threshold

# Wilson Coefficients:

$$\mathcal{L} = \sum_i \frac{c_i}{\Lambda^{d_i-4}} \, O_i$$

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$$\longrightarrow \qquad \boxed{\begin{aligned} c_i(\mu) &= c_i(\mu_0)\,\exp\left\{\int_{\alpha_0}^{\alpha} \frac{d\alpha}{\alpha}\,\frac{\gamma_{O_i}(\alpha)}{\beta(\alpha)}\right\} \\ &= c_i(\mu_0)\,\left[\frac{\alpha(\mu^2)}{\alpha(\mu_0^2)}\right]^{\gamma_{O_i}^{(1)}/\beta_1}\left\{1+\cdots\right\}\end{aligned}}$$

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