High energy evolution in QCD

Simone Marzani

School of Physics
University of Edinburgh

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Outline

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The theory of strong interactions

- QCD is a non-Abelian gauge theory with gauge group $SU(3)_c$.
- The coupling constant is a decreasing function of the energy: *asymptotic freedom and confinement*.
  (Gross, Politzer, Wilckez).
- Computations of strong processes (at sufficiently high energy) are feasible in PT.

Factorization theorem

The factorization theorem (DIS)

$$\sigma = \int \frac{dz}{z} \hat{\sigma}\left(\frac{x}{z}, Q^2\right) G(z, Q^2)$$

- Convolution between hard partonic cross section and PDFs.
- The cross section is affected by $\alpha_s^n ln^n \frac{Q^2}{\mu^2}$.
- These collinear logs are resummed if $G(x, Q^2)$ is the solution of GLAP evolution equation.

GLAP evoultion

- The Gribov-Lipatov-Altarelli-Parisi equation is an integro-differential one in (x, t) space:

$$\frac{\partial}{\partial t}\begin{pmatrix} q_i(x,t) \\ g(x,t) \end{pmatrix} = \alpha_s(t)\sum_{q_j,\bar{q}_j}\int_x^1 \frac{d\xi}{\xi}\begin{pmatrix} q_j(\xi,t) \\ g(\xi,t) \end{pmatrix}$$

$$\times \begin{pmatrix} P_{q_i q_j}\left(\frac{x}{\xi},\alpha_s(t)\right) & P_{q_i g}\left(\frac{x}{\xi},\alpha_s(t)\right) \\ P_{g q_j}\left(\frac{x}{\xi},\alpha_s(t)\right) & P_{gg}\left(\frac{x}{\xi},\alpha_s(t)\right) \end{pmatrix}$$

where $t = ln\frac{Q^2}{\mu^2}$ and $P_{a,b}(x,\alpha_s)$ are the splitting functions.

- The study of the GLAP equation is simplified in Mellin space:

$$\hat{f}(N,t) = \int_0^1 \frac{dx}{x} x^N f(x,t)$$

- Convolutions become products and GLAP equation an ordinary differential one.

$$\frac{d}{dt} G(N,t) = \gamma(\alpha_s(t), N) G(N,t)$$

  $G(N,t)$ is the eigenvector with the biggest eigenvalue γ in the singlet sector.

- The anomalous dimension are known at NNLO accuracy. (Vogt, Moch, Vermaseren).

Outline

1. The perturbative regime of quantum chromodynamics
   - Hard processes in QCD and Altarelli-Parisi evolution
   - The high energy limit and the BFKL equation
2. BFKL evolution from GLAP
   - Naive duality
   - Running coupling case
3. Conclusions

Small-x physics

- PT works in QCD in the presence of a hard scale (eg Q^2 in DIS, m_f in heavy flavour production).
- GLAP equation resums large logs of this scale.
- Define x as the ratio of the hard scale and the collision energy.
- In the high energy limit ($s \to \infty$) cross sections are affected by large logs of x.
- We need a formalism to compute cross sections in the small-x limit.

Cross sections in the high energy limit

The k_T factorization theorem (DIS)

$$\sigma = \int \frac{dz}{z} d^2 k_T \hat{\sigma}\left(\frac{x}{z}, k_T^2\right) \mathcal{G}(z, k_T^2)$$

- $\hat{\sigma}\left(\frac{x}{z}, k_T^2\right)$ is the off-shell partonic cross section.
- $\mathcal{G}(z, k_T^2)$ is the parton density not integrated over transverse momenta.

The BFKL equation

- The evolution of $\mathcal{G}$ is described by BFKL equation:

$$\frac{d}{d\xi}\mathcal{G}(\xi, M) = \chi(M, \alpha_s)\mathcal{G}(\xi, M)$$

- This equation resums (at LO) $\alpha_s^n \xi^n = \alpha_s^n ln^n \frac{1}{x}$
- Already Mellin transformed with respect to $\left(\frac{Q^2}{\mu^2}\right)^M$
- The introduction of the running coupling is nontrivial:

$$\alpha_s(t) = \frac{\alpha_s}{1 + \alpha_s \beta_0 t} \implies \hat{\alpha}_s = \frac{\alpha_s}{1 - \alpha_s \beta_0 \frac{\partial}{\partial M}}$$

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BFKL kernel

- The BFKL kernel is known at NLO accuracy:

$$\chi(M, \hat{\alpha}_s) = \hat{\alpha}_s \chi_0(M) + \hat{\alpha}_s^2 \chi_1(M) + \dots$$

- The LO contribution is symmetric with respect $M \leftrightarrow 1 - M$ (exchange of the initial virtualities)
- This symmetry is broken at NLO: running coupling effect.
- It can be restored by a symmetric choice of the running (by a suitable order of the operators).
- Unstable expansion due to large unresummed Q^2 logs: double leading resummation (see Altarelli, Ball, Forte).

Outline

Duality relations

- In the fixed coupling case, if χ and γ are related by

$$\begin{aligned} \chi(\gamma(N,\alpha_s),\alpha_s) &= N \\ \gamma(\chi(M,\alpha_s),\alpha_s) &= M \end{aligned}$$

  BFKL and GLAP equation admit the same solution, provided that BC are suitably chosen.
- The proof of this statement is easily performed in the double Mellin (N, M) space.
- It is clear that we can recover information about χ from γ and viceversa.

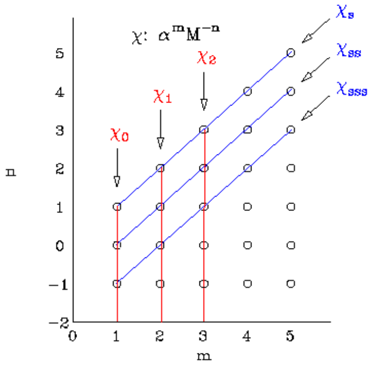
The target is the computation of the NNLO BFKL kernel χ_2 in the collinear approximation from the anomalous dimensions.

- Computation of the naive dual through fixed coupling duality relations.
- Analysis of the running coupling case.
- Inclusion of *kinematic variable* contributions.

χ_0 and χ_1 are also computed in this approximation and their expressions are compared to the complete ones.

Explicit computations

- Starting point: anomalous dimensions up to NNLO
- Computation through duality of the first three terms of the expansion of χ in powers of α_s at fixed $\frac{\alpha_s}{M}$.
- Extraction of the contributions to the standard expansion up to NNLO.



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The running coupling case

- The introduction of the running coupling in BFKL is problematic and the extension of the naive duality to the general case in nontrivial.
- BFKL equation still admits a GLAP type solution, but naive duality has to be corrected by so called *running coupling contributions*.

Algebraic approach

- Let us consider the GLAP equation in (N, M) space at fixed running:

$$MG(N,M) = \gamma(\alpha_s, N{\alpha_s}^{-1})G(N,M)$$

- With running coupling this algebraic equation become an operatorial one:

$$MG(N,M) = \gamma(\hat{\alpha}_s, N\hat{\alpha}_s^{-1})G(N,M)$$

- It is of the form:

$$\hat{q}\ G(N,M) = \hat{p}\ G(N,M), \quad \hat{q} = \gamma(\hat{\alpha}_s, N\hat{\alpha}_s^{-1})\,, \hat{p} = M$$

- Given $f(\hat{q})$, we want the function $g(\hat{p})$, such that:

$$f(\hat{q})\ G(N,M) = g(\hat{p})\ G(N,M)\,,$$

- $g \neq f$ since $\hat{p}$ and $\hat{q}$ do not commute.

- Baker-Campbell-Hausdorff formula enables to express g in terms of f and commutators:

$$f(\hat{q})G(N,M) = (f(\hat{p}) - \frac{1}{2}[\hat{p},\hat{q}]f''(\hat{p}) + \frac{1}{6}[\hat{q},[\hat{q},\hat{p}]]f'''(\hat{p}) + + \frac{1}{8}([\hat{p},\hat{q}])^2 f^{IV}(\hat{p}) + \dots)G(N,M)\,.$$

- If we apply to GLAP equation the function $f = \chi$ such that $\chi(\gamma(\hat{\alpha}_s, N\hat{\alpha}_s^{-1})) = N$, we recover the BFKL equation:

$$NG(N,M) = [\chi(\hat{\alpha},M) + \Delta\chi(\hat{\alpha},M))]\, G(N,M)$$

- RC contributions completely expressed in terms of derivative of the naive dual χ and multiple commutators of $\gamma(\hat{\alpha}_s, N\hat{\alpha}_s^{-1})$ and M.

Explicit computation of RC contributions

- Compute the commutators at the order in $\hat{\alpha}_s$ required.
- Consider the 1-loop accuracy for the running coupling:

$$\hat{\alpha}_s^{-1} = \frac{1}{\alpha_s} - \beta_0 \frac{\partial}{\partial M} + \beta_1 \left(-\alpha_s \beta_0 \frac{\partial}{\partial M} - \frac{1}{2} (\alpha_s \beta_0)^2 \frac{\partial^2}{\partial M^2} \right) + O(\alpha_s^3)\,.$$

- The result for the NNLO contribution is:

$$\begin{aligned}\tilde{\chi}_2(M) &= \chi_2(M) + \frac{1}{4}\beta_0^2 \frac{\chi_0 (\chi_0'')^2}{(\chi_0')^2} - \frac{1}{2}\beta_0\beta_1 \frac{\chi_0 \chi_0''}{\chi_0'} \\ &+ \frac{1}{24}\beta_0^2 \frac{(\chi_0)^2}{(\chi_0')^4} \left(12 \left(\chi_0''\right)^3 - 2\chi_0'\chi_0''\chi_0''' + 3\left(\chi_0'\right)^2 \chi_0{}^{IV} \right) \\ &- \frac{1}{2}\beta_0 \frac{\chi_0 \chi_1''}{\chi_0'} - \beta_0 \frac{\chi_1 \chi_0''}{\chi_0'} + \frac{1}{2}\beta_0 \frac{\chi_0 \chi_0'' \chi_1'}{(\chi_0')^2}\,.\end{aligned}$$

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Further contributions

Three more contributions have to be included:

- Unintegrated parton density
- Kinematic variables
  These transformations are easily implemented in the operatorial formalism: straightforward computation of the contributions to the kernel.
- Factorization scheme

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Factorization schemes

- There is a mismatching of normalization between the scheme $\overline{MS}$ and Q^0
- The relation between the two scheme is known only at NLO (Catani, Hautmann 1994)
- The unknown NNLO function should be free of poles in $M = 0$, not affecting the collinear approximation of χ_2
- A full description of the scheme change with the operator analysis has not been understood yet.
- We are testing the argument doing Catani's computation at NNLO.

Symmetrization

- We have the collinear approximation (around $M = 0$) of the BFKL kernel at NNLO accuracy in symmetric variables.
- The symmetry between the gluon virtualities in the Mellin space is $M \leftrightarrow 1 - M$: we can extend the result in the region $M \sim 1$:

$$\begin{aligned} \chi(\hat{\alpha}_s, M) &= \hat{\alpha}_s \chi_0^{sym}(M) + \hat{\alpha}_s^2 \chi_1^{sym}(M) + \hat{\alpha}_s^3 \chi_2^{sym}(M) + \\ &+ \chi_0^{sym}(1-M)\hat{\alpha}_s + \chi_1^{sym}(1-M)\hat{\alpha}_s^2 + \\ &+ \chi_2^{sym}(1-M)\hat{\alpha}_s^3 \,. \end{aligned}$$

The BFKL kernel up to order α_s^3

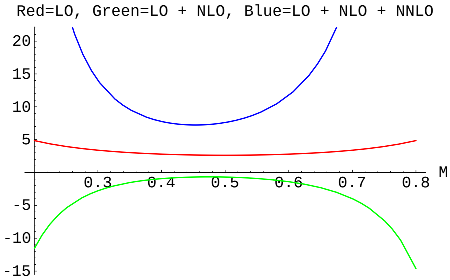
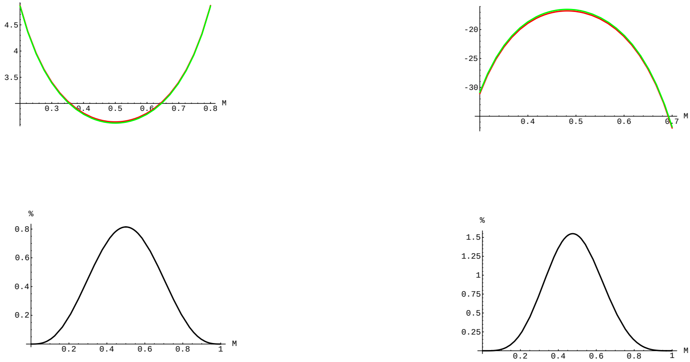


Figure: Plots of collinear approximation of the BFKL kernel. The blue line is the major result: $\chi^{coll} = \alpha_s \chi_0 + \alpha_s^2 \chi_1 + \alpha_s^3 \chi_2$, $\alpha_s = 0.2$.

Comparison of LO and NLO



Discrepancy

$\Delta = 100\frac{\chi_i - \chi_i^{coll}}{\chi_i} < 1.5$, in both cases.

Summary and conclusions

- Introduction to PT in QCD: factorization and GLAP evolution.
- Description of the high energy limit of QCD and the BFKL equation.
- Connection between GLAP and BFKL evolution: duality relations.
- Use of these relation to extract the high energy kernel from the anomalous dimensions up to NNLO.
- The scheme change at NNLO: work in progress.