LATTICE QCD AND FLAVOUR PHYSICS

LNF Spring School "Bruno Touschek"

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INFN - TOR VERGATA

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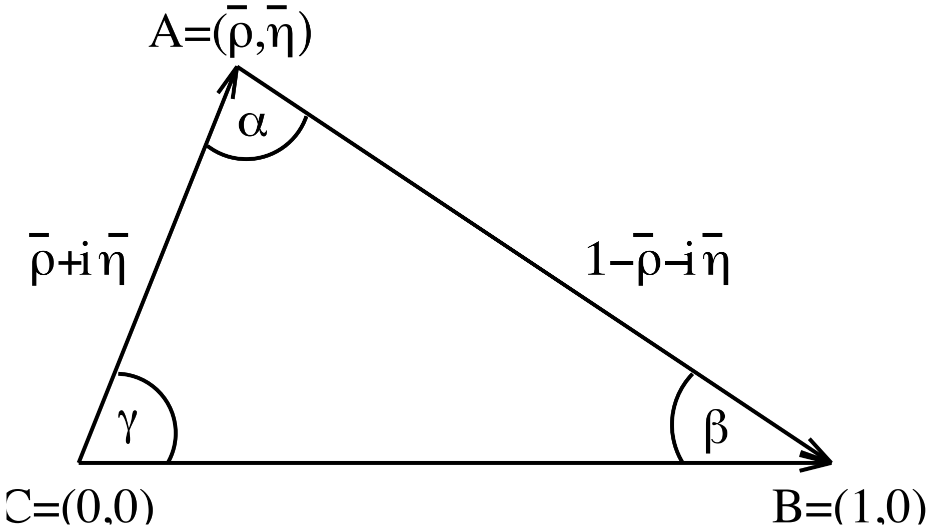
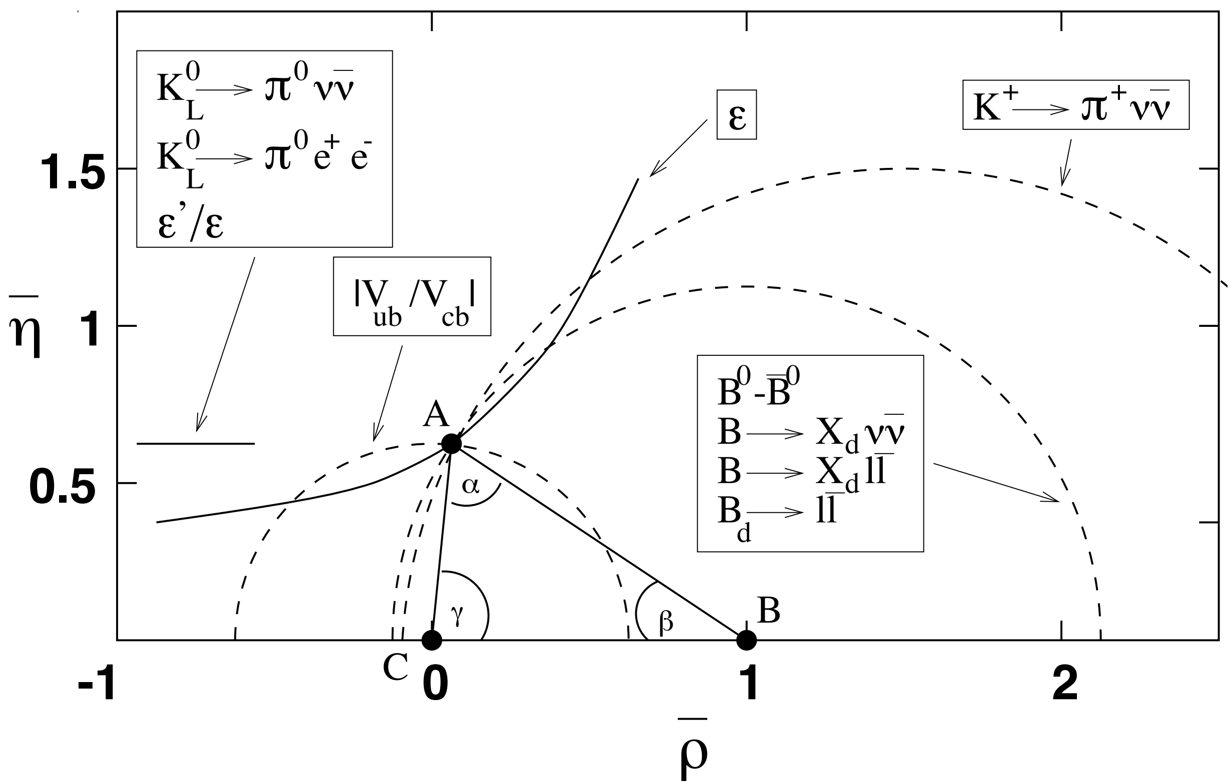
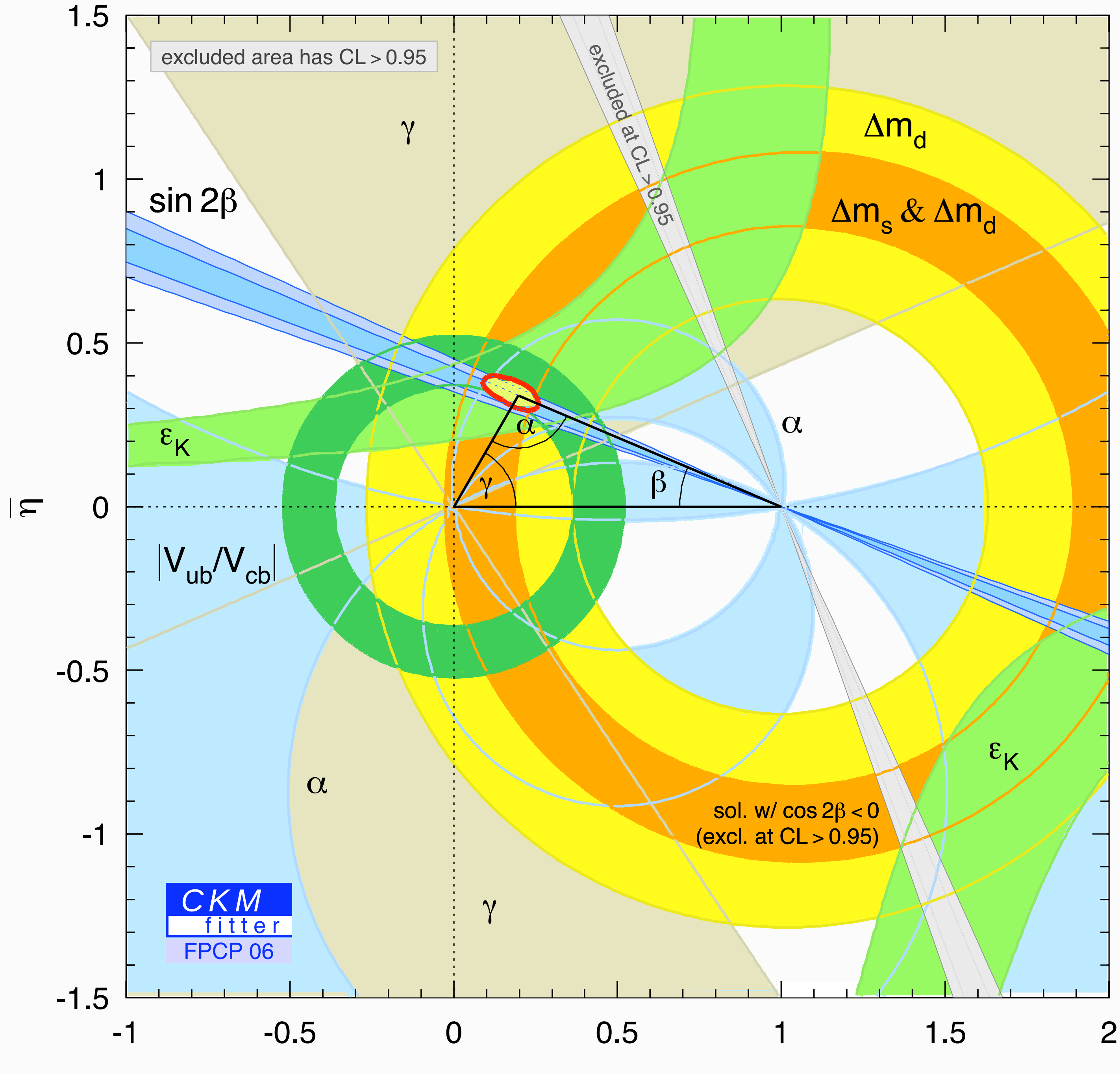
Prelude

- QCD: a central issue in the Standard Model (SM)
- Strongly interacting matter comes in 3 generations of quarks & leptons
- Weak interaction asymmetries: *CP*-violation under intensive study
- Test subtler properties of SM
- Hope to see signatures of Physics beyond SM
- Experiments (strange sector): CERN, FNAL, ...
- Experiments (bottom sector): CERN, DESY, FNAL, KEK, ...
- Experiments (charm sector): Frascati, FNAL, KEK
- Theory: Dortmund, Dubna, Lund, Montpelier, Munich, Rome, Taipei, Trieste, Valencia, ...
- Main difficulty: control of strong interaction effects at low energies (non-perturbative QCD)



- several processes to check UT
- "Gold plated" decay $B_d \to J/\Psi + K_s$ gives sin(β) [Belle - BaBar]
- ϵ-hyperbola: $K^0 - \bar{K}^0 (\Delta S = 2)$
- AB-side (Δ M_d): $B_d^0 - \bar{B}_d^0 (\Delta B = 2)$
- ϵ' / ϵ





excluded area has CL > 0.95
excluded at CL > 0.95
γ
Δm_d
sin 2β
Δm_s & Δm_d
α
ε_K
α
γ
β
|V_ub/V_cb|
ε_K
sol. w/ cos 2β < 0
(excl. at CL > 0.95)
α
γ
CKM fitter
FPCP 06
1.5
1
0.5
0
-0.5
-1
-1.5
-1
-0.5
0
0.5
1
1.5
2
η̄
ρ̄

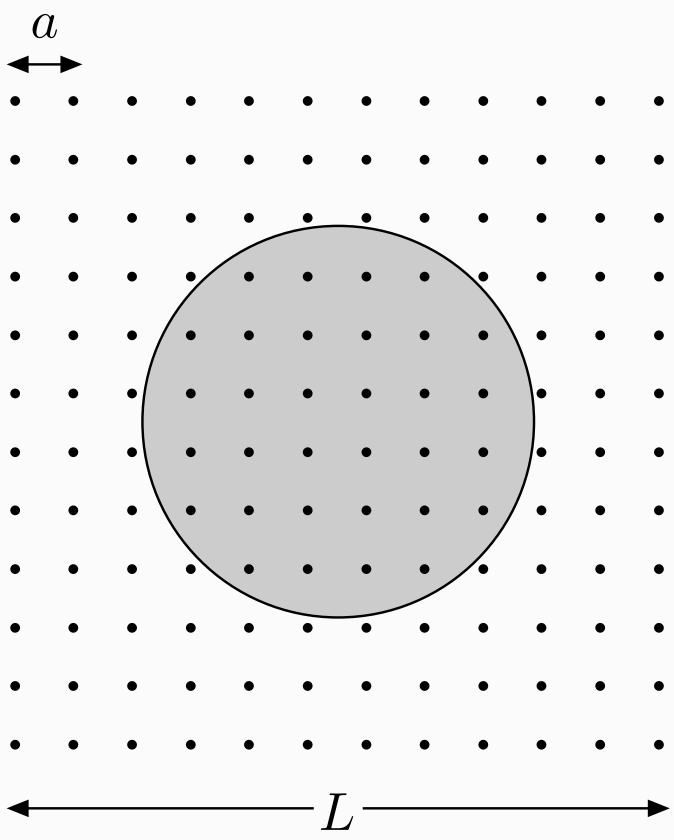
Lattice basics

Lattice themes

- discretization of spacetime and QCD length scales
- hadron masses and WME from the lattice
- lattice actions, fermion doubling
- renormalization & improvement
  - non-perturbative renormalization
  - RG-running and step scaling function
- heavy flavours on the lattice
  - HQET, NRQCD
  - more step scaling functions

Lattice basics

- Regularize QCD by discretizing space-time:
  - hypercube with lattice spacing *a* (UV cutoff) ...
  - ... and linear extension *L* (IR cutoff)
- PI is now well-defined for bare theory and can be computed; we can do experimental QCD at finite UV cutoff



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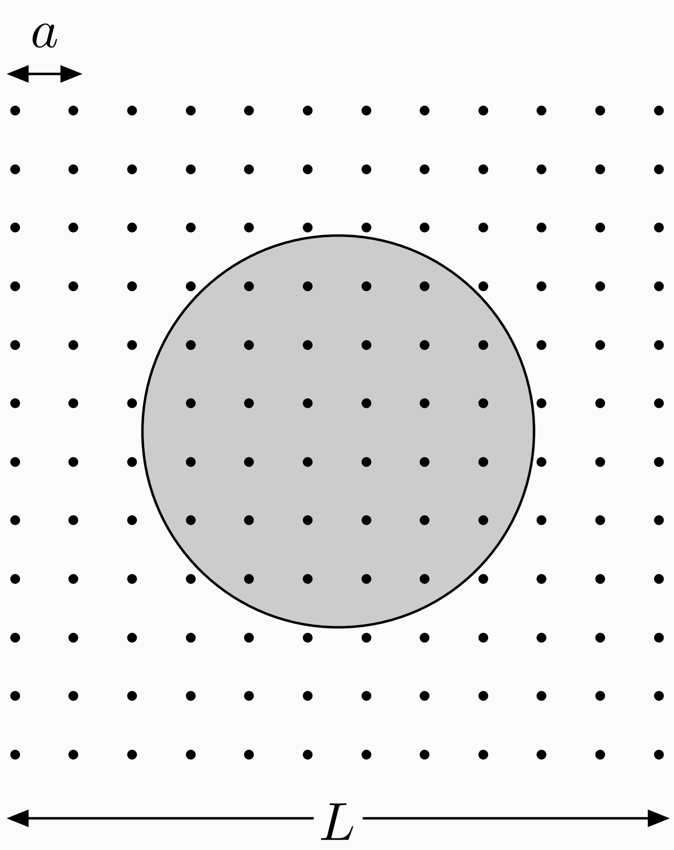


- scales (e.g. hadron masses) must satisfy

$$L^{-1} << m_H << a^{-1}$$

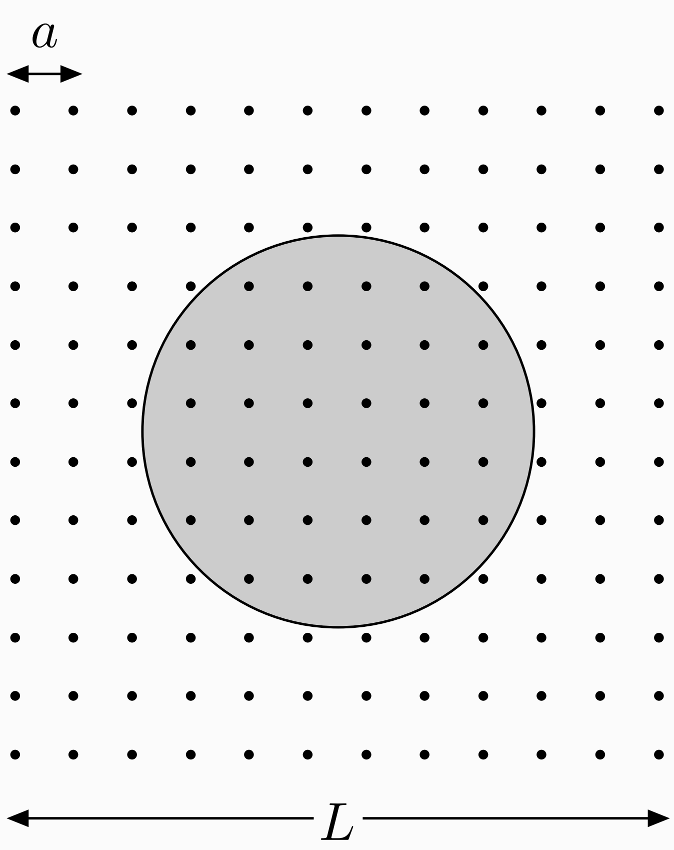
- must also ensure

$$\Lambda_{QCD} << a^{-1}$$

- [N.B. $\Lambda_{QCD} \sim 300$ *MeV*]

Practical difficulties

- present day computers can tackle $a \sim 0.04\ fm$ and $L \sim 2\ fm$; i.e. $L/a \sim 50$ lattice sites
- we have $O(50^4)$ degrees of freedom
- $a^{-1} \sim 5\ GeV$ and $L^{-1} \sim 100\ MeV$
- OK for strange and charm mesons



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- we have $O(50^4)$ degrees of freedom
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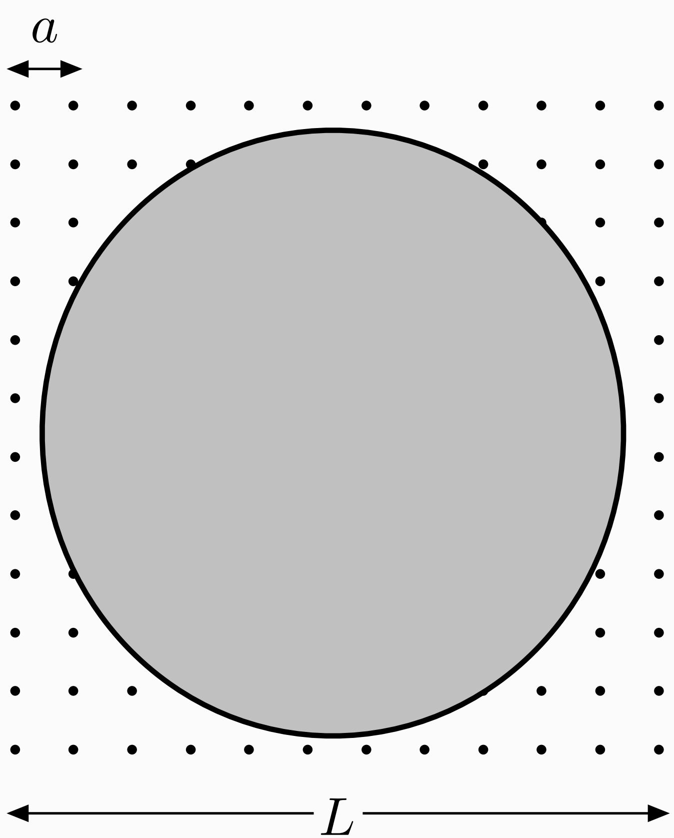
- "Goldstone" mesons $m_\pi \sim 150\ MeV$ afflicted by finite volume effects



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$$L^{-1} \ll m_H \ll a^{-1}$$

- compute in range $m_s/8 < m_q < m_s/2$ and extrapolate to light quark values
- use functional form suggested by χPT in the extrapolation
- ensure $m_H L > 4$

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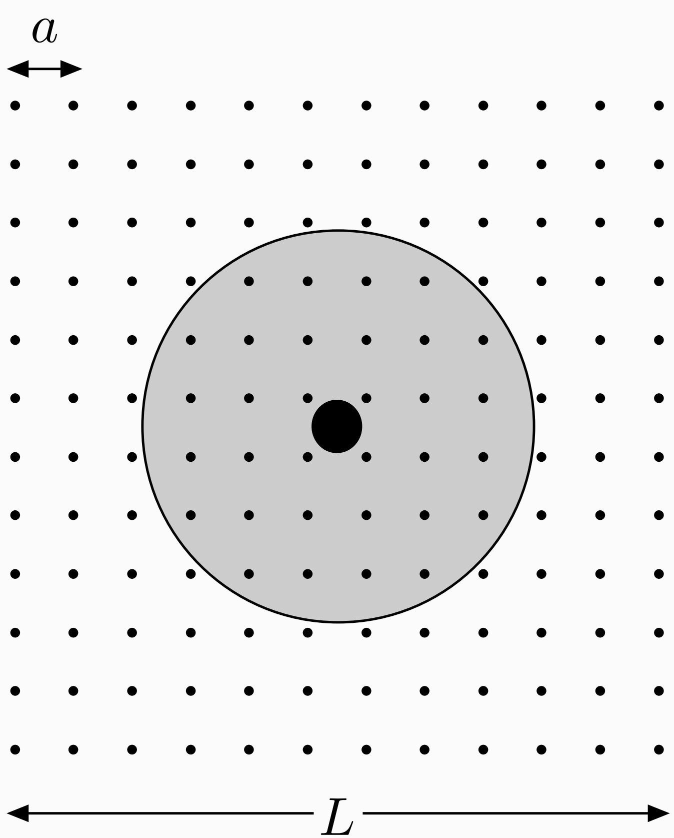
- we have $O(50^4)$ degrees of freedom
- $a^{-1} \sim 5\ GeV$ and $L^{-1} \sim 100\ MeV$
- heavy mesons $m_B \sim 5\ GeV$ afflicted by finite size effects



- scales (e.g. hadron masses) must satisfy

$$L^{-1} \ll m_H \ll a^{-1}$$

- compute in range $m_c < m_q < 1.5\ m_c$ and extrapolate to bottom quark values
- using results suggested by HQET or NRQCD interpolate charm up to bottom region

Lattice correlation functions

- in the lattice PI framework, we compute **bare** correlation functions of the form:

$$< 0|\ Q(x_1, \cdots, x_n)\ |0 > = \ \frac{1}{Z} \int \mathcal{D}A_\mu \mathcal{D}\bar{\psi}\mathcal{D}\psi\ \ \exp[-S^{latt}]\ Q(x_1, \cdots, x_n)$$

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- the formalism is set up in Euclidean space-time; i.e. $i\,S \rightarrow -\,S^{latt}$
- this ensures real & bounded exponential factor
- correlation function can be computed numerically (Monte Carlo weighted averages)
- use $\exp[-\,S^{latt}\,]$ as probability weight to generate a configuration ensemble
- compute observable on this ensemble
- process characterized by **statistical error**; this is the least source of worry
- "easily" controlled by increasing configuration ensemble N_{conf} (NB: $\epsilon \sim 1\,/\,\sqrt{N_{conf}}$)

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- compute observable on this ensemble
- how does this work with Grassmann (fermionic) variables?

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- lattice (bare QCD) action in general has the form:

$$S^{latt} \ = \ a^4 \sum \left\{ [F_{\mu\nu}F_{\mu\nu}]^{latt} + \bar{\psi}[\not{D}^{latt} + m]\psi \right\}$$

- integrate Grassmann degrees of freedom:

$$< 0|\ Q(x_1, \cdots, x_n)\ |0> = \frac{1}{Z} \int \mathcal{D}A_\mu\ \ \exp[-S^{glue}]\ \ \det[\not{D}^{latt} + m]\ \tilde{Q}(x_1, \cdots, x_n)$$

- the non-local determinant is the costly part

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- the non-local determinant corresponds to internal fermion loops (sea quarks)

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- popular shortcut id to set $\det[D^{latt}+m]=1$; i.e. sea quarks are infinitely heavy.
- This is the QUENCHED APPROXIMATION which has been (and still is) a principal source of **uncontrolled** errors

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- we are currently at the end of the quenched era, in the middle of *N_f=2 and N_f=2+1*, aiming at *N_f=2+1+1*

From correlation functions to hadron masses and matrix elements

- How do we obtain matrix elements and hadronic masses (i.e. **bare** low energy quantities)?
- Consider the lattice correlation function:

$$
\begin{aligned}
C_Q(t) &= \sum_{\vec{x}} < 0|\; Q(x)\; Q(0)\; |0 > \\
&\sim \sum_{s} < 0|\; Q(0)\; |s > < s|\; Q(0)\; |0 > \exp[-m_s t] \\
&\to \;|< 0|\; Q(0)\; |G > |^2 \exp[-m_G t] \;+\; \cdots
\end{aligned}
$$

- the states $|s>$ are those with the quantum numbers of $Q(x)$
- m_s are the corresponding hadronic masses; m_G the ground state
- $< 0 \,|\, Q \,|\, G >$ is the vacuum-to-G **bare** WME of operator Q
- higher excited states (same quantum numbers) drop out in the large-t limit

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- example: the operator *Q* is the charged axial current $\quad Q \;\rightarrow\; A_0 \;=\; \bar{u}\gamma_0\gamma_5 d$
- the state |*G*> is the charged pion; $m_G \rightarrow m_\pi$
- the matrix element defines the pion decay contant $\quad < \; 0|\; A_0\; |\,\pi\; >= \; f_\pi\; m_\pi$

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- masses and matrix elements are computed from first principles in a model independent way
- the computation is clean in principle, but systematic errors abound (see later)

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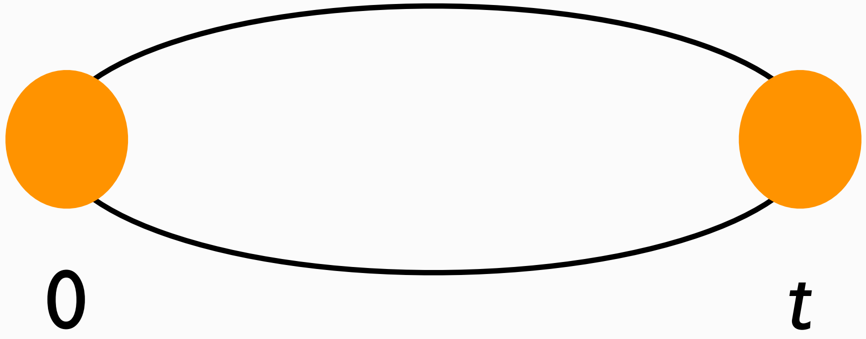
NB: gluon and sea quarks not drawn



From correlation functions to hadron masses and matrix elements

- More general WMEs are obtained from more complicated correlation functions:

$$
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C(t_x, t_y) \quad &= \quad \sum_{\vec{x}\vec{y}} < 0|\ H_2(y)\ Q(0)\ H_1(x)\ |0> \\
&\sim \quad \exp[-m_2 t_y]\ \exp[m_1 t_x] \\
&\times \quad < 0|\ H_2(0)\ |H_2> \ < H_2|\ Q(0)\ |H_1> \ < H_1|\ H_1(0)\ |0>
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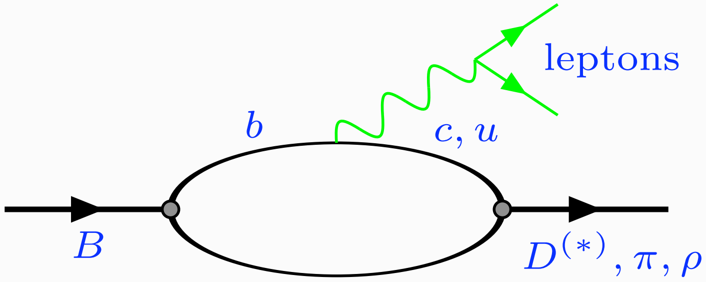
$$\times \quad < 0|\ H_2(0)\ |H_2 > \ < H_2|\ Q(0)\ |H_1 > \ < H_1|\ H_1(0)\ |0 >$$

Example1:

B→D+l+ν decays have WME <D | Jμ | B >

$$Q \ \equiv J_\mu = \ \bar{b}\gamma_\mu c$$
$$H_1 \ \equiv P_B = \ \bar{d}\gamma_5 b$$
$$H_2 \ \equiv P_D = \ \bar{c}\gamma_5 d$$



$\Rightarrow \quad V_{cb}\ ,\ V_{ub}$

Renormalization and improvement

- the lattice formalism is a **bare** QFT
- computation results are **bare** WMEs at fixed UV cutoff $1/a$ (i.e. fixed $g_0^2(a)$)
- must renormalize them in order to get continuum physics
- UV cutoff is present at all stages of a computation and it is gradually increased

$$< f|\ Q_R(\mu)\ |i> = \lim_{a\to 0}\left[\ Z_Q(a\mu, g_0^2)\ \ < f|\ Q(g_0^2)\ |i>\ \ + \mathcal{O}(a)\ \right]$$

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renormalized WME depends on dressed coupling, masses and scale

bare WME depends on bare coupling and masses

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continuum limit obtained gradually by successive simulations

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bare WME depends on bare coupling and masses

discretization effects due to cutoff finiteness contaminate all computations

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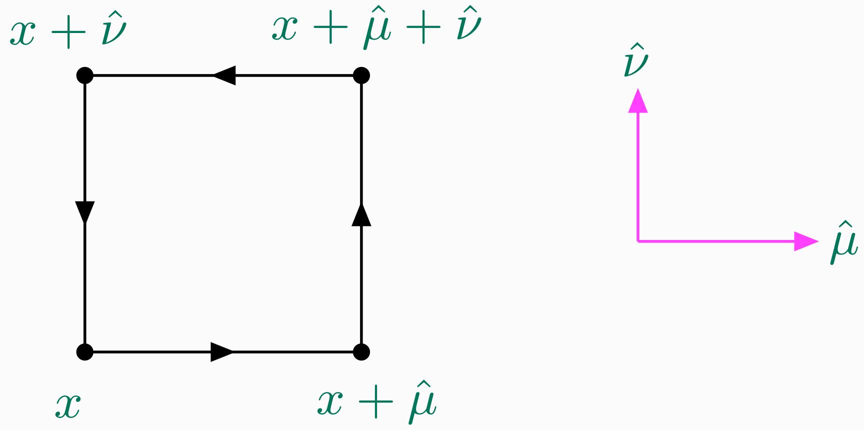
$$< f|\ Q_R(\mu)\ |i> = \lim_{a\rightarrow 0} \Big[\ Z_Q(a\mu, g_0^2)\ \ < f|\ Q(g_0^2)\ |i>\ \ + \mathcal{O}(a)\ \Big]$$

- lattice renormalization can be done either in PT or non-perturbatively (NP)
- lattice PT is tedious and **badly convergent**; at say LO, it introduces large $\mathcal{O}(g_0^4)$ errors in Z_Q
- NP methods introduce $\mathcal{O}(a)$ **discretization errors** is Z_Q ; as also the bare WME has $\mathcal{O}(a)$ effects, this is preferable to PT
- better still: attempt to "help" continuum extrapolation by reducing all discretization errors to $\mathcal{O}(a^2)$ [Symanzik improvement; see later]

Lattice actions

Lattice regularization: gluons

- write a lattice action which reduces to bare QCD when $a \to 0$
- **naive** continuum limit: bare quantities g_0, m_q are kept fixed
- **true** continuum limit: physical quantities g_R, m_H are kept fixed
- arbitrariness in choice of action; it should give QCD in **naive** cont. limit
- naive discretization of gluonic action is not gauge invariant
- Wilson: degree of freedom is the link variable $U_\mu(x) \in SU(3)$



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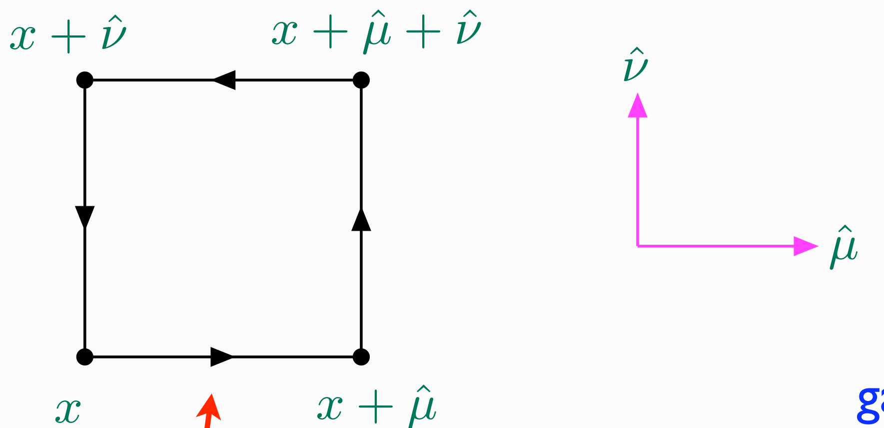


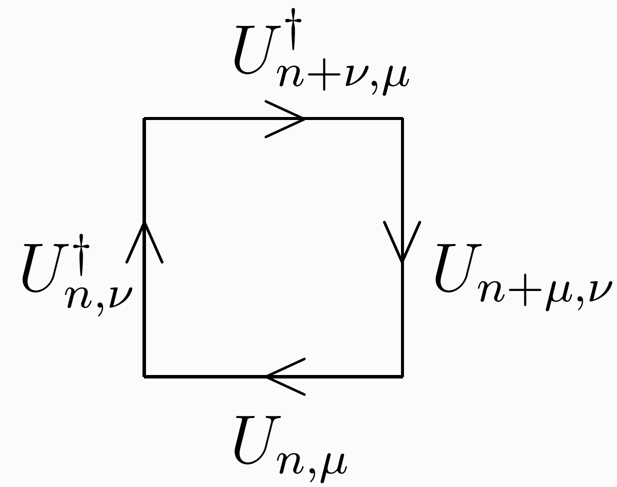
gauge field (gluon)

$$U_\mu(x) = \exp[\, i\, g_0\, a\, A_\mu(x)\,] \approx 1 + i\, g_0\, a\, A_\mu(x) + \ldots$$

- basic gauge invariant element is the trace of the plaquette

$$P_{\mu\nu}(x) = U_\mu(x)\, U_\nu(x+\mu)\, U_\mu(x+\mu+\nu)^\dagger\, U_\nu(x+\mu)^\dagger \quad \in SU(3)$$

$$P_{\mu\nu}(x)^\dagger = I - i\, g_0^2\, a^2\, F_{\mu\nu}(x)^2 + \mathcal{O}(a^2)$$



- Wilson action:

$$\frac{6}{g_0^2} \sum_P \mathrm{Re}\,\mathrm{Tr}\, P_{\mu\nu} \;\sim\; a^4 \sum_x F_{\mu\nu}(x)\, F_{\mu\nu}(x) \;+\; \mathcal{O}(a^2)$$

- N.B. higher order discretization effects

- gauge invariance is maintained

- Lorenz symmetry reduced to hypercubic rotations by $\pi/2$ and translations by a

- generally, it is not the only symmetry lost on the lattice (chiral, SUSY); a central issue is symmetry recovery in the true continuum limit

- recovery of Lorentz symmetry appears to be straightforward while recovery of **chiral** symmetry is intricate

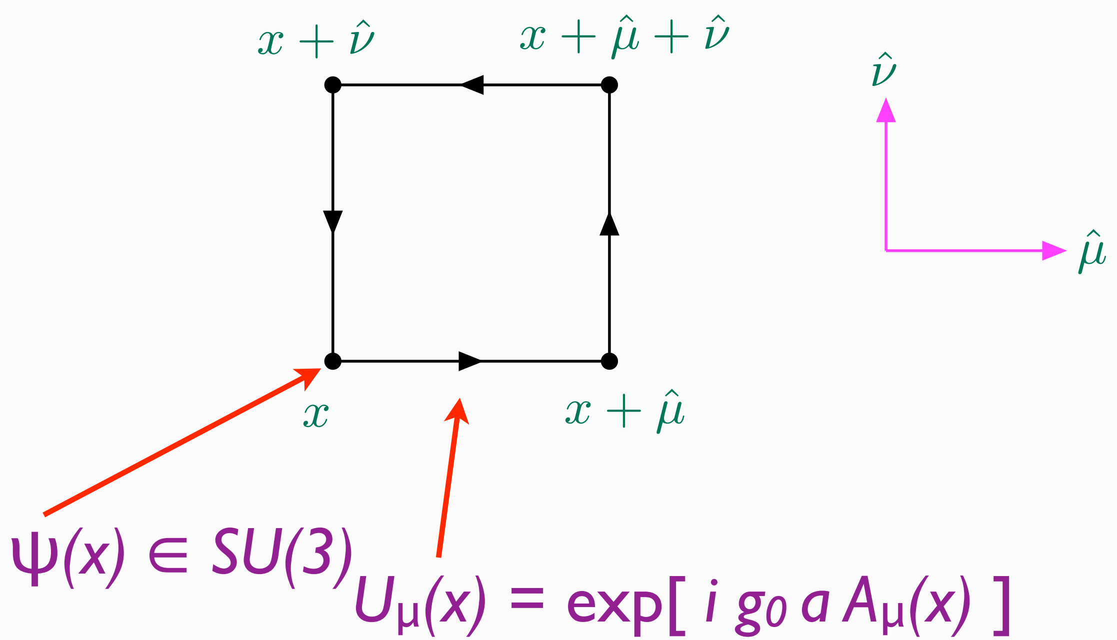
Lattice regularization: naive fermions

- write a lattice action which reduces to bare QCD when $a \to 0$
- naive discretization of fermionic derivative (free quarks)

$$\partial_\mu^{latt} \psi(x) \; = \; \frac{1}{2}\left[\psi(x+\mu) - \psi(x-\mu)\right]$$

- naive fermionic Euclidean action

$$S^{ferm} \; = \; a^4 \sum \bar{\psi}(x) \left[\gamma_\mu \partial_\mu^{latt} \; + \; M\right] \psi(x)$$



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- naive free fermion propagator (limited in first Brillouin zone-BZ): $-\pi/a < p_\mu \leq \pi/a$

$$S(p;M) \;=\; \frac{\left[-\sum_\mu \gamma_\mu \sin(ap_\mu)/a\right] \;+\; M}{\left[\sum_\mu \sin(ap_\mu)^2/a^2\right] \;+\; M^2}$$

- "correct" naive continuum limit

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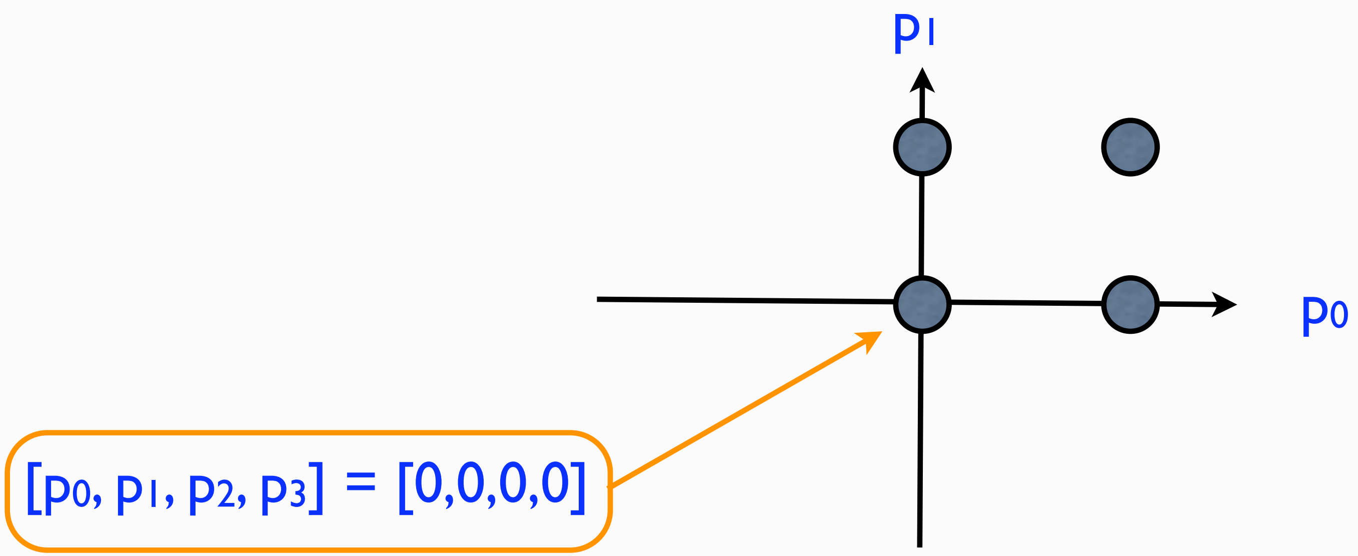
- wrong true continuum limit due to fermion doublers (i.e. 16 poles in the first BZ)



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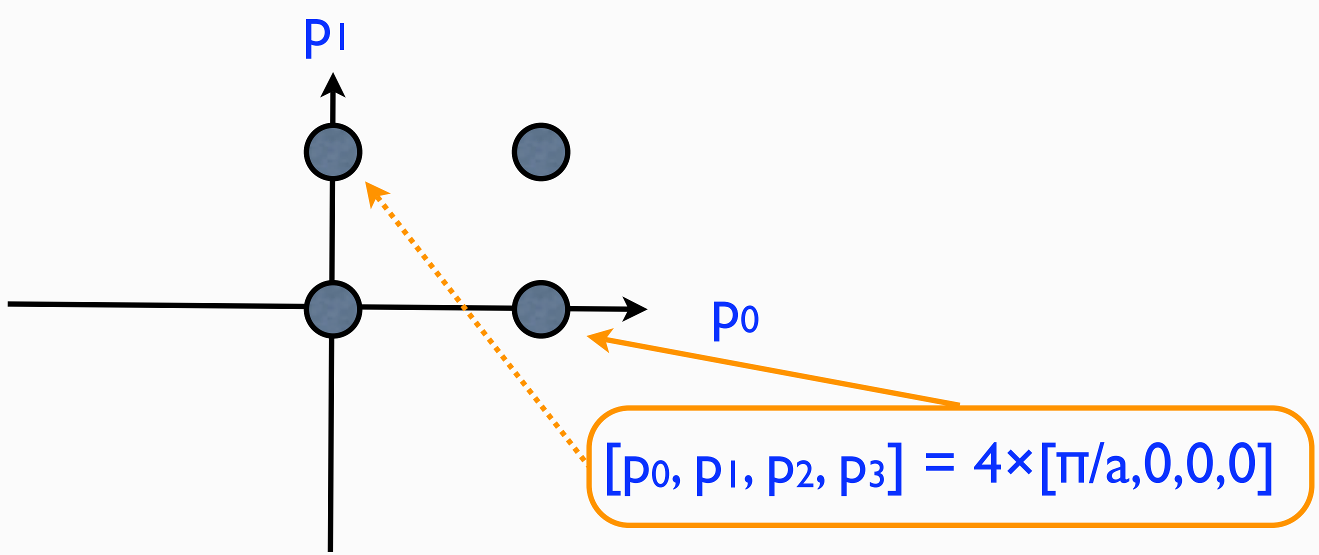

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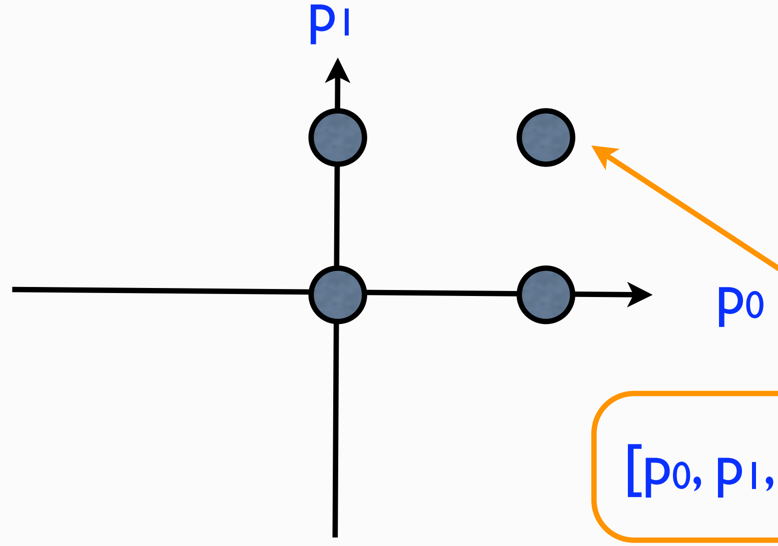

$[p_0, p_1, p_2, p_3] = 6\times[\pi/a,\pi/a,0,0]$

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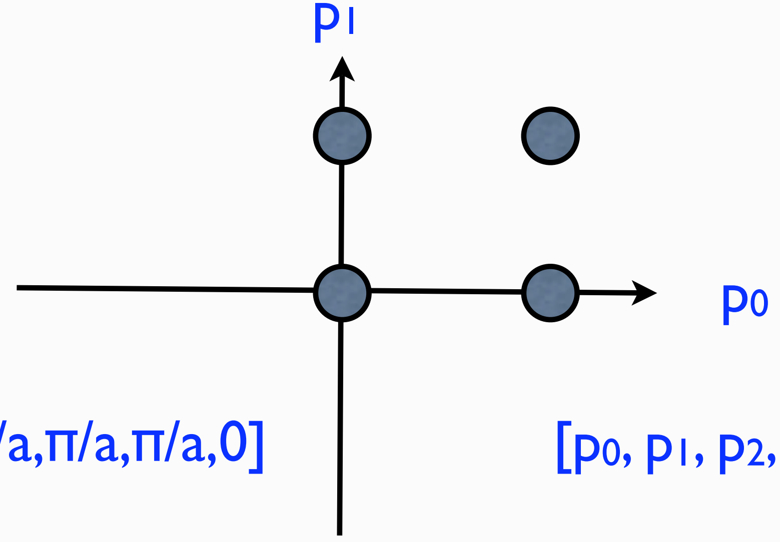

$[p_0, p_1, p_2, p_3] = 4\times[\pi/a, \pi/a, \pi/a, 0]$

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Fermion "doubling"

- the problem is general: for any lattice fermion action (free massless case)

$$S^{ferm} = a^4 \sum_{x,y} \bar{\psi}(x)\ D(x-y)\ \psi(y)$$

- the lattice Dirac operator should satisfy:
  - Locality: $D(x\text{-}y) < C \exp[- \kappa\ |x\text{-}y|\ ]$
  - Continuum limit: $D(p) = \gamma_\mu\ p_\mu + \mathcal{O}(a\ p^2)$
  - No doublers: $D(p)$ invertible for $p_\mu \neq 0$
  - chiral symmetry: $D(x)\ \gamma_5 + \gamma_5\ D(x) = 0$
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- Wilson fermions: introduce irrelevant (D=5) operator in the action, which breaks chiral symmetry, recovered in the **true** continuum limit.

Wilson fermions

- Add a chiral breaking irrelevant (D=5) term to the (free) action (Wilson term):

$$D(x-y) \rightarrow \gamma_\mu \hat{\partial}_\mu - \frac{1}{2} a \hat{\partial}_\mu \hat{\partial}_\mu + m_0$$

- ultra-local operator, fairly cheap to compute
- flavour is as in the continuum
- the origin of Brillouin zone [0,0,0,0] corresponds to physical fermion
- the other 15 corners of Brillouin zone correspond to fermions of mass ~1/a
- Disadvantage: chiral symmetry lost!!! Must be recovered in the **true** continuum limit
- Consequence: renormalization gets much more complicated and χal limit hard

fermions with chirality OK

$$m_q = Z_m(g_0^2, a\mu)\, m_0$$

$$[\, A_\mu \,]_R = [\, A_\mu \,]_0$$

Wilson fermions

$$m_q = Z_m(g_0^2, a\mu) \left[m_0 - \frac{f(g_0^2)}{a} \right]$$

$$[\, A_\mu \,]_R = Z_A(g_0^2)\, [\, A_\mu \,]_0$$

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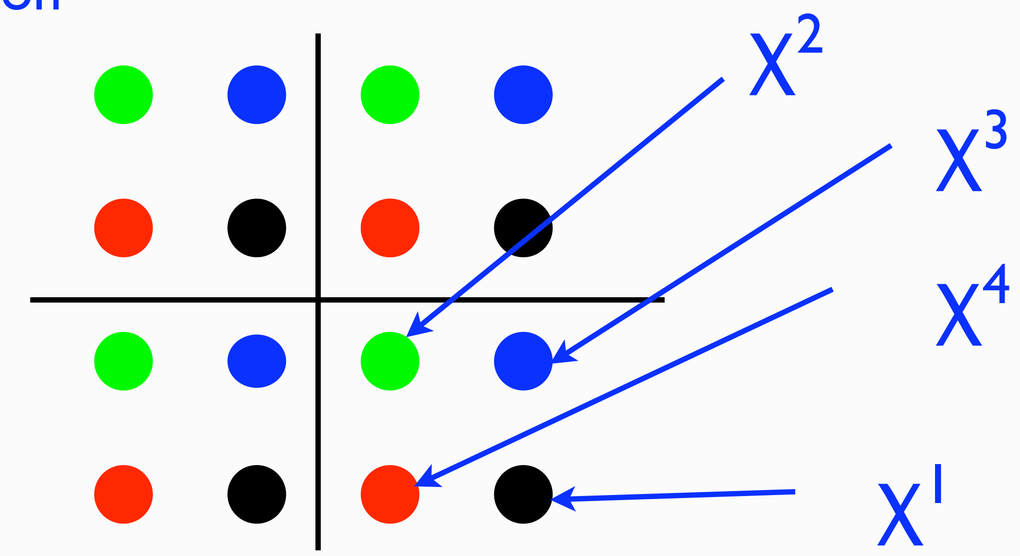
  - chiral symmetry: $D(x)\, \gamma_5 + \gamma_5\, D(x) = 0$
- Nielsen-Ninomyia theorem: all 4 properties cannot be satisfied simultaneously
- staggered fermions: dilute 16 spinorial degrees of freedom on hypercube points. Retain a reduced U(1) chiral symmetry. Loose "flavour transparency"

Staggered fermions

- distribute 16 spinorial degrees of freedom on each hypercube vertex

$$D(x - y) \rightarrow \eta_\mu(x)\,\hat{\partial}_\mu + m_0$$

- $\eta_\mu(x) = \pm 1$ is an (even-odd) site dependent sign
- ultra-local operator, very cheap to compute
- 4 physical fermions (4-spinors) constructed from staggered single spinors & $\eta_\mu(x)$
- theory describes 4 degenerate flavours, or 1 flavour + 3 tastes !!
- chiral symmetry is reduced to the U(1) group; adequate in many cases for straightforward renormalization



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- the fermion determinant $\det[D^{latt}+m]$ for a single staggered field describes 1 flavour + 3 tastes; to get the physical reality (i.e. non-degenerate flavours) people take its fourth root. Do you lose locality? Probably not, but the discussion is intricate and on-going; see Sharpe, PoS(LAT2006)022

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  - chiral symmetry: $D(x)\,\gamma_5 + \gamma_5\, D(x) = O(a)$
- Nielsen-Ninomyia theorem: all 4 properties cannot be satisfied simultaneously
- Ginsparg-Wilson fermions: break chirality mildly to $\mathcal{O}(a)$, give up strict locality; costly in practice. Known as overlap fermions

Lattice setup: GW (chiral) fermions

Ginsparg-Wilson fermions:

$$\gamma_5 D + D\gamma_5 = \bar{a}\, D\gamma_5 D\,, \qquad \bar{a} = \frac{a}{1+s}$$

Ginsparg, Wilson 1982

Kaplan; Neuberger; Hasenfratz, Laliena, Niedermayer; ...

Our choice: Neuberger-Dirac operator.

$$D_{\mathrm{N}} = \frac{1}{\bar{a}} \left\{ 1 - \frac{A}{(A^{\dagger}A)^{1/2}} \right\}, \qquad A = 1 - aD_{\mathrm{w}}$$

Neuberger 1997

Numerical treatment challenging and expensive.

Giusti, Hoelbling, Lüscher, Wittig 2002

Lattice setup: GW (chiral) fermions

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Kaplan; Hasenfratz, Laliena, Niedermayer; Neuberger; ...

Lattice QCD action enjoys an <u>exact chiral symmetry</u>:

$$\delta\psi = i\epsilon\hat{\gamma}_5\psi\,, \quad \hat{\gamma}_5 = \gamma_5(\mathbf{1} - \bar{a}\,D)$$

$$\delta\bar{\psi} = i\epsilon\bar{\psi}\gamma_5$$

Lüscher 1998

Renormalisation and mixing patterns as in the formal continuum theory, provided:

$$\psi \;\rightarrow\; \tilde{\psi} = (\mathbf{1} - \tfrac{1}{2}\bar{a}\,D)\,\psi\,, \qquad \bar{\psi} \;\rightarrow\; \bar{\psi}$$

In particular, there is no dangerous mixing with lower dim. operators.

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- Nielsen-Ninomyia theorem: all 4 properties cannot be satisfied simultaneously
- Domain wall fermions: An equivalent formulation to GW fermions: introduce a fifth dimension; the 4-D lattice is a hypersurface (a defect) where both chiralities merge. Fairly costly (computationally); chirality is recovered at infinitely large DW

O(a) Symanzik improvement

- systematic way to improve approach to continuum limit by eliminating $\mathcal{O}(a)$ effects

$$< f\,|\,Q_R(\mu)\,|\,i\ > = \lim_{a\to 0}\ \left[Z_Q(a\mu, g_0^2)\ < f\,|\,Q(g_0^2)\,|\,i\ > \ + \mathcal{O}(a^2)\ \right]$$

- the gauge action is already free of $\mathcal{O}(a)$ effects
- same is true for staggered fermion action, but $\mathcal{O}(a^2)$ effects are very big
  - they can be reduced by modifying the action in RG-inspired ways
- also GW fermions start from $\mathcal{O}(a^2)$ discretization errors
- Wilson fermions suffer from $\mathcal{O}(a)$ effects
- this is a big disadvantage

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- modify Wilson fermion action by adding counterterms of the form [$a \times Q_{\mathrm{dim}=5}$]

- one such counterterm is adequate for the improvement of physical (“on-shell) quantities (hadron masses)

$$\mathcal{L}^{\mathrm{Wilson}} \;\to\; \mathcal{L}^{\mathrm{Wilson}} \;+\; a\; c_{SW}(g_0^2)\;\left[\bar{\psi}\sigma_{\mu\nu}F_{\mu\nu}\psi\right]$$

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Clover term

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$$A_\mu \rightarrow \left[1 + b_A(g_0^2)\, am_q\right]\left[A_\mu + ac_A(g_0^2)\,\partial_\mu P\right]$$

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$\boxed{\text{adequate in the chiral limit}}$

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Twistəd mass QCD

ALPHA Frezzotti, Grassi, Sint & Weisz, JHEP08(2001)058

- Wilson fermions appear to suffer from contorted renormalizations and $\mathcal{O}(a)$ effects

- one can cure everything in one go: tmQCD

$$D(x-y) \;\rightarrow\; \gamma_\mu \hat{\partial}_\mu \;-\; \frac{1}{2} a \hat{\partial}_\mu \hat{\partial}_\mu \,+\, m_0 \,+\, i\gamma_5 \tau_3 \mu_0$$

Break flavour symmetry in non-trivial direction in flavour space → preserve different subgroup.
No free lunch: break P,T, flavour symmetries.

- universality implies that this is equivalent to QCD in the continuum limit, provided the **twist angle** $\tan(\alpha) = \mu_R / m_R$ is fixed

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- Control of chiral symmetry breaking allows for simpler renormalisation properties of many operators and allows a closer approach to the chirla limit → "mimic" exact chiral symmetry.

ALPHA Frezzotti, Grassi, Sint & Weisz, JHEP08(2001)058 Pena, Sint, AV JHEP09(2004)069

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- universality implies that this is equivalent to QCD in the continuum limit, provided the **twist angle** $\tan(\alpha) = \mu_R/m_R$ is fixed
- at **twist angle** $\pi/2$, many quantities are automatically improved

Frezzotti, Rossi JHEP08(2004)007

Renormalization

Hadronic renormalization scheme

- for simplicity consider a lattice theory with isospin symmetry and 3 flavours
- the 3 bare parameters are g_0, $m_q = m_u = m_d < m_s$
- they run with the lattice spacing a in the RG sense
- the hadronic scheme renormalization conditions are simply stated: tune all 3 bare parameters so as to ensure that 3 physical quantities are fixed to their (experimentally) known values

$$\frac{a\, m_P}{m_P^{\exp}} = a(g_0^2; m_q; m_s) \qquad \frac{a\, m_\pi}{a m_P} = \frac{m_\pi^{\exp}}{m_P^{\exp}} \qquad \frac{a\, m_K}{a m_P} = \frac{m_K^{\exp}}{m_P^{\exp}}$$

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fixed by experiment

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$\boxed{\text{computed non-perturbatively}}$

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lattice calibration

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- all other physical quantities (hadronic masses) can now be predicted (i.e. computed) since QCD is a renormalizable theory
- predictions must be repeated at smaller couplings g_0 i.e. smaller lattice spacings (asymptotic freedom)
- NB: this is explicitly non-perturbative and yields QCD mass spectrum

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- all other physical quantities (hadronic masses) can now be predicted (i.e. computed) since QCD is a renormalizable theory
- predictions must be repeated at smaller couplings g_0 i.e. smaller lattice spacings (asymptotic freedom)
- several "practical" problems have induced variants of this procedure

Operator renormalization scheme

- there are also quantities which depend on ("run with") renormalization scale μ (e.q. renormalized coupling and quark masses, operator WME etc.)
- need to impose extra renormalization conditions (renormalization schemes)
- these can be standard PT schemes (e.g. MOM or even MS) with lattice regularization
- but lattice PT is badly converging
  - example 1: MILC collaboration found that the strange quark mass was raised by 14% once its renormalization constant, known in 1-loop PT, was calculated at 2-loops
  - example 2: Göckeler et al. found that the strange quark mass was raised by 24% once its renormalization constant, known in 1-loop PT, was calculated by a NP method
- two NP renormalization schemes have been devised for these renormalizations
  - RI/MOM scheme — G.Martinelli et al. Nucl.Phys.B445(1995)81
  - Schrödinger Functional (SF) scheme — M.Lüscher et al. Nucl.Phys.B478(1996)365

Operator RG-running

- suppose a quantity *Q(μ)* (quark mass, operator WME) is renormalized in a NP scheme

$$Q_R(\mu) \;=\; \lim_{a\to 0}\; Z_Q(g_0^2,\, a\mu)\; Q(g_0^2)$$

- if you use a hadronic scheme, the renormalization scale is going to be low μ~ m_H

- you need to know *Q(μ)* at a larger scale either for conventional reasons (i.e. people are used to MS-scheme quark masses $m_q(\mu)$ with μ~2GeV) or for matching with perturbative scales, as in the OPE:

$$Q^{\rm phys} \;=\; \sum\; C_W(\mu)\;\lim_{a\to 0}\;\left[\; Z_Q(g_0^2, a\mu)\;\; < f|\, Q(g_0^2)\,|i > \;\right]$$

Operator RG-running

- suppose a quantity $Q(\mu)$ (quark mass, operator WME) is renormalized in a NP scheme

$$Q_R(\mu) \;=\; \lim_{a \to 0} \; Z_Q(g_0^2,\, a\mu)\; Q(g_0^2)$$

- if you use a hadronic scheme, the renormalization scale is going to be low $\mu \sim m_H$

- you need to know $Q(\mu)$ at a larger scale either for conventional reasons (i.e. people are used to MS-scheme quark masses $m_q(\mu)$ with $\mu \sim 2$GeV) or for matching with perturbative scales, as in the OPE:

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Wilson coefficients calculated in PT long-distance effects

renormalization scale must be large; say 10GeV

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must be smaller than 1
for avoiding discretization errors

Operator RG-running

- suppose a quantity $Q(\mu)$ (quark mass, operator WME) is renormalized in a NP scheme

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$$Q^{\mathrm{phys}} \;=\; \sum\; C_W(\mu)\; \lim_{a\to 0}\; \left[\; Z_Q(g_0^2, a\mu)\;\; < f|\, Q(g_0^2)\, |i > \;\right]$$

- if we wish to compute everything at one go (a single lattice) we must also ensure that $m_H\, L >> 1$, in order to avoid finite size errors

- i.e. we must satisfy $L >> 1/m_H \sim 1/(0.15\ \mathrm{GeV}) >> 1/\mu \sim 1/(10\ \mathrm{GeV}) >> a$

- IMPOSSIBLE on present day resources

Operator RG-running

- suppose a quantity $Q(\mu)$ (quark mass, operator WME) is renormalized in a NP scheme

$$Q_R(\mu) \; = \; \lim_{a \to 0} \; Z_Q(g_0^2, \, a\mu) \; Q(g_0^2)$$

- if you use a hadronic scheme, the renormalization scale is going to be low $\mu \sim m_H$

- you need to know $Q(\mu)$ at a larger scale either for conventional reasons (i.e. people are used to MS-scheme quark masses $m_q(\mu)$ with μ~2GeV) or for matching with perturbative scales, as in the OPE:

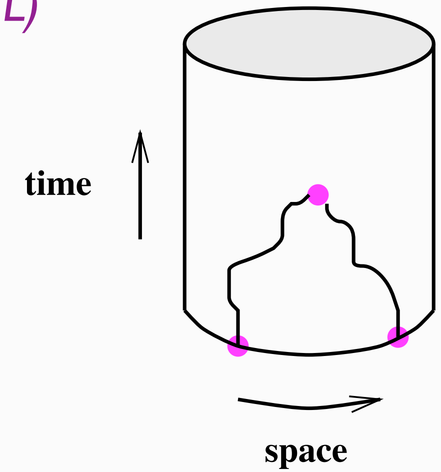
$$Q^{\text{phys}} \; = \sum \; C_W(\mu) \; \lim_{a \to 0} \; \left[\; Z_Q(g_0^2, a\mu) \;\; < f| \, Q(g_0^2) \, |i > \; \right]$$

- need to compute the renormalized WME at a hadronic (low) scale μ_{min} and then do RG-running all the way to a perturbative (high) scale μ_{max}

- the SF scheme, combined with finite size techniques, is the only one used so far for this RG-running

Schrödinger Functional (SF) and step scaling function (SSF)

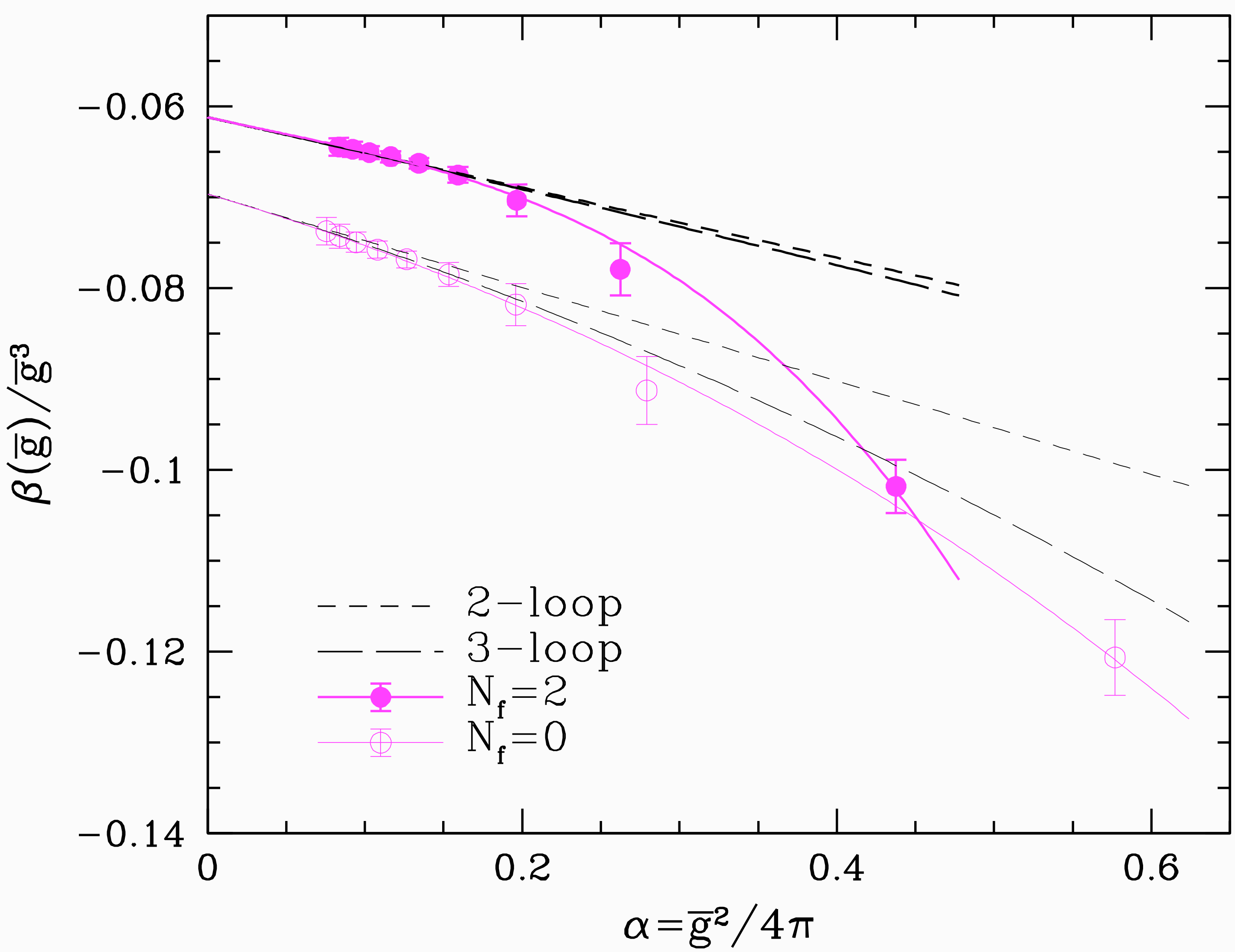
- work in finite (small) physical volumes, as they are OK for the determination of renormalization constants, which are local UV quantities
- use finite size as renormalization scale; i.e. $\mu = 1/L$ (this is NOT the L of the WME)
- impose **SF** renormalization condition on a suitably constructed lattice (Dirichlet b.c.'s in time; periodic in space), which allows us to work with **mass-independent** renormalization scheme and compute $Z(g_0^2, a/L)$
- this is done at a hadronic scale $\mu_{min} = 1/L_{max}$
- we run all the way to $\mu_{max} = 1/L_{min}$ in stepwise manner, through the **step scaling function**



$$\sigma_Q[g_R^2] = \left.\frac{Q_R(g_R^2; 2L)}{Q_R(g_R^2; L)}\right|_{g_R^2=\text{fixed}}$$

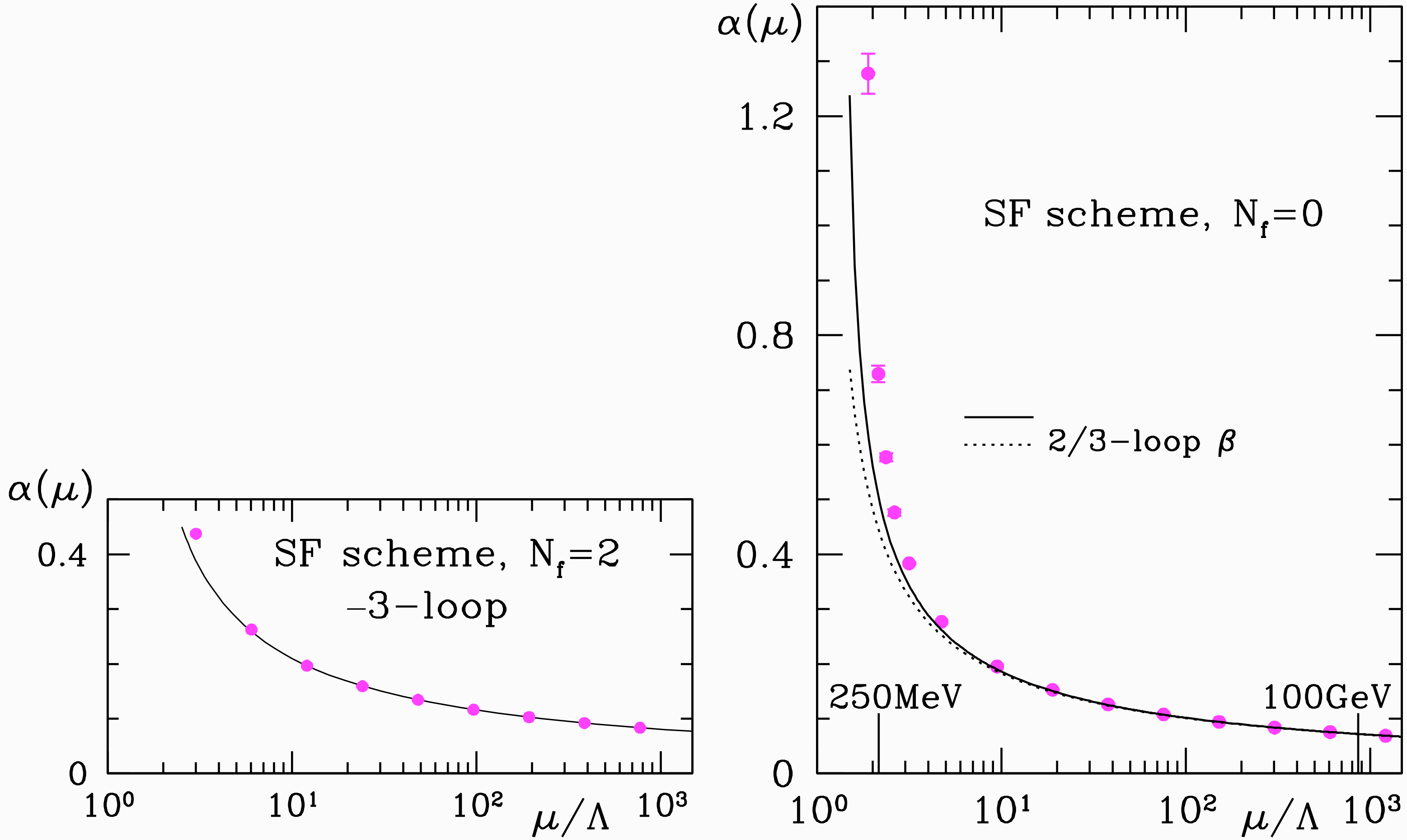
- a discrete version of the anomalous dimension, for a change of scale by a factor of 2
- compute this iteratively until you arrive at μ_{max} which amount to RG-running
- it is a continuum quantity; can be calculated on the lattice with modest lattice volumes

Schrödinger Functional (SF) and step scaling function (SSF)



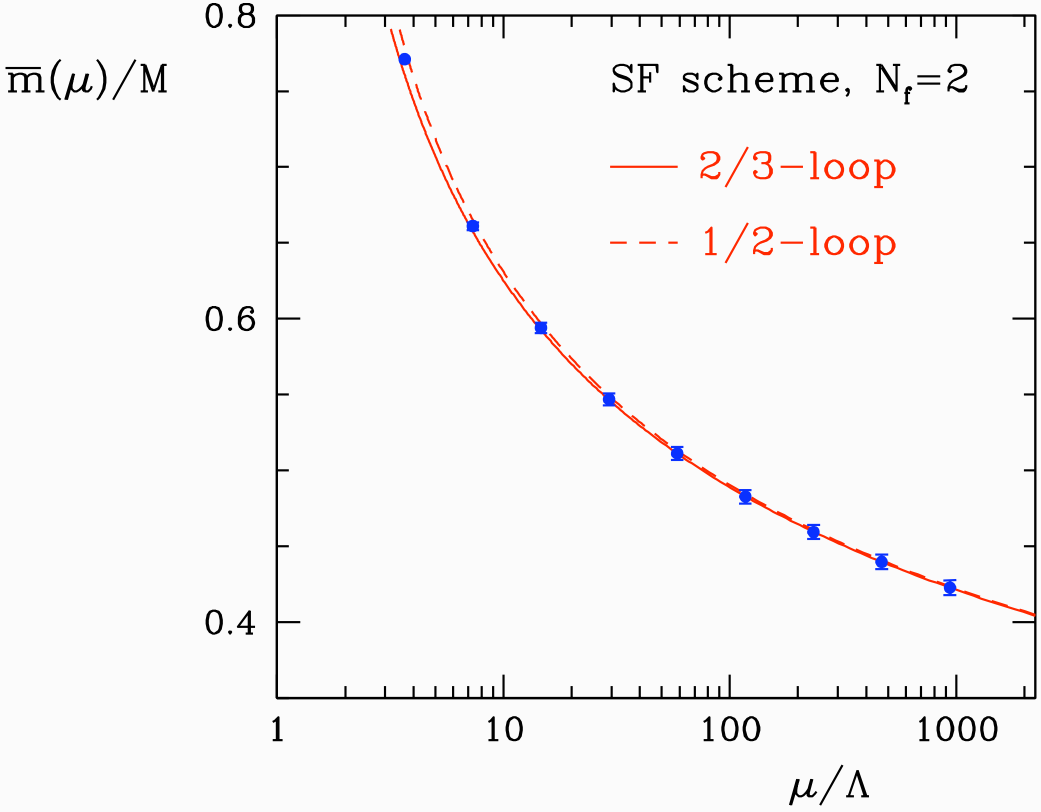
Schrödinger Functional (SF) and step scaling function (SSF)

ALPHA, J. Heitger et al. Nucl.Phys.Proc.Suppl.106 (2002) 859



Schrödinger Functional (SF) and step scaling function (SSF)

ALPHA M. Della Morte et al. Nucl.Phys.B729(2005)117



NB: these results depend on N_f, but NOT on the quark mass values
(mass independent renormalization, carried out explicitly at zero quark mass)