Cancellation in the transverse CSR force—again

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I will show that the logarithmically diverging term in the transverse force is cancelled by the contribution from the potential term. The derivation follows that of R. Li from JLab (see her latest paper at EPAC 2002), but simpler (I think).

Let us start our analysis by considering two ultrarelativistic particles ($v = c$, or $\gamma \to \infty$) of charge q moving in a circle of radius R. Those results are not applicable at the distance from the source particle $\Delta s \sim R/\gamma^3$. There is no Coulomb field in this limit. We can calculate both the longitudinal and transverse forces in this case assuming that the distance between the particles Δs is much smaller than R, or the angular separation $\psi = \Delta s/R \ll 1$.

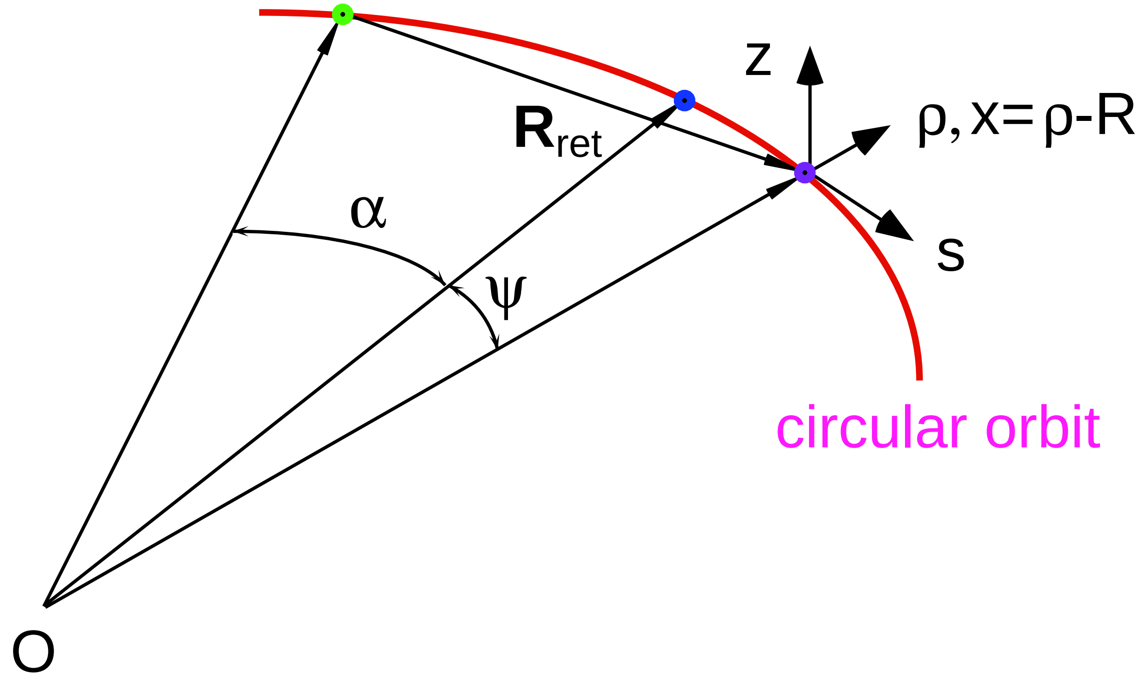


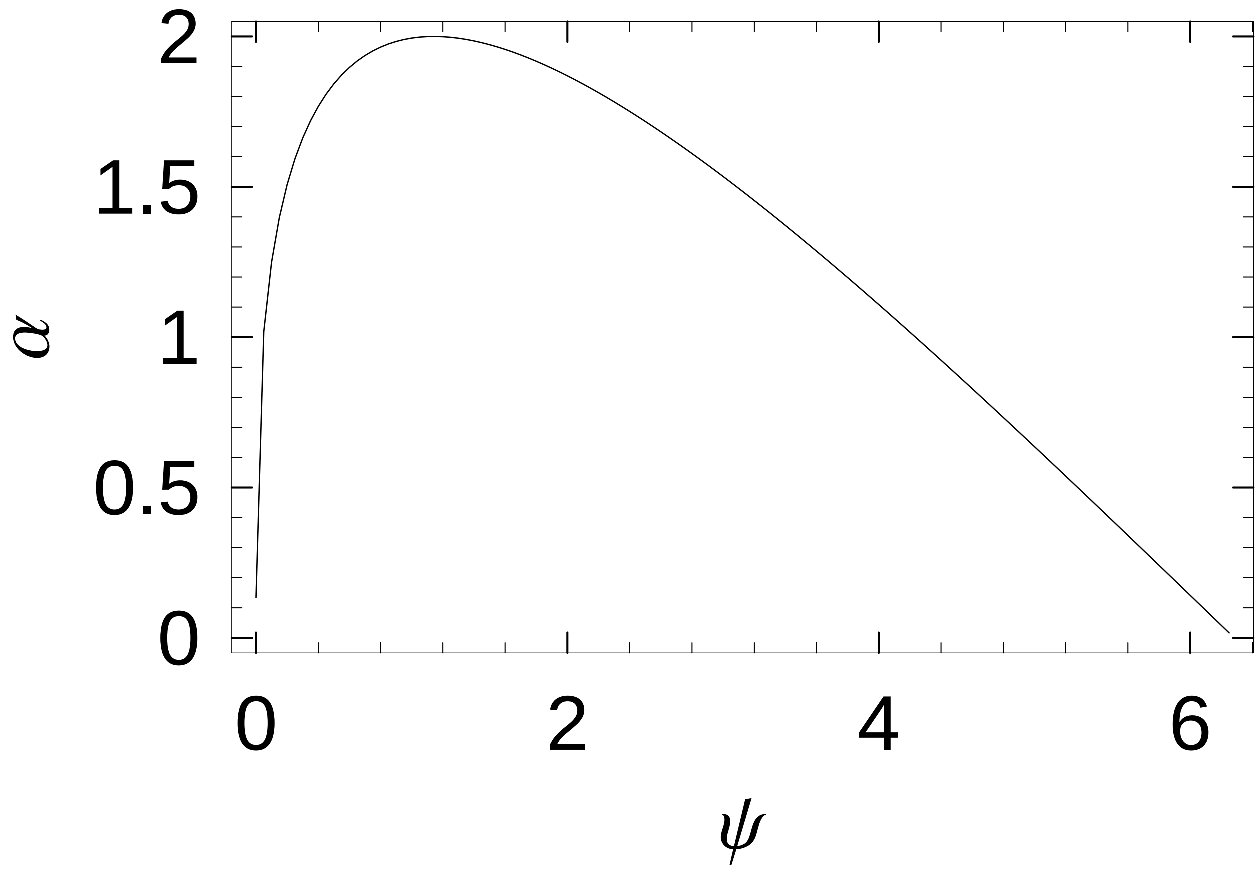
Figure 1: Coordinate system. Shown are locations of the particle at time t, the point P, the observation point in front of the particle at an angle ψ, and the point where the radiation occurred at the retarded time (at an angle α behind the particle). Vector $\boldsymbol{R}_{\rm ret}$ connects the radiation point with the observation point.

Expressions for the scalar potential ϕ and the vector potential $\boldsymbol{A}$ follow directly from the Lienard-Wiehert relations:

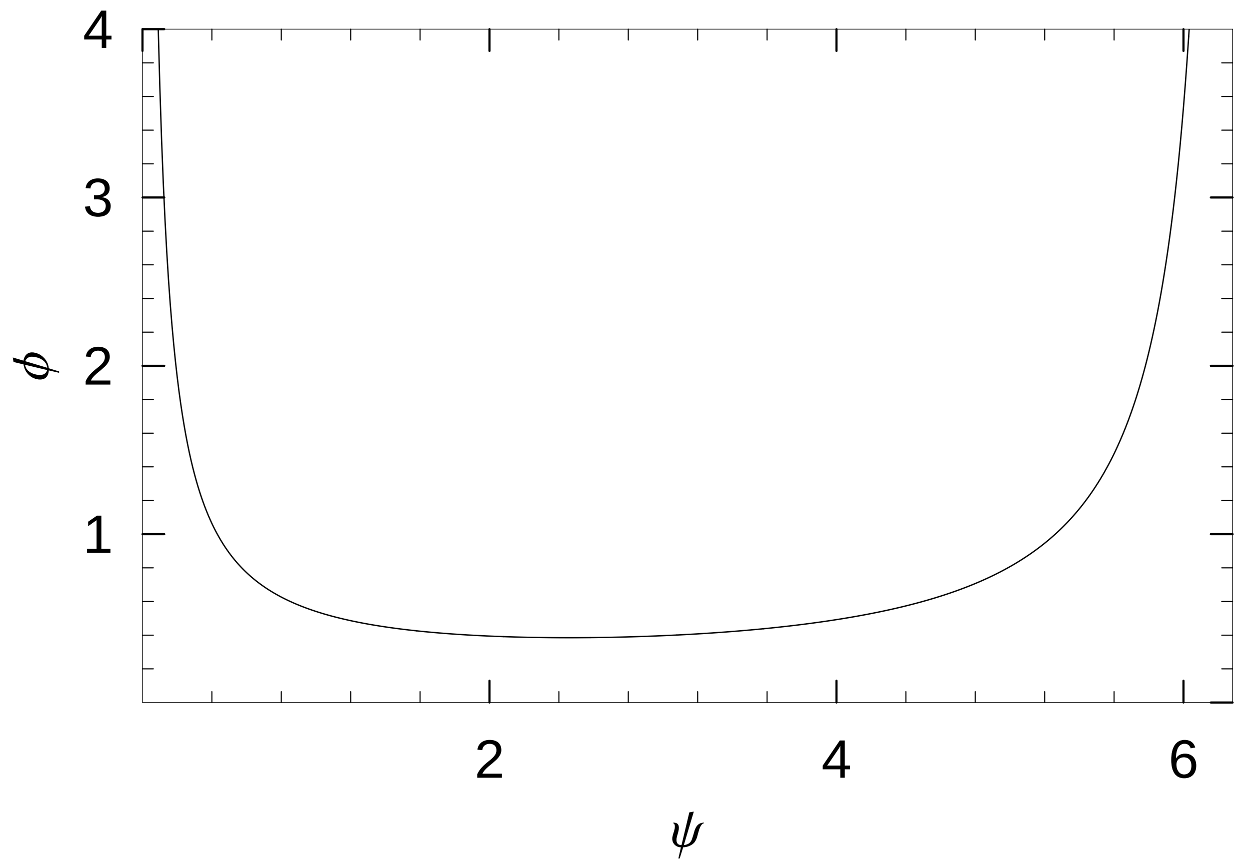
$$\begin{aligned}
\phi &= \frac{q}{R}\frac{1}{\alpha - \sin(\alpha+\psi)} \\
A_\psi &= \frac{q}{R}\frac{\cos(\alpha+\psi)}{\alpha - \sin(\alpha+\psi)} \\
A_\rho &= \frac{q}{R}\frac{\sin(\alpha+\psi)}{\alpha - \sin(\alpha+\psi)}
\end{aligned}$$

where α is the retarted position of the source particle that satisfy the following equation

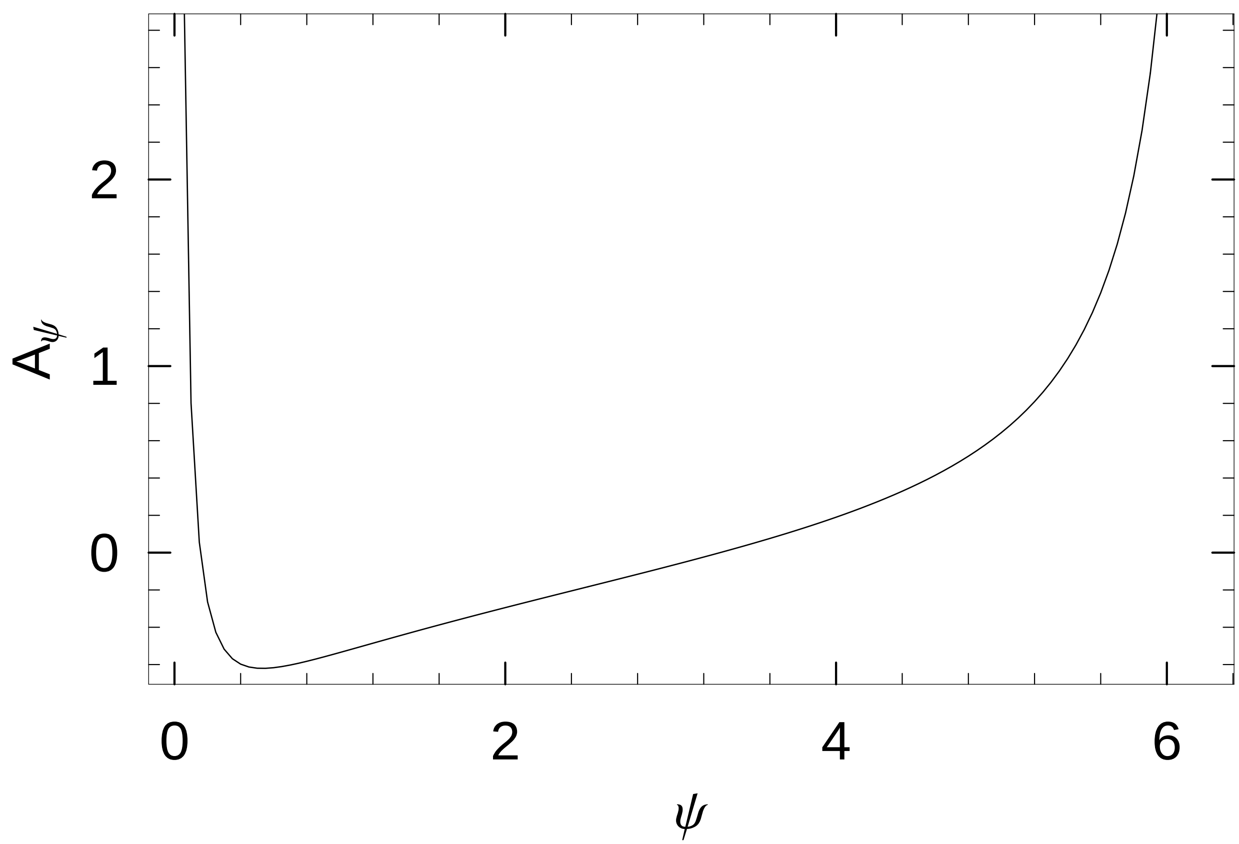
$$\alpha = 2\left|\sin\left(\frac{\alpha+\psi}{2}\right)\right|$$



α
2
1.5
1
0.5
0
0
2
4
6
ψ



4
3
2
1
φ
2
4
6
ψ

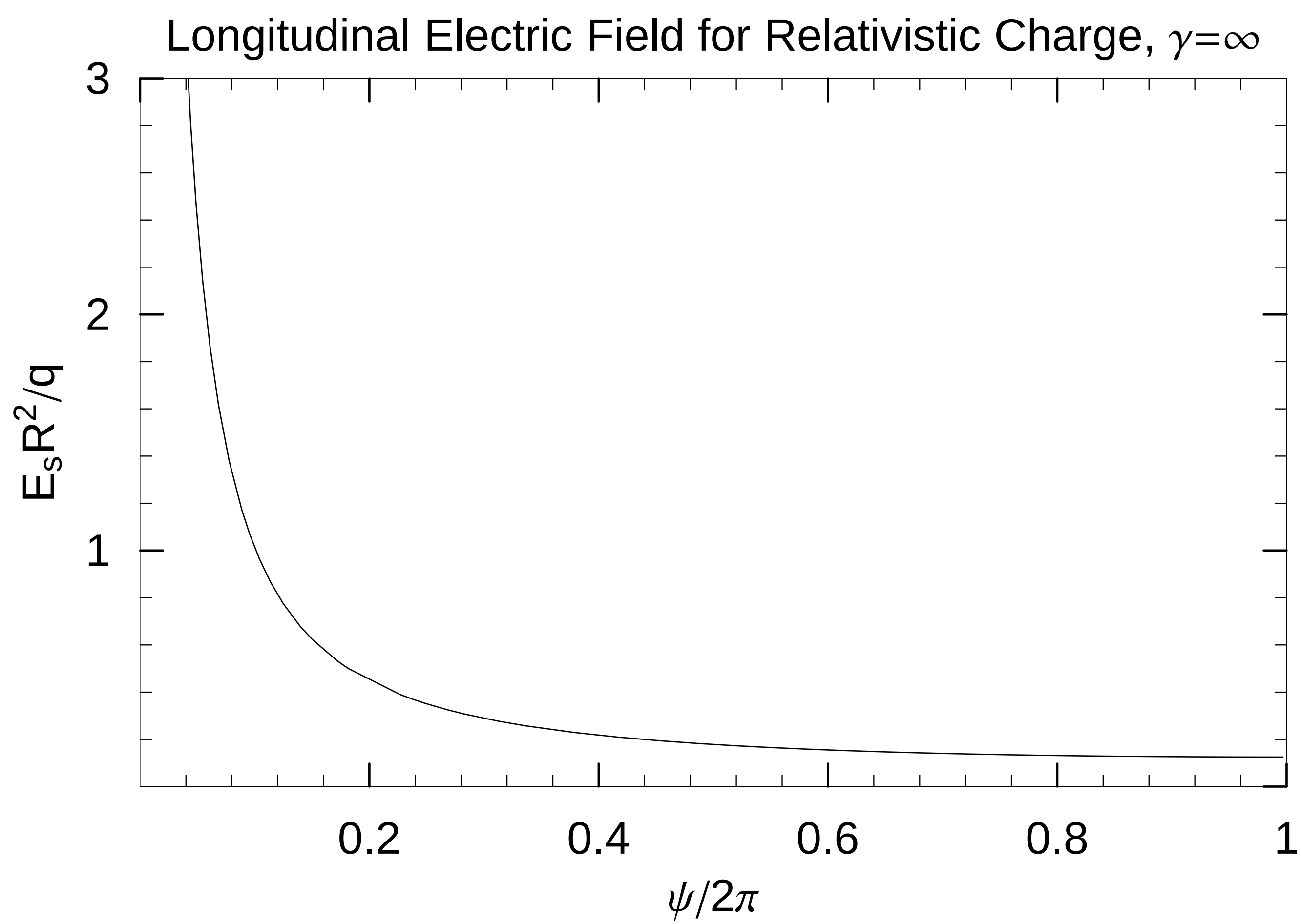


A$_\psi$
2
1
0
0
2
4
6
ψ

One can also write expressions for any component of the electric and magnetic fields,

$$\begin{aligned} E_s(\psi) &= \frac{q}{R^2}\frac{2\sin\frac{1}{2}(\alpha+\psi)}{[\alpha-\sin(\alpha+\psi)]^3} \\ &\times \left\{2(\sin\frac{1}{2}(\alpha+\psi))^3\left[\cos\frac{1}{2}(\alpha+\psi)-\cos(\alpha+\psi)\right]\right. \\ &- \left.\sin(\alpha+\psi)[\alpha-\sin(\alpha+\psi)]\right\}, \end{aligned}$$

$$\begin{aligned} E_\rho(\psi) &= \frac{q}{R^2}\frac{2\sin\frac{1}{2}(\alpha+\psi)}{[\alpha-\sin(\alpha+\psi)]^3} \\ &\times \left\{2(\sin\frac{1}{2}(\alpha+\psi))^3\left[\sin\frac{1}{2}(\alpha+\psi)-\sin(\alpha+\psi)\right]\right. \\ &+ \left.\cos(\alpha+\psi)[\alpha-\sin(\alpha+\psi)]\right\}, \\ H_z(\psi) &= E_s\sin\frac{1}{2}(\alpha+\psi)-E_\rho\cos\frac{1}{2}(\alpha+\psi), \end{aligned}$$


Longitudinal Electric Field for Relativistic Charge, $\gamma=\infty$
E_sR^2/q
3
2
1
0.2
0.4
0.6
0.8
1
$\psi/2\pi$

We are interested in the limit $|\psi| \ll 1$ $(2\pi - \psi \to -\psi)$, then $\alpha \ll 1$, and the above equations can be solved using the Taylor expansion of the expressions for the potential. Here are the results for positive ψ (test particle in front of the source), in units q/R:

$$\begin{aligned} \alpha &\approx (24)^{1/3}\psi^{1/3} \\ \phi &= \frac{1}{3\psi} + \frac{1}{5 \cdot 3^{1/3}\psi^{1/3}} \\ A_\psi &= \frac{1}{3\psi} - \frac{3^{5/3}}{5\psi^{1/3}} \\ A_\rho &= \frac{2}{3^{2/3}\psi^{2/3}} \end{aligned}$$

For negative ψ (test particle behind the source):

$$\begin{aligned} \phi &= \frac{1}{|\psi|} \\ A_\psi &= \frac{1}{|\psi|} \\ A_\rho &= 0 \end{aligned}$$

Let us now calculate the longitudinal filed, $|\boldsymbol{\beta}| = 1$, taking into account that $\phi(R\psi - ct)$ and $\boldsymbol{A}(R\psi - ct)$

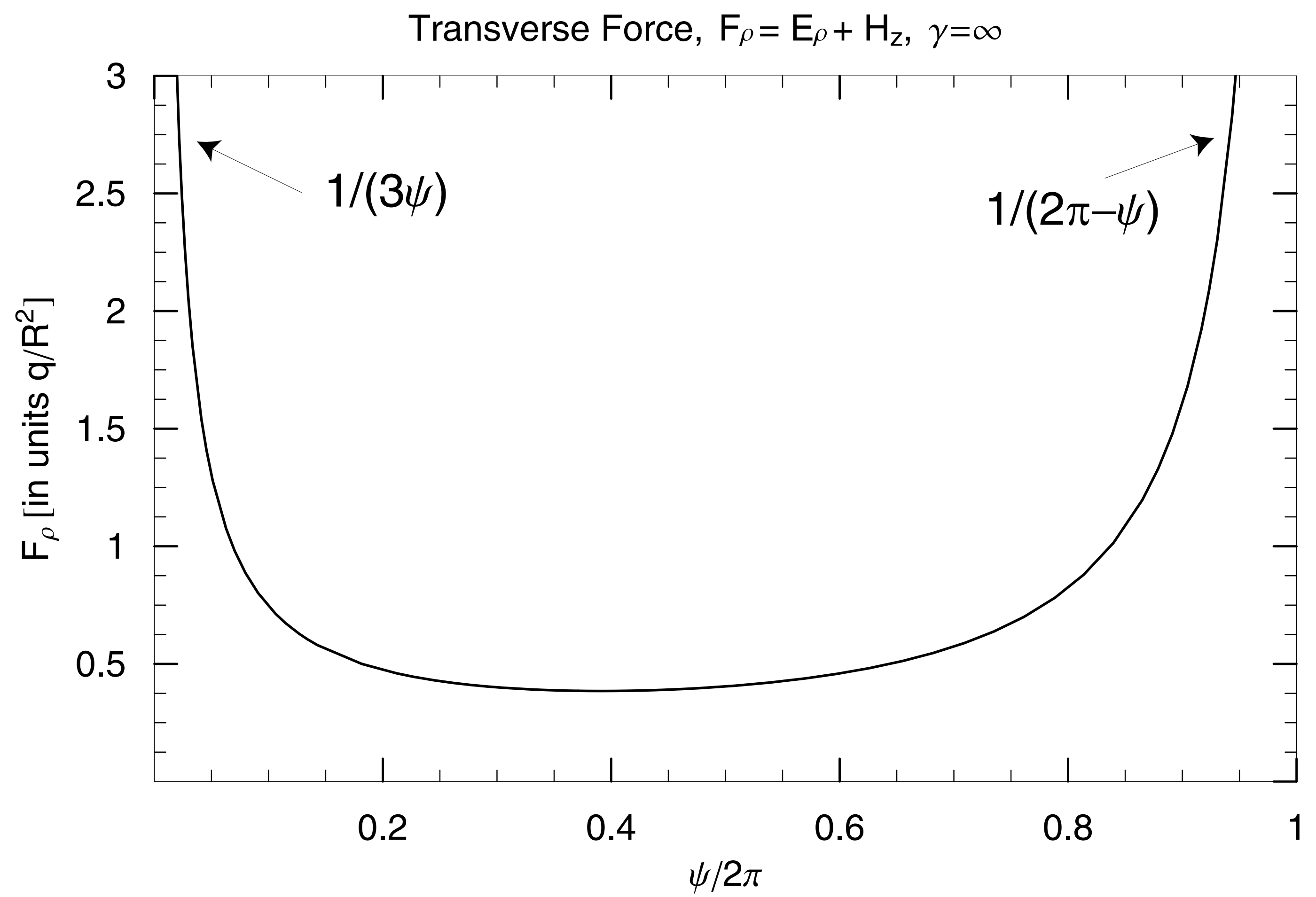
$$
\begin{aligned}
E_s &= \boldsymbol{\beta}\boldsymbol{E} \\
&= \boldsymbol{\beta}\left(-\frac{1}{c}\frac{\partial \boldsymbol{A}}{\partial t} - \nabla\phi\right) \\
&= \left(-\frac{1}{c}\frac{\partial A_\psi}{\partial t} - \frac{1}{R}\frac{\partial \phi}{\partial \psi}\right) \\
&= \frac{1}{R}\frac{\partial A_\psi}{\partial \psi} - \frac{1}{R}\frac{\partial \phi}{\partial \psi} \\
&= \frac{1}{R}\frac{\partial (A_\psi - \phi)}{\partial \psi}
\end{aligned}
$$

The ψ^{-1} singularity cancels out and only $\psi^{-1/3}$ singularity is left

$$E_s = \frac{1}{R}\frac{\partial (A_\psi - \phi)}{\partial \psi} = -\frac{q}{R^2}\frac{2}{3^{1/3}}\frac{\partial}{\partial \psi}\frac{1}{\psi^{1/3}}$$

Let us now calculate the transverse force (per unit charge)

$$F_\rho = E_\rho + H_z$$



For small $|\psi|$

$$\begin{aligned} F_\rho &= \frac{q}{R^2}\frac{1}{3\psi} \qquad \text{for} \qquad \psi > 0 \\ F_\rho &= \frac{q}{R^2}\frac{1}{|\psi|} \qquad \text{for} \qquad \psi < 0 \end{aligned}$$

If we want to integrate this force and find the force in the beam, with some density distribution $\lambda(z)$, we have to deal with the singularity

$$F_\rho^{(bunch)}(z) = \int dz' \lambda(z') F_\rho(z - z')$$

To resolve the singularity one has to take into account the transverse size of the beam, $\sigma_\perp$, then we get *centrifugal* force

$$F_\rho^{(bunch)} \sim \frac{q}{R\sigma_z} \ln\left(\frac{R}{\sigma_\perp}\right)$$

Derbenev and Shiltsev derived in 1996 an effective *centripital* force (in steady state)

$$F_\rho^{(bunch)} = -\frac{2q\lambda}{R} \sim \frac{q}{R\sigma_z}$$

The problem, however, is in the dynamic equation of motion of the beam electrons

$$x'' + Kx = \frac{qF_\rho + \Delta E/R}{E}$$

We want to solve this equation for each electron in the bunch located at coordinate z, $F_\rho(z)$, $\Delta E(z)$. If initially, $\Delta E = 0$, then the energy deviation can be generated when the beam enters the magnet (transient regime) and can cancel part of $F_\rho(z)$.

That is exactly what happens!

I want to use potentials derived for the circle of radius R in the transient regime assuming that $R(s)$. I will call this *local approximation*. This can be done if the retardation length $(24R(s)^2\Delta s)^{1/3}$ is much smaller than $R(s)$. Since I am interested in $\Delta s \to 0$, that is justified.

Let us calculate ΔE generated in the transient region (when the beam enters the magnet)

$$\begin{aligned}\Delta E &= q\int dt\,\boldsymbol{v}\cdot\boldsymbol{E} \\ &= -q\int dt\,\boldsymbol{v}\cdot\left(\frac{1}{c}\frac{\partial \boldsymbol{A}}{\partial t}+\nabla\phi\right) \\ &= -q\int dt\,\left(\boldsymbol{\beta}\frac{\partial \boldsymbol{A}}{\partial t}+\boldsymbol{v}\nabla\phi\right)\end{aligned}$$

$$\boldsymbol{v}\nabla\phi = \frac{d\phi}{dt}-\frac{\partial\phi}{\partial t}\,, \qquad \boldsymbol{\beta}\frac{\partial \boldsymbol{A}}{\partial t}=\frac{\partial}{\partial t}\boldsymbol{\beta}\boldsymbol{A}$$

$$\begin{aligned}\Delta E &= -q\int dt\left(\frac{d\phi}{dt}-\frac{\partial\phi}{\partial t}+\frac{\partial}{\partial t}\boldsymbol{\beta}\boldsymbol{A}\right) \\ &= -q\phi-q\int dt\,\frac{\partial(A_\psi-\phi)}{\partial t}\end{aligned}$$

The first term cancels singularities in the transverse force. The net result—the log term is gone.