Chiral Perturbation Theory

- Sigma Model
- Goldstone Theorem
- Chiral Symmetry
- Effective Goldstone Theory
- Explicit Symmetry Breaking
- Higher Orders
- Resonance Chiral Theory

SIGMA MODEL: $\quad \mathbf{\Phi}^{\mathsf{T}} \equiv (\sigma, \vec{\pi})$

$$\mathcal{L}_\sigma = \frac{1}{2}\,\partial_\mu \mathbf{\Phi}^{\mathsf{T}} \partial^\mu \mathbf{\Phi} \;-\; \frac{\lambda}{4}\,\left(\mathbf{\Phi}^{\mathsf{T}}\mathbf{\Phi} - v^2\right)^2$$

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Global Symmetry: $\quad \mathbf{O(4)} \sim \mathbf{SU(2)} \otimes \mathbf{SU(2)}$

- $v^2 < 0$: $\quad m_\Phi^2 \,=\, -\lambda\, v^2$
- $v^2 > 0$: $\quad \langle 0|\sigma|0\rangle = v \quad , \quad \langle 0|\vec{\pi}|0\rangle = 0$

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SSB: $\quad \mathbf{O(4)} \to \mathbf{O(3)} \quad$ [$\frac{4\times 3}{2} - \frac{3\times 2}{2} \; = \;$ 3 broken generators]

$$\mathcal{L}_\sigma = \frac{1}{2}\left\{\partial_\mu\hat{\sigma}\,\partial^\mu\hat{\sigma} + \partial_\mu\vec{\pi}\,\partial^\mu\vec{\pi} - M^2\hat{\sigma}^2\right\} - \frac{M^2}{2v}\,\hat{\sigma}\left(\hat{\sigma}^2 + \vec{\pi}^2\right) - \frac{M^2}{8v^2}\left(\hat{\sigma}^2 + \vec{\pi}^2\right)^2$$

$$\hat{\sigma} \equiv \sigma - v \qquad ; \qquad M^2 = 2\,\lambda\, v^2$$

3 Massless Goldstone Bosons

1) $\boldsymbol{\Sigma}(x) \equiv \sigma(x)\, \mathbf{I}_2 + i\, \vec{\tau}\, \vec{\pi}(x)$; $\langle \mathbf{A} \rangle \equiv \mathrm{Tr}\,(\mathbf{A})$

$$\mathcal{L}_\sigma = \frac{1}{4} \langle \partial_\mu \boldsymbol{\Sigma}^\dagger \partial^\mu \boldsymbol{\Sigma} \rangle - \frac{\lambda}{16} \left(\langle \boldsymbol{\Sigma}^\dagger \boldsymbol{\Sigma} \rangle - 2\, v^2 \right)^2$$

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2) $\boldsymbol{\Sigma}(x) \,\equiv\, [v + S(x)]\;\mathbf{U}(x)$; $\mathbf{U} \equiv \exp\left\{i\,\frac{\vec{\tau}\,\vec{\phi}}{v}\right\} \longrightarrow\; g_R\,\mathbf{U}\,g_L^\dagger$

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Derivative Golstone Couplings

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Derivative Golstone Couplings

3) $E \ll M \sim v$:

$$\boxed{\mathcal{L}_\sigma \approx \frac{v^2}{4}\, \langle \partial_\mu \mathbf{U}^\dagger\, \partial^\mu \mathbf{U} \rangle}$$

SYMMETRY REALIZATIONS

Symmetry **G** $\{T_a\}$ $\Longleftrightarrow$ **Conserved charges** $\mathcal{Q}_a$

Noether Theorem: $\partial_\mu j^\mu_a = 0 \quad ; \quad \mathcal{Q}_a = \int d^3x \, j^0_a(x) \quad ; \quad \frac{d}{dt}\,\mathcal{Q}_a = 0$

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- Degenerate Multiplets
- Linear Representation

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Nambu–Goldstone

$$\mathcal{Q}_a\,|\,0\,\rangle \neq 0$$

- Spontaneously Broken Symmetry
- Massless Goldstone Bosons
- Non-Linear Representation

GOLDSTONE THEOREM

$$\mathcal{Q} = \int d^3x \, j^0(x) \quad ; \quad \partial_\mu j_a^\mu = 0 \quad ; \quad \exists\, \mathcal{O} : \; v(t) \equiv \langle 0 | \left[\mathcal{Q}(t), \mathcal{O} \right] | 0 \rangle \neq 0$$

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$$\exists\, |n\rangle : \; \langle 0 | \mathcal{O} | n \rangle \, \langle n | j^0 | 0 \rangle \neq 0 \quad ; \quad E_n \, \delta^{(3)}(\vec{p}_n) = 0 \quad ; \quad M_n = 0$$

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$$= \sum_n \int d^3x\; \left\{\mathrm{e}^{-ip_n\cdot x}\, \langle 0|j^0(0)|n\rangle\langle n|\mathcal{O}|0\rangle \,-\, \mathrm{e}^{ip_n\cdot x}\, \langle 0|\mathcal{O}|n\rangle\langle n|j^0(0)|0\rangle\right\}$$

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$$\begin{aligned} v(t) &= \sum_n \int d^3x\, \left\{ \langle 0 | j^0(x) | n \rangle \langle n | \mathcal{O} | 0 \rangle - \langle 0 | \mathcal{O} | n \rangle \langle n | j^0(x) | 0 \rangle \right\} \\ &= \sum_n \int d^3x\, \left\{ \mathrm{e}^{-ip_n\cdot x} \langle 0 | j^0(0) | n \rangle \langle n | \mathcal{O} | 0 \rangle - \mathrm{e}^{ip_n\cdot x} \langle 0 | \mathcal{O} | n \rangle \langle n | j^0(0) | 0 \rangle \right\} \\ &= (2\pi)^3 \sum_n \delta^{(3)}(\vec{p}_n)\, \left\{ \mathrm{e}^{-iE_n t} \langle 0 | j^0(0) | n \rangle \langle n | \mathcal{O} | 0 \rangle - \mathrm{e}^{iE_n t} \langle 0 | \mathcal{O} | n \rangle \langle n | j^0(0) | 0 \rangle \right\} \neq 0 \end{aligned}$$

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$$= (2\pi)^3 \sum_n \delta^{(3)}(\vec{p}_n) \left\{ \mathrm{e}^{-iE_n t} \langle 0 | j^0(0) | n \rangle \langle n | \mathcal{O} | 0 \rangle - \mathrm{e}^{iE_n t} \langle 0 | \mathcal{O} | n \rangle \langle n | j^0(0) | 0 \rangle \right\} \neq 0$$

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$$\frac{d}{dt} v(t) = 0 = -i\,(2\pi)^3 \sum_n \delta^{(3)}(\vec{p}_n)\, E_n \left\{ \mathrm{e}^{-iE_n t} \langle 0|j^0(0)|n\rangle\langle n|\mathcal{O}|0\rangle + \mathrm{e}^{iE_n t} \langle 0|\mathcal{O}|n\rangle\langle n|j^0(0)|0\rangle \right\} \qquad \square$$

CHIRAL SYMMETRY

$\mathbf{q} \equiv \begin{pmatrix} u \\ d \\ s \end{pmatrix}$; $\mathbf{m_q = 0}$ (Chiral Limit)

$$\mathcal{L}^0_{QCD} = -\frac{1}{4}\, G^{\mu\nu}_a G^a_{\mu\nu} + \bar{\mathbf{q}}_L\, i\,\gamma^\mu D_\mu\, \mathbf{q}_L + \bar{\mathbf{q}}_R\, i\,\gamma^\mu D_\mu\, \mathbf{q}_R$$

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$$q = \left(\frac{1-\gamma_5}{2}\right) q + \left(\frac{1+\gamma_5}{2}\right) q \equiv q_L + q_R$$

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- $\mathcal{L}^0_{QCD}$ **invariant under** $\mathbf{G \equiv SU(3)_L \otimes SU(3)_R}$**:**

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- **The** 0^- **octet is nearly massless:** $\mathbf{m_\pi \approx 0}$

- **The vacuum is not invariant (SSB):** $\langle 0 | \left(\bar{\mathbf{q}}_L \mathbf{q}_R + \bar{\mathbf{q}}_R \mathbf{q}_L \right) | 0 \rangle \neq 0$

8 Massless 0^- Goldstone Bosons

Noether QCD Currents: $\quad G \equiv SU(3)_L \otimes SU(3)_R$

$$J_X^{a\mu} = \bar{\mathbf{q}}_X \, \gamma^\mu \, \frac{\lambda^a}{2} \, \mathbf{q}_X \quad ; \quad \mathcal{Q}_X^a = \int d^3x \, J_X^{a0}(x) \qquad (a = 1, \cdots, 8 \; ; \; X = L, R)$$

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Dynamical Symmetry Breaking:

- 8 Pseudoscalar Goldstones $\pi^a = (\pi, K, \eta)$

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Dynamical Symmetry Breaking:

- 8 Pseudoscalar Goldstones $\qquad \pi^a = (\pi, K, \eta)$
- $\mathcal{Q}_A^a = \mathcal{Q}_R - \mathcal{Q}_L \qquad ; \qquad \mathcal{O}^b = \bar{\mathbf{q}} \, \gamma_5 \, \lambda^b \, \mathbf{q}$

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Energy Scale	Fields	Effective Theory
M_W	W, Z, γ, g τ, μ, e, ν_i t, b, c, s, d, u	Standard Model
	$\big\downarrow$ OPE	
$\lesssim m_c$	$\gamma, g\ ;\ \mu, e, \nu_i$ s, d, u	$\mathcal{L}_{\rm QCD}^{(n_f=3)}$, $\mathcal{L}_{\rm eff}^{\Delta S=1,2}$
	$\big\downarrow$ $N_C \to \infty$	
M_K	$\gamma\ ;\ \mu, e, \nu_i$ π, K, η	χPT

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$$\longrightarrow \quad \boxed{\mathcal{L}_2 \;=\; \frac{f^2}{4}\, \langle \partial_\mu \mathbf{U}^\dagger\, \partial^\mu \mathbf{U} \rangle}$$

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$$+ \frac{1}{6f^2}\left\{\left(\pi^+ \overleftrightarrow{\partial}_\mu\, \pi^-\right)\left(\pi^+ \overleftrightarrow{\partial}^\mu \pi^-\right) + 2\left(\pi^0 \overleftrightarrow{\partial}_\mu\, \pi^+\right)\left(\pi^- \overleftrightarrow{\partial}^\mu \pi^0\right) + \cdots\right\}$$

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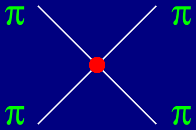
Chiral Symmetry Determines the Interaction:



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$t \equiv (p'_+ - p_+)^2$

Weinberg

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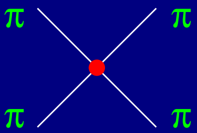

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Non-Linear Lagrangian: $2\pi \to 2\pi, 4\pi, \cdots$ related

EXPLICIT SYMMETRY BREAKING

$$\mathcal{L}_{QCD} \equiv \mathcal{L}^0_{QCD} + \bar{\mathbf{q}}\,(\not{\mathbf{v}} + \gamma_5\,\not{\mathbf{a}})\,\mathbf{q} - \bar{\mathbf{q}}\,(\mathbf{s} - i\,\gamma_5\,\mathbf{p})\,\mathbf{q}$$

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$$\mathbf{s} = \mathcal{M} + \cdots \quad ; \quad \mathcal{M} \equiv \mathrm{diag}(m_u, m_d, m_s)$$

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Local $\mathrm{SU(3)_L} \otimes \mathrm{SU(3)_R}$ Symmetry:

$$\mathbf{q}_L \to g_L\,\mathbf{q}_L \qquad \mathbf{l}_\mu \to g_L\,\mathbf{l}_\mu\,g_L^\dagger + i\,g_L\,\partial_\mu g_L^\dagger$$

$$\mathbf{r}_\mu \to g_R\,\mathbf{r}_\mu\,g_R^\dagger + i\,g_R\,\partial_\mu g_R^\dagger$$

$$\mathbf{q}_R \to g_R\,\mathbf{q}_R \qquad (\mathbf{s} + i\,\mathbf{p}) \to g_R\,(\mathbf{s} + i\,\mathbf{p})\,g_L^\dagger$$

Lowest–Order Effective Lagrangian

$$\boxed{\mathcal{L} = \frac{f^2}{4} \langle D_\mu \mathbf{U} \, D^\mu \mathbf{U}^\dagger + \chi \, \mathbf{U}^\dagger + \mathbf{U} \, \chi^\dagger \rangle}$$

$$D_\mu \mathbf{U} = \partial_\mu \mathbf{U} - i \, \mathbf{r}_\mu \, \mathbf{U} + i \, \mathbf{U} \, \mathbf{l}_\mu$$

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$$\langle 0 |\, (J^\mu_A)_{12}\, | \pi^+(p) \rangle = i\, \sqrt{2}\, f\; p^\mu \quad \longrightarrow \quad \boxed{f = f_\pi \approx 92.4\ \mathrm{MeV}}$$

$(\pi^+ \to \mu^+ \nu_\mu)$

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QUARK MASSES: $\frac{f^2}{4}\,\langle \chi\, \mathbf{U}^\dagger + \mathbf{U}\, \chi^\dagger \rangle \quad\longrightarrow\quad \mathcal{L}_m = -B_0\, \langle \mathcal{M}\, \Phi^2 \rangle$

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Isospin limit: $\mathbf{m_u = m_d = \hat{m}}$

$$\frac{M_\pi^2}{2\,\hat{m}} = \frac{M_K^2}{\hat{m} + m_s} = \frac{3\, M_\eta^2}{2\,\hat{m} + 4\, m_s} = B_0$$

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- **Gell-Mann–Okubo:** $4\,M_K^2 = M_\pi^2 + 3\,M_\eta^2$

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$$\left. + \;(m_d + m_s)\, K^0 \bar{K}^0 \;+\; \frac{1}{6}\,(m_u + m_d + 4\, m_s)\,\eta^2 \;+\; \frac{1}{\sqrt{3}}\,(m_u - m_d)\,\pi^0\eta \right\}$$

Isospin limit: $\mathbf{m_u = m_d = \hat{m}}$

$$\frac{M_\pi^2}{2\,\hat{m}} \;=\; \frac{M_K^2}{\hat{m} + m_s} \;=\; \frac{3\, M_\eta^2}{2\,\hat{m} + 4\, m_s} \;=\; B_0$$

- **Gell-Mann–Okubo:** $4\, M_K^2 = M_\pi^2 + 3\, M_\eta^2$
- **Gell-Mann–Oakes–Renner:** $f^2\, M_\pi^2 \;=\; -\hat{m}\,\langle 0|\bar{u}\, u + \bar{d}\, d|0\rangle$

QUARK MASS RATIOS

Dashen Theorem:

$$\boxed{\left(M^2_{K^0} - M^2_{K^\pm}\right)_{\mathrm{em}} = \left(M^2_{\pi^0} - M^2_{\pi^\pm}\right)_{\mathrm{em}} + \mathcal{O}(e^2 p^2)}$$

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Proof: $e^2 \langle \mathcal{Q}_{\mathrm{R}} \mathrm{U} \mathcal{Q}_{\mathrm{L}} \mathrm{U}^\dagger \rangle = -\frac{2e^2}{f^2} \left(\pi^+\pi^- + K^+K^-\right) + \mathcal{O}(\phi^2)$; $\mathcal{Q}_X \to g_X\, \mathcal{Q}_X\, g_X^\dagger$ $\square$

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$$\frac{m_d - m_u}{m_d + m_u} = \frac{\left(M^2_{K^0} - M^2_{K^\pm}\right) - \left(M^2_{\pi^0} - M^2_{\pi^\pm}\right)}{M^2_{\pi^0}} \approx 0.29$$

QUARK MASS RATIOS

Dashen Theorem:

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$$\frac{m_s - m_u}{m_u + m_d} = \frac{M_{K^0}^2 - M_{\pi^0}^2}{M_{\pi^0}^2} \approx 12.6$$

QUARK MASS RATIOS

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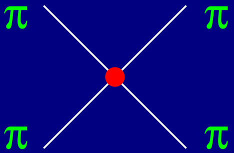
$\longrightarrow$ $\boxed{\mathbf{m_u : m_d : m_s = 0.55 : 1 : 20.3}}$ Weinberg

$$\frac{f^2}{4}\,\langle \chi\,\mathbf{U}^\dagger + \mathbf{U}\,\chi^\dagger \rangle \;=\; -B_0\,\langle\,\mathcal{M}\,\Phi^2\rangle \;+\; \frac{B_0}{6\,f^2}\,\langle\,\mathcal{M}\,\Phi^4\rangle \;+\;\cdots$$

$$\frac{f^2}{4} \langle \chi \mathbf{U}^\dagger + \mathbf{U} \chi^\dagger \rangle = -B_0 \langle \mathcal{M} \Phi^2 \rangle + \frac{B_0}{6 f^2} \langle \mathcal{M} \Phi^4 \rangle + \cdots$$



$$T \left(\pi^+ \pi^0 \to \pi^+ \pi^0 \right) = \frac{t - M_\pi^2}{f_\pi^2}$$

$t \equiv (p'_+ - p_+)^2$

Weinberg

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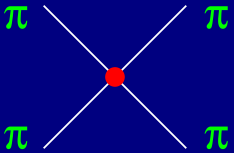

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Weinberg

$\mathcal{L}_2$ ⟷ **Current Algebra** **60's**

Chiral Power Counting

$\mathbf{U}$	$\mathcal{O}(p^0)$
$D_\mu \mathbf{U}\,,\ \mathbf{l}_\mu\,,\ \mathbf{r}_\mu$	$\mathcal{O}(p^1)$
$\chi\,,\ \mathbf{F}^{\mu\nu}_{L,R}$	$\mathcal{O}(p^2)$

$$\mathbf{F}^{\mu\nu}_L \equiv \partial^\mu \mathbf{l}^\nu - \partial^\nu \mathbf{l}^\mu - i\ [\mathbf{l}^\mu, \mathbf{l}^\nu]$$

$$\mathbf{F}^{\mu\nu}_R \equiv \partial^\mu \mathbf{r}^\nu - \partial^\nu \mathbf{r}^\mu - i\ [\mathbf{r}^\mu, \mathbf{r}^\nu]$$

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General connected diagram with N_d vertices of $\mathcal{O}(p^d)$ and L loops:

$$D \;=\; 2\,L + 2 + \sum_d N_d\,(d-2)$$ Weinberg

Chiral Power Counting

$\mathbf{U}$	$\mathcal{O}(p^0)$
$D_\mu \mathbf{U}\,,\ \mathbf{l}_\mu\,,\ \mathbf{r}_\mu$	$\mathcal{O}(p^1)$
$\chi\,,\ \mathbf{F}^{\mu\nu}_{L,R}$	$\mathcal{O}(p^2)$

$$\mathbf{F}_L^{\mu\nu} \equiv \partial^\mu \mathbf{l}^\nu - \partial^\nu \mathbf{l}^\mu - i\ [\mathbf{l}^\mu, \mathbf{l}^\nu]$$

$$\mathbf{F}_R^{\mu\nu} \equiv \partial^\mu \mathbf{r}^\nu - \partial^\nu \mathbf{r}^\mu - i\ [\mathbf{r}^\mu, \mathbf{r}^\nu]$$

General connected diagram with N_d vertices of $\mathcal{O}(p^d)$ and L loops:

$$D = 2L + 2 + \sum_d N_d\,(d-2) \qquad \text{Weinberg}$$

- $D = 2:$ $\quad L = 0\ ,\ d = 2$
- $D = 4:$ $\quad L = 0\ ,\ d = 4\ ,\ N_4 = 1$
  $\quad L = 1\ ,\ d = 2$

$\mathcal{O}(\mathbf{p^4})$ χPT

i) $\mathcal{L}_4$ **at tree level** (Gasser–Leutwyler)

$$
\begin{aligned}
\mathcal{L}_4 \;=\; & L_1 \,\langle D_\mu U^\dagger D^\mu U\rangle^2 \,+\, L_2\,\langle D_\mu U^\dagger D_\nu U\rangle\,\langle D^\mu U^\dagger D^\nu U\rangle \\
+\; & L_3\,\langle D_\mu U^\dagger D^\mu U D_\nu U^\dagger D^\nu U\rangle \,+\, L_4\,\langle D_\mu U^\dagger D^\mu U\rangle\,\langle U^\dagger\chi + \chi\dagger U\rangle \\
+\; & L_5\,\langle D_\mu U^\dagger D^\mu U\left(U^\dagger\chi + \chi^\dagger U\right)\rangle \,+\, L_6\,\langle U^\dagger\chi + \chi^\dagger U\rangle^2 \\
+\; & L_7\,\langle U^\dagger\chi - \chi^\dagger U\rangle^2 \,+\, L_8\,\langle \chi^\dagger U\chi^\dagger U + U^\dagger\chi U^\dagger\chi\rangle \\
-\; & i\,L_9\,\langle F_R^{\mu\nu} D_\mu U D_\nu U^\dagger + F_L^{\mu\nu} D_\mu U^\dagger D_\nu U\rangle \,+\, L_{10}\,\langle U^\dagger F_R^{\mu\nu} U F_{L\mu\nu}\rangle
\end{aligned}
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$\mathcal{O}(\mathbf{p}^4)$ χPT

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\end{aligned}$$

ii) $\mathcal{L}_2$ **at one loop** **(unitarity):** $\qquad T_4 \sim p^4 \left\{ a \,\log(p^2/\mu^2) + b(\mu)\right\}$

- Chiral Logarithms unambiguously predicted

$\mathcal{O}(\mathbf{p}^4)\quad \chi\mathbf{PT}$

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\end{aligned}
$$

ii) $\mathcal{L}_2$ **at one loop (unitarity):** $\quad T_4 \sim p^4\left\{a\,\log(p^2/\mu^2) + b(\mu)\right\}$

- Chiral Logarithms unambiguously predicted
- L_i's fixed by QCD dynamics. 1–loop divergences $\longrightarrow$ $L_i^r(\mu)$

$\mathcal{O}(\mathbf{p}^4)$ χPT

i) $\mathcal{L}_4$ **at tree level** (Gasser–Leutwyler)

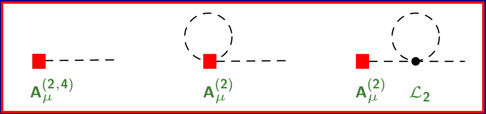
$$\boxed{\begin{aligned}\mathcal{L}_4 \;=\;& L_1 \,\langle D_\mu U^\dagger D^\mu U\rangle^2 + L_2\,\langle D_\mu U^\dagger D_\nu U\rangle\,\langle D^\mu U^\dagger D^\nu U\rangle \\ &+ L_3\,\langle D_\mu U^\dagger D^\mu U D_\nu U^\dagger D^\nu U\rangle + L_4\,\langle D_\mu U^\dagger D^\mu U\rangle\,\langle U^\dagger \chi + \chi \dagger U\rangle \\ &+ L_5\,\langle D_\mu U^\dagger D^\mu U \left(U^\dagger\chi + \chi^\dagger U\right)\rangle + L_6\,\langle U^\dagger\chi + \chi^\dagger U\rangle^2 \\ &+ L_7\,\langle U^\dagger\chi - \chi^\dagger U\rangle^2 + L_8\,\langle \chi^\dagger U \chi^\dagger U + U^\dagger\chi U^\dagger\chi\rangle \\ &- i\,L_9\,\langle F_R^{\mu\nu} D_\mu U D_\nu U^\dagger + F_L^{\mu\nu} D_\mu U^\dagger D_\nu U\rangle + L_{10}\,\langle U^\dagger F_R^{\mu\nu} U F_{L\mu\nu}\rangle\end{aligned}}$$

ii) $\mathcal{L}_2$ **at one loop** **(unitarity):** $T_4 \sim p^4 \left\{a \,\log(p^2/\mu^2) + b(\mu)\right\}$

- Chiral Logarithms unambiguously predicted
- L_i's fixed by QCD dynamics. 1–loop divergences $\longrightarrow$ $L_i^r(\mu)$

iii) **Wess–Zumino–Witten term** **(chiral anomaly):** $\pi^0, \eta \to \gamma\gamma$

Meson Decay Constants



$$\mu_P \equiv \frac{M_P^2}{32\pi^2 f^2} \log\left(\frac{M_P^2}{\mu^2}\right)$$

$$f_\pi = f \left\{1 - 2\mu_\pi - \mu_K + \frac{4M_\pi^2}{f^2} L_5^r(\mu) + \frac{8M_K^2 + 4M_\pi^2}{f^2} L_4^r(\mu)\right\}$$

$$f_K = f \left\{1 - \frac{3}{4}\mu_\pi - \frac{3}{2}\mu_K - \frac{3}{4}\mu_{\eta_8} + \frac{4M_K^2}{f^2} L_5^r(\mu) + \frac{8M_K^2 + 4M_\pi^2}{f^2} L_4^r(\mu)\right\}$$

$$f_{\eta_8} = f \left\{1 - 3\mu_K + \frac{4M_{\eta_8}^2}{f^2} L_5^r(\mu) + \frac{8M_K^2 + 4M_\pi^2}{f^2} L_4^r(\mu)\right\}$$

$$\boxed{\frac{f_K}{f_\pi} = 1.22 \pm 0.01 \;\longrightarrow\; L_5^r(M_\rho) = (1.4 \pm 0.5)\cdot 10^{-3} \;\longrightarrow\; \frac{f_{\eta_8}}{f_\pi} = 1.3 \pm 0.05}$$

$O(p^4)$ χPT COUPLINGS

i	$L_i^r(M_\rho) \times 10^3$	Source	Γ_i
1	0.4 ± 0.3	K_{e4}, $\pi\pi \to \pi\pi$	$3/32$
2	1.4 ± 0.3	K_{e4}, $\pi\pi \to \pi\pi$	$3/16$
3	-3.5 ± 1.1	K_{e4}, $\pi\pi \to \pi\pi$	0
4	-0.3 ± 0.5	Zweig rule	$1/8$
5	1.4 ± 0.5	F_K/F_π	$3/8$
6	-0.2 ± 0.3	Zweig rule	$11/144$
7	-0.4 ± 0.2	GMO, $L_{5,8}$	0
8	0.9 ± 0.3	$M_{K^0} - M_{K^+}$, L_5, $(m_s - \hat{m})/(m_d - m_u)$	$5/48$
9	6.9 ± 0.7	$\langle r^2 \rangle_V^\pi$	$1/4$
10	-5.5 ± 0.7	$\pi \to e\nu\gamma$	$-1/4$

- $$L_i = L_i^r(\mu) + \Gamma_i \, \frac{\mu^{D-4}}{32\pi^2} \left\{ \frac{2}{D-4} + \gamma_E - \log(4\pi) - 1 \right\}$$

$O(p^4)$ χPT COUPLINGS

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- $\Lambda_\chi \sim 1\,\mathrm{GeV}$ $\Rightarrow$ $L_i \sim \frac{f_\pi^2/4}{\Lambda_\chi^2} \sim 2 \times 10^{-3}$

$O(p^4)$ χPT COUPLINGS

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- $\Lambda_\chi \sim 1\,\mathrm{GeV}$ ➡ $L_i \sim \frac{f_\pi^2/4}{\Lambda_\chi^2} \sim 2 \times 10^{-3}$
- χPT Loops $\sim$ $1/(4\pi f_\pi)^2$

$O(p^6)$ χ**PT**

i) $\mathcal{L}_6 = \sum_i C_i\, O_i^{p^6}$ **at tree level** (Bijnens–Colangelo–Ecker, Fearing–Scherer)

$90+4$ $[53+4]$ terms in SU(3) [SU(2)] χPT (even-intrinsic parity only)

ii) $\mathcal{L}_4$ **at one loop,** $\mathcal{L}_2$ **at two loops** (Bijnens–Colangelo–Ecker)

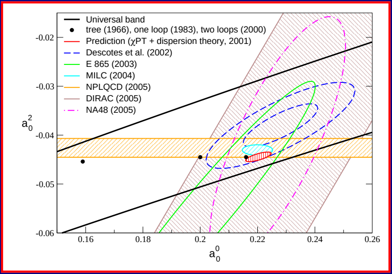
Double chiral logarithms

Many Calculations: M_ϕ, f_ϕ, $\gamma\gamma \to \pi\pi$, $\pi\pi \to \pi\pi$, $\pi K \to \pi K$, K_{l4},
$\pi \to e\,\bar{\nu}_e \gamma$, $F_V(s)$, $F_S(s)$, $\Pi_{V,A}(s)$, $\cdots$

Amoros–Bijnens–Dhonte–Talavera, Bellucci–Gasser–Sainio, Bürgui, Bijnens–Colangelo–Ecker–Gasser–Sainio, Descotes-Genon–Girlanda–Knecht–Moussallam–Stern–Fuchs, Ananthanarayan–Colangelo–Gasser–Leutwyler, Post–Schilcher, Golowich–Kambor, . . .

$\mathcal{O}(\mathbf{p^6})$ Analysis of $\pi\pi \to \pi\pi$

Colangelo et al.



χPT, analyticity, crossing, unitarity

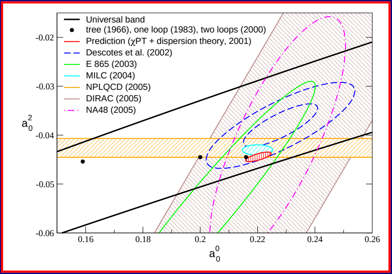
$$a_0^0 = 0.220 \pm 0.005$$

$$a_0^2 = -0.0444 \pm 0.0010$$

$$\delta_0^0(M_K^2) - \delta_0^2(M_K^2) = (47.7 \pm 1.5)^\circ$$

$\mathcal{O}(\mathbf{p^6})$ Analysis of $\pi\pi \longrightarrow \pi\pi$

Colangelo et al.



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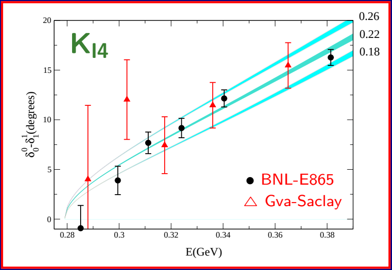


$$a_0^0 = 0.221 \pm 0.026 \quad \longrightarrow \quad |\bar{l}_3| \lesssim 16$$

GMOR accounts for $\sim$ 94% of M_π

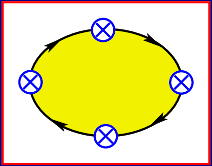
$$M_\pi^2 = M^2 \left\{ 1 - \frac{1}{2} \frac{M^2}{(4\pi F)^2} \bar{\ell}_3 + O(M^4) \right\}$$

$$M^2 = (m_u + m_d) \frac{|\langle \bar{u} u \rangle|}{f^2}$$

Energy Scale	Fields	Effective Theory
M_W	t, b, c $s, d, u\ ;\ G^a$	$\mathrm{QCD}^{N_f=6}$
$\lesssim\ m_c$	$s, d, u\ ;\ G^a$	$\mathrm{QCD}^{N_f=3}$
Λ_χ	V, A, S, P π, K, η	$\mathrm{R}\chi\mathrm{T}$
$\lesssim\ M_K$	π, K, η	$\chi\mathrm{PT}^{N_f=3}$
$\lesssim\ M_\pi$	π	$\chi\mathrm{PT}^{N_f=2}$

LARGE–N_C COUNTING RULES

$g_s \sim 1/\sqrt{N_C}$; $\alpha_s \sim 1/N_C$; $\langle T\,(J_1 \cdots J_n)\rangle \sim N_C$

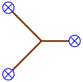
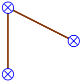


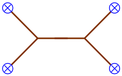
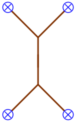
- Dominance of planar gluonic exchanges
- Non-planar diagrams suppressed by $1/N_C^2$
- Internal quark loops suppressed by $1/N_C$

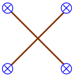
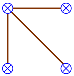
Colour Confinement $\Longrightarrow$ $J\,|0\rangle \sim |1 \text{ Meson}\rangle$

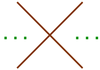
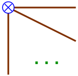
$$\boxed{\langle J(k)\,J(-k)\rangle = \sum_n \frac{f_n^2}{k^2 - M_n^2}}$$

- **Infinite** number of mesons $(\sim \ln k^2)$
- $f_n = \langle 0|J|n\rangle \sim \sqrt{N_C}$; $M_n \sim O(1)$
- Mesons are **free**, **stable** and **non-interacting**

$\langle J\,J\,J \rangle \;=\; \sum$  $+ \;\sum$ 

$\langle J\,J\,J\,J \rangle \;=\; \sum$  $+ \;\sum$ 

$+ \;\sum$  $+ \;\sum$ 

 $\sim N_C^{1-\frac{n}{2}}$  $\sim N_C^{1-\frac{n}{2}}$

Crossing + Unitarity $\longrightarrow$ **Tree Approximation to some Local Effective Meson Lagrangian**

RχT

(Ecker, Gasser, Pich, de Rafael)

Resonance Nonet Multiplets: **V**(1^{--}) , **A**(1^{++}) , **S**(0^{++}) , **P**(0^{-+})

$$\mathcal{L}_2^V = \frac{F_V}{2\sqrt{2}} \langle V_{\mu\nu} f_+^{\mu\nu} \rangle + \frac{i\, G_V}{\sqrt{2}} \langle V_{\mu\nu} u^\mu u^\nu \rangle$$

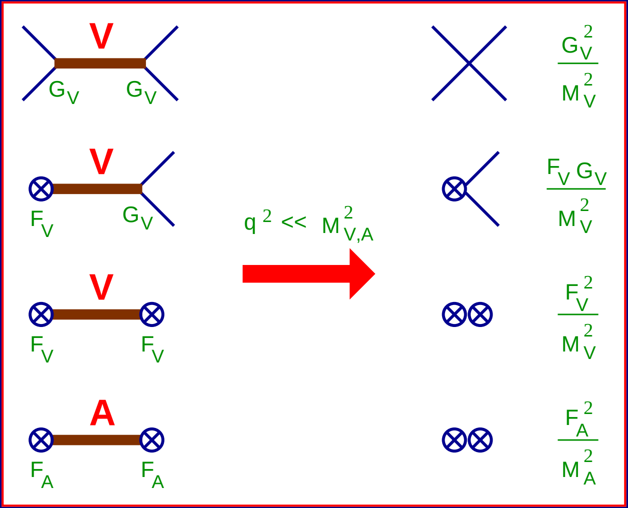
$$\mathcal{L}_2^A = \frac{F_A}{2\sqrt{2}} \langle A_{\mu\nu} f_-^{\mu\nu} \rangle$$

$$\mathcal{L}_2^S = c_d \langle S\, u^\mu u^\nu \rangle + c_m \langle S\, \chi_+ \rangle$$

$$\mathcal{L}_2^P = i\, d_m \langle P\, \chi_- \rangle$$

$$u_\mu = i\, u^\dagger D_\mu U\, u^\dagger = u_\mu^\dagger \qquad ; \qquad U = u^2$$

$$f_\pm^{\mu\nu} = u\, F_L^{\mu\nu} u^\dagger \pm u^\dagger F_R^{\mu\nu} u \quad ; \quad \chi_\pm = u^\dagger \chi\, u^\dagger \pm u\, \chi^\dagger u$$



$$\left\{ \mathbf{V}\,[M_V, G_V, F_V]\;,\; \mathbf{A}\,[M_A, F_A] \right\} \longleftrightarrow \left\{ \mathbf{S}\,[M_S, c_d, c_m]\;,\; \mathbf{P}\,[M_P, d_m] \right\}$$

$\mathbf{O(N_C)}$:

$$2\,L_1 \;=\; L_2 \;=\; \sum_i \frac{G_{V_i}^2}{4\,M_{V_i}^2} \qquad ; \qquad L_3 \;=\; \sum_i \left\{ -\frac{3\,G_{V_i}^2}{4\,M_{V_i}^2} + \frac{c_{d_i}^2}{2\,M_{S_i}^2} \right\}$$

$$L_5 \;=\; \sum_i \frac{c_{d_i}\,c_{m_i}}{M_{S_i}^2} \qquad ; \qquad L_8 \;=\; \sum_i \left\{ \frac{c_{m_i}^2}{2\,M_{S_i}^2} - \frac{d_{m_i}^2}{2\,M_{P_i}^2} \right\}$$

$$L_9 \;=\; \sum_i \frac{F_{V_i}\,G_{V_i}}{2\,M_{V_i}^2} \qquad ; \qquad L_{10} \;=\; \frac{1}{4}\sum_i \left\{ \frac{F_{A_i}^2}{M_{A_i}^2} - \frac{F_{V_i}^2}{M_{V_i}^2} \right\}$$

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$O(1)$:

$$2\,L_1 - L_2 = L_4 = L_6 = 0 \quad ; \quad L_7 = -\frac{\tilde{d}_m^{\,2}}{2\,M_{\eta_1}^2}$$

O(N$_C$) :

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O(1) : $$2\,L_1 - L_2 \;=\; L_4 \;=\; L_6 \;=\; 0 \qquad ; \qquad L_7 \;=\; -\frac{\tilde{d}_m^{\,2}}{2\,M_{\eta_1}^2}$$

BUT $$M_{\eta_1}^2 \;\sim\; O\left(\frac{1}{N_C},\,\mathcal{M}\right)$$

SHORT–DISTANCE CONSTRAINTS

Vector Form Factor $\langle\pi|v_\mu|\pi\rangle$**:**

$$F_V(t) \,=\, 1 \,+\, \sum_i \frac{F_{V_i}\, G_{V_i}}{f^2}\, \frac{t}{M_{V_i}^2 - t}$$

$$\lim_{t\to\infty} F_V(t) = 0 \qquad \Longrightarrow \qquad \boxed{\sum\nolimits_i \, F_{V_i}\, G_{V_i} \,=\, f^2}$$

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Axial Form Factor $\langle\gamma|a_\mu|\pi\rangle$:
$$G_A(t) = \sum_i \left\{ \frac{2\, F_{V_i}\, G_{V_i} - F_{V_i}^2}{M_{V_i}^2} + \frac{F_{A_i}^2}{M_{A_i}^2 - t} \right\}$$

$$\lim_{t\to\infty} G_A(t) = 0 \quad \longrightarrow \quad \boxed{\sum_i \left(2\, F_{V_i}\, G_{V_i} - F_{V_i}^2\right) / M_{V_i}^2 = 0}$$

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Weinberg Sum Rules:

$$\Pi_{LR}(t) = -\frac{f^2}{t} + \sum_i \frac{F_{V_i}^2}{M_{V_i}^2 + t} - \sum_i \frac{F_{A_i}^2}{M_{A_i}^2 + t}$$

$$\lim_{t\to\infty} t\; \Pi_{LR}(t) = 0$$

$$\lim_{t\to\infty} t^2\, \Pi_{LR}(t) = 0$$

$$\longrightarrow \quad \boxed{\begin{array}{c} \sum_i \left(F_{V_i}^2 - F_{A_i}^2\right) = f^2 \\ \sum_i \left(M_{V_i}^2\, F_{V_i}^2 - M_{A_i}^2\, F_{A_i}^2\right) = 0 \end{array}}$$

Scalar FF: $$F^S_{K\pi}(s) = 1 + \sum_i \frac{4c_{m_i}}{f^2}\left[c_{d_i} + (c_{m_i} - c_{d_i})\,\frac{M_K^2 + M_\pi^2}{M_{S_i}^2}\right]\frac{s}{M_{S_i}^2 - s}$$

$$\lim_{s\to\infty} F^S_{K\pi}(s) = 0 \quad \Longrightarrow \quad \boxed{4\sum_i c_{d_i} c_{m_i} = f^2 \quad ; \quad \sum_i \frac{c_{m_i}}{M_{S_i}^2}\,(c_{m_i} - c_{d_i}) = 0}$$

Scalar FF:

$$F^S_{K\pi}(s) = 1 + \sum_i \frac{4c_{m_i}}{f^2}\left[c_{d_i} + (c_{m_i} - c_{d_i})\frac{M_K^2 + M_\pi^2}{M_{S_i}^2}\right]\frac{s}{M_{S_i}^2 - s}$$

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S − P Sum Rules:

$$\Pi_{SS-PP}(t) = 16 B_0^2 \left\{\sum_i \frac{c_{m_i}^2}{M_{S_i}^2 + t} - \sum_i \frac{d_{m_i}^2}{M_{P_i}^2 + t} - \frac{f^2}{8\,t}\right\}$$

$$\lim_{t\to\infty} t\ \Pi_{SS-PP}(t) = 0 \quad \longrightarrow \quad \boxed{8\sum_i \left(c_{m_i}^2 - d_{m_i}^2\right) = f^2}$$

Scalar FF:

$$F^S_{K\pi}(s) = 1 + \sum_i \frac{4c_{m_i}}{f^2}\left[c_{d_i} + (c_{m_i} - c_{d_i})\,\frac{M_K^2 + M_\pi^2}{M_{S_i}^2}\right]\frac{s}{M_{S_i}^2 - s}$$

$$\lim_{s\to\infty} F^S_{K\pi}(s) = 0 \quad \longrightarrow \quad \boxed{4\sum_i c_{d_i} c_{m_i} = f^2 \quad ; \quad \sum_i \frac{c_{m_i}}{M_{S_i}^2}\,(c_{m_i} - c_{d_i}) = 0}$$

S − P Sum Rules:

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$$\lim_{t\to\infty} t\;\Pi_{SS-PP}(t) = 0 \quad \longrightarrow \quad \boxed{8\sum_i (c_{m_i}^2 - d_{m_i}^2) = f^2}$$

Pseudoscalar Nonet:

$$\mathcal{L}_2 \doteq \frac{f^2}{4}\,\langle \chi\, U^\dagger + U\,\chi^\dagger\rangle \;\sim\; -i\,\frac{f}{\sqrt{24}}\,\eta_1\,\langle\chi_-\rangle \quad \longrightarrow \quad \boxed{\tilde{d}_m = -\frac{f}{\sqrt{24}}}$$

1–Resonance Approximation: (Ecker, Gasser, Leutwyler, Pich, de Rafael)

$$F_V = 2\,G_V = \sqrt{2}\,F_A = \sqrt{2}\,f \quad ; \quad M_A = \sqrt{2}\,M_V \quad ; \quad d_m = \frac{1}{2\sqrt{2}}\,f$$

$$c_m = c_d = \frac{1}{2}\,f$$ (Jamin, Oller, Pich)

$$M_P \approx \sqrt{2}\,M_S$$

$$\longrightarrow \quad \boxed{\begin{array}{c} \mathbf{2\,L_1 = L_2 = \frac{1}{4}\,L_9 = -\frac{1}{3}\,L_{10} = \frac{f^2}{8\,M_V^2}} \\[2ex] \mathbf{L_3 = -\frac{3\,f^2}{8\,M_V^2} + \frac{f^2}{8\,M_S^2} \quad ; \quad L_5 = \frac{f^2}{4\,M_S^2}} \\[2ex] \mathbf{L_8 = \frac{f^2}{8\,M_S^2} - \frac{f^2}{16\,M_P^2} \quad ; \quad L_7 = -\frac{f^2}{48\,M_{\eta_1}^2}} \end{array}}$$

L_i'S FROM RESONANCE EXCHANGE

i	$10^3 \cdot L_i^r(M_\rho)$	V	A	S	η_1	Total	Total[b)]
1	0.4 ± 0.3	0.6	0	0	0	0.6	0.9
2	1.4 ± 0.3	1.2	0	0	0	1.2	1.8
3	-3.5 ± 1.1	-3.6	0	0.6	0	-3.0	-4.3
4	-0.3 ± 0.5	0	0	0	0	0.0	0.0
5	1.4 ± 0.5	0	0	1.4[a)]	0	1.4	2.1
6	-0.2 ± 0.3	0	0	0	0	0.0	0.0
7	-0.4 ± 0.2	0	0	0	-0.3	-0.3	-0.3
8	0.9 ± 0.3	0	0	0.9[a)]	0	0.9	0.8
9	6.9 ± 0.7	6.9[a)]	0	0	0	6.9	7.2
10	-5.5 ± 0.7	-10.0	4.0	0	0	-6.0	-5.4

[a)] Input

[b)] Short-Distance Constraints

Recent Work

- **Three–Point Functions:**

  Moussallam, Knecht–Nyffeler, Ruiz-Femenía–Pich–Portolés, Cirigliano–Ecker–Eidemüller–Pich–Portolés, Bijnens–Gámiz–Lipartia–Prades

  → **Constraints on $\mathcal{O}(p^6)$ χPT couplings**

- **Vertices with 2 or 3 Resonances. $\mathcal{O}(p^4)$ Couplings:**

  Cirigliano–Ecker–Eidemüller–Kaiser–Pich–Portolés

- **More Resonance Multiplets:** Minimal Hadronic Ansatz

  Knecht–Peris–Perrottet–Phily–de Rafael, Ruiz-Femenía–Portolés

- **$1/N_C$ Corrections:**

  - **Resonance Widths** Guerrero–Pich, Gómez-Dumm–Pich–Portolés
  - **Unitarity Corrections** Final State Interactions

    Guerrero–Pich, Pallante–Pich–Scimemi, Cillero–Pich–Portolés, Jamin–Oller–Pich, Oller et al., . . .
  - **Quantum Loops in RχT** Cillero–Rosell–Pich, Portolés–Rosell–Ruiz-Femenía

SUMMARY

- **Chiral Symmetry** constraints the low-energy dynamics
- Useful tool for quantitative non-perturbative analyses at low energies
- Many applications:
  - χPT, RχT, Baryons, HMχPT, ...
  - Isospin breaking, electromagnetism, weak decays ...
  - (Partially) Quenched χPT, finite size effects, ... (lattice)
- Loops (unitarity) unambiguously predicted: **Chiral logarithms**
- QCD dynamics encoded in the **low-energy chiral couplings**
- The $\mathbf{N_C} \to \infty$ limit provides good estimates of the **LECs**

Challenge: **Rigorous control of subleading $1/\mathrm{N_C}$ corrections**