QFEL

Wigner function approach

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The way up

- from classical to quantum regime
- from Schrödinger to Wigner
- from 1D to 3D (and back to 2D)
- focus on energy output
- relative importance of different terms

Introducing the 1D Wigner function model

Schoedinger formulation in 1D

Longitudinal momentum states decomposition

$$\frac{\partial c_n}{\partial \bar{z}} = -i\left(\frac{n^2}{2\bar{\rho}^{3/2}} + n\bar{\delta}\right)c_n - \left(Ac_{n-1} + A^* c_{n+1}\right)$$

$$\frac{\partial A}{\partial \bar{z}} + \frac{\partial A}{\partial z_1} = b(z_1;\bar{z}) = \sum_{n=-\infty}^{\infty} c_n c_{n-1}^*$$

$$\Psi(\theta, z_1, \bar{z}) = \sum_{n=-\infty}^{\infty} c_n(z_1, \bar{z}) e^{in\theta}$$

Model equations in 1D

discrete Wigner function

$$\frac{\partial w_s^k}{\partial \bar{z}} = -ik\left(\frac{s}{\bar{\rho}} + \bar{\delta}\right) w_s^k + \bar{\rho} A \left(w_{s+1/2}^{k-1} - w_{s-1/2}^{k-1}\right) + \bar{\rho} A^* \left(w_{s+1/2}^{k+1} - w_{s-1/2}^{k+1}\right)$$

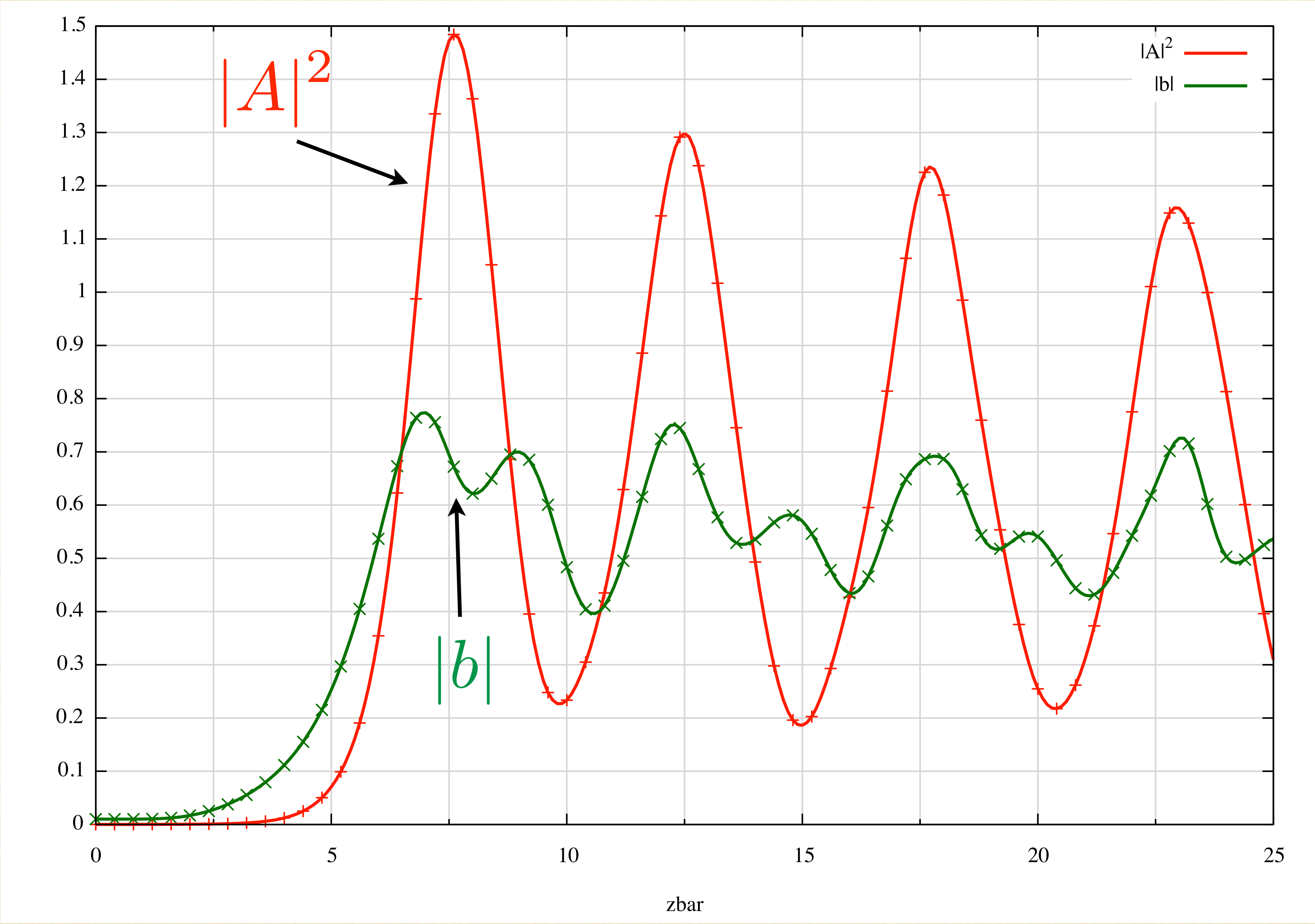
$$\frac{\partial A}{\partial \bar{z}} = B(\bar{z}) = \sum_{m=-\infty}^{+\infty} w_{m+1/2}^{1}$$

classical scaling

$$w_s(\theta) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} w_s^k \, e^{ik\theta}$$

Steady state 1D in the classical regime

$\bar{\rho} = 5$



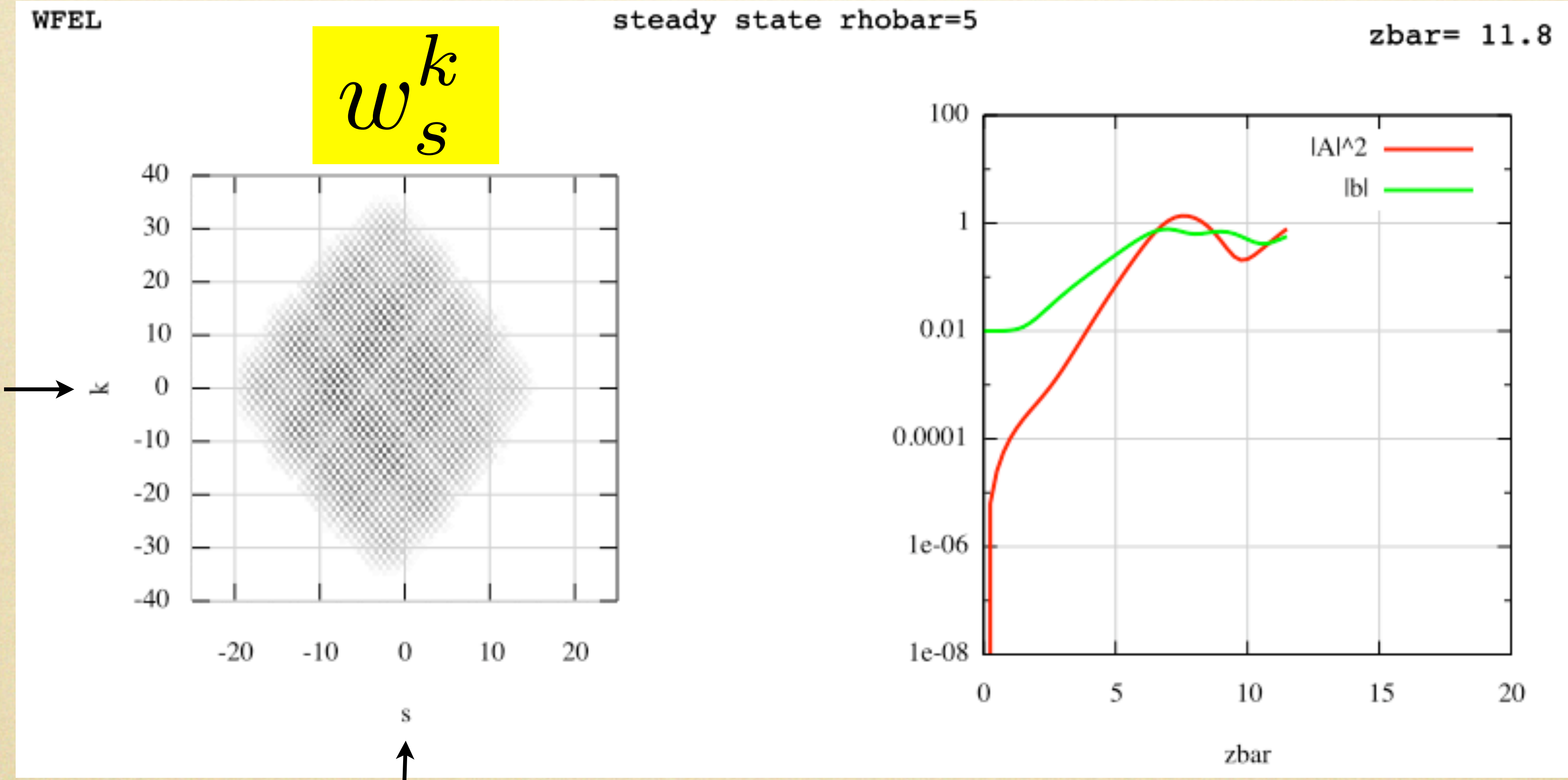
solid lines = Wigner function model

crosses = Schrödinger equation model

Population of momentum and bunching states

$$\bar{\rho} = 5$$

Fourier components k



half-integer index s of the small Wigner function

Model equations in 1D

$$\frac{\partial w_s^k}{\partial \hat{z}} = -ik\left(\frac{s}{\hat{\rho}} + \hat{\delta}\right) w_s^k + \hat{A}\left(w_{s+1/2}^{k-1} - w_{s-1/2}^{k-1}\right) + \hat{A}^*\left(w_{s+1/2}^{k+1} - w_{s-1/2}^{k+1}\right)$$

$$\frac{\partial \hat{A}}{\partial \hat{z}} = B(\hat{z}) = \sum_{m=-\infty}^{+\infty} w_{m+1/2}^{1}$$

$$\bar{z} \quad \longrightarrow \quad \hat{z} = \sqrt{\bar{\rho}}\,\bar{z}$$

$$A \quad \longrightarrow \quad \hat{A} = \sqrt{\bar{\rho}}\,A$$

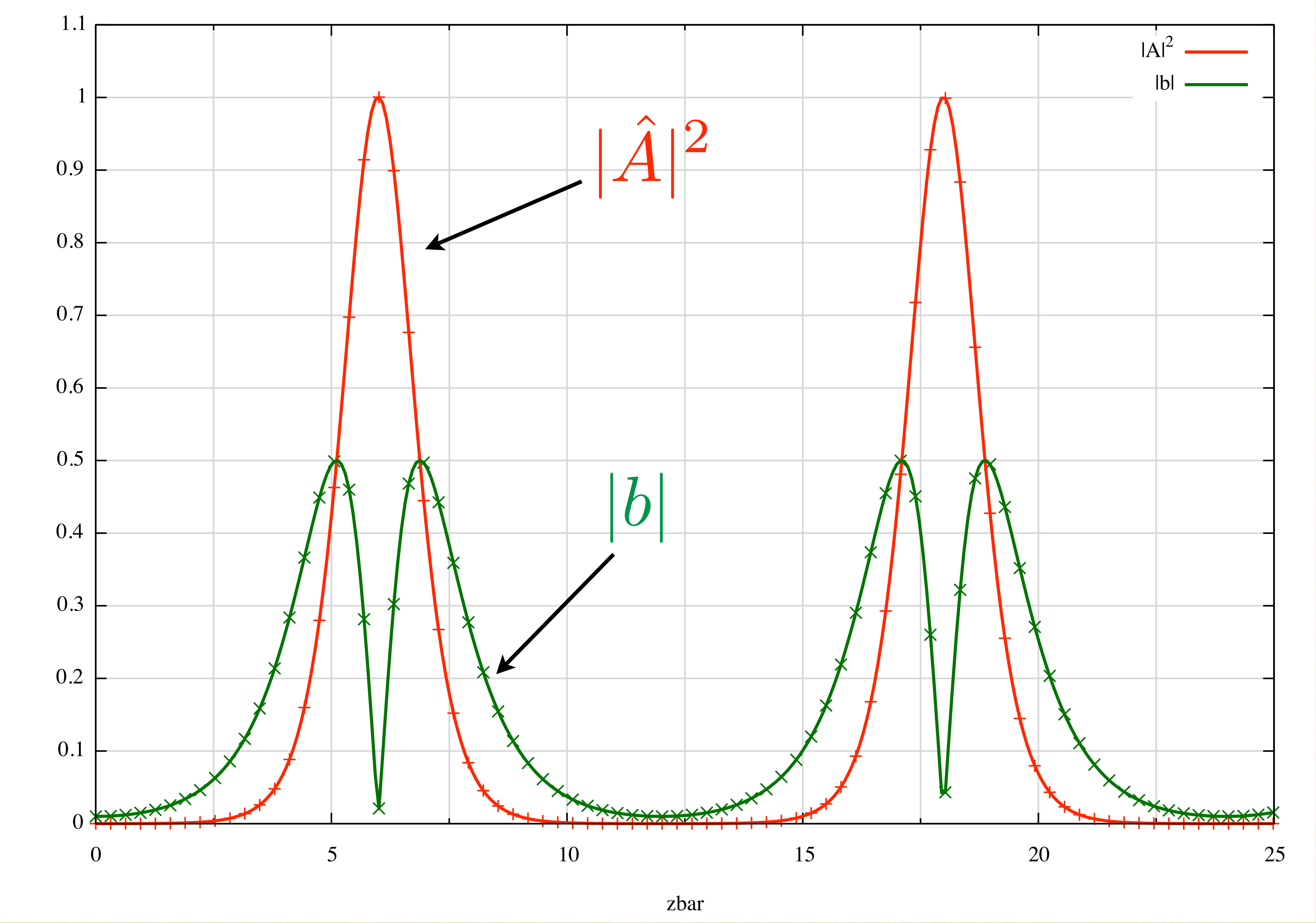
$$\bar{\rho} \quad \longrightarrow \quad \hat{\rho} = \sqrt{\bar{\rho}}\,\bar{\rho}$$

$$\bar{\delta} \quad \longrightarrow \quad \hat{\delta} = \frac{\bar{\delta}}{\sqrt{\bar{\rho}}}$$

quantum scaling

Steady state 1D in the quantum regime

$$\bar{\rho} = 0.1$$



$$\bar{\delta} = 5$$

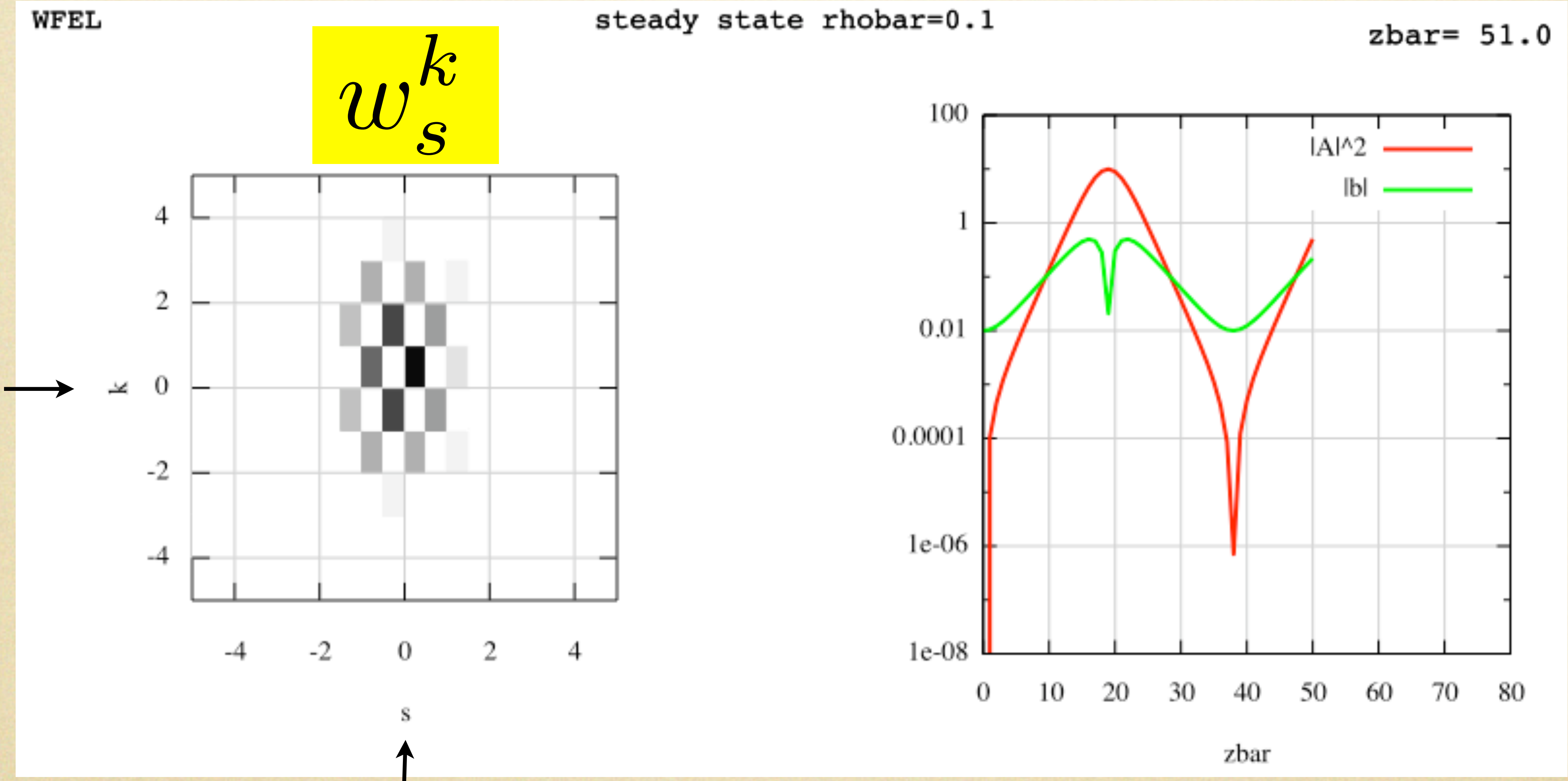
solid lines = Wigner function model

crosses = Schrödinger equation model

Population of momentum and bunching states

$$\bar{\rho} = 0.1$$

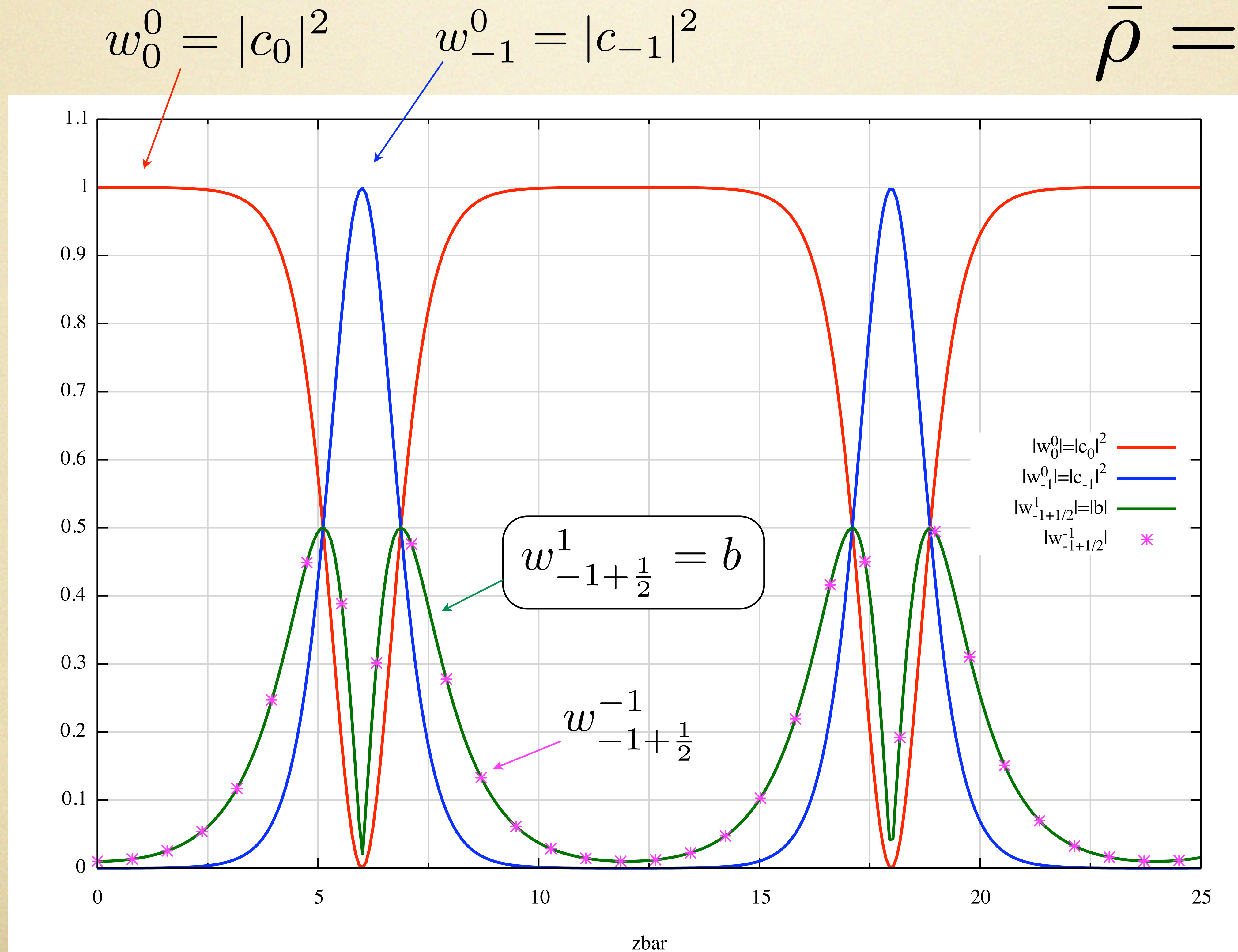
Fourier components k



half-integer index s of the small Wigner function

The 4 components of the discrete Wigner function in the two level approximation

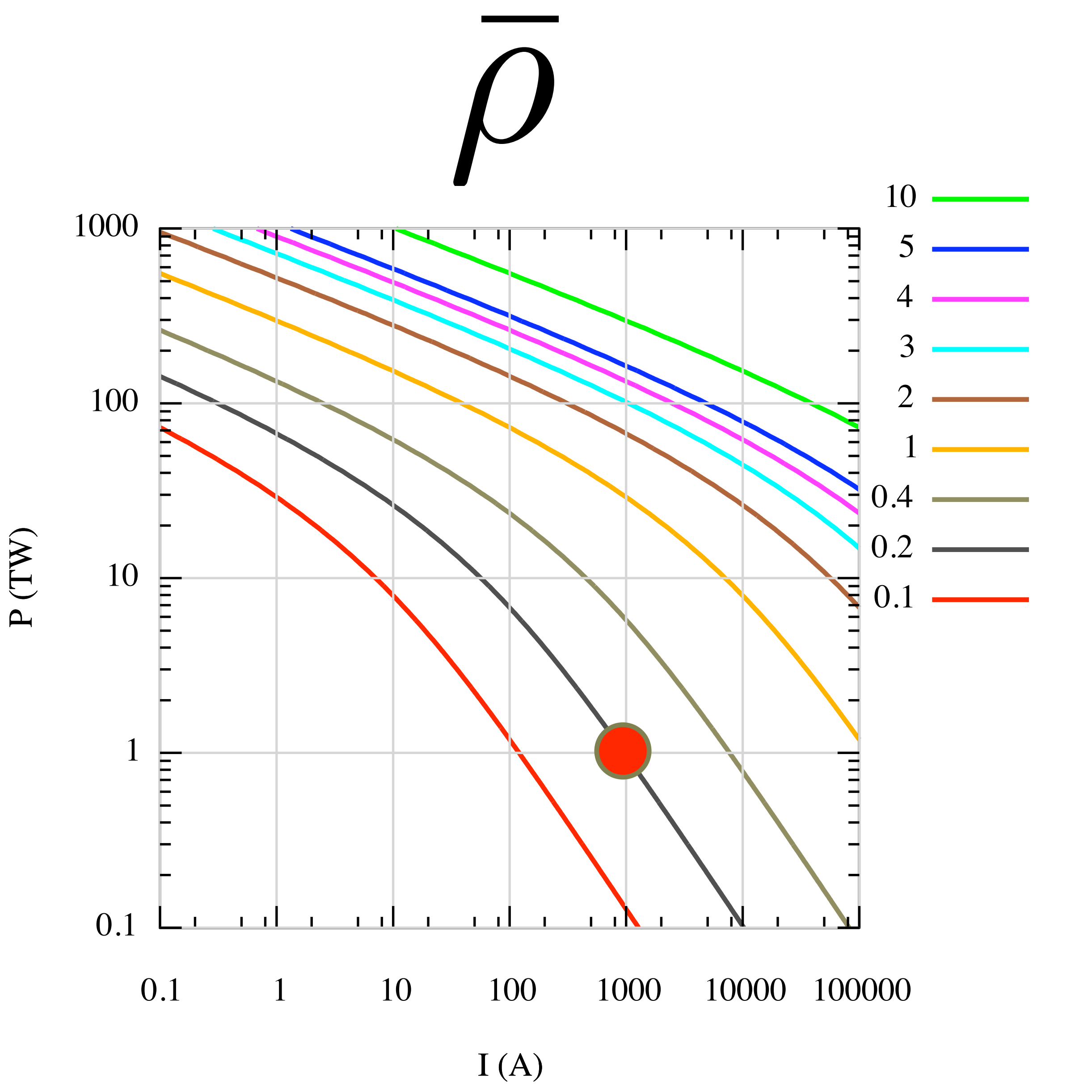
$\bar{\rho} = 0.1$

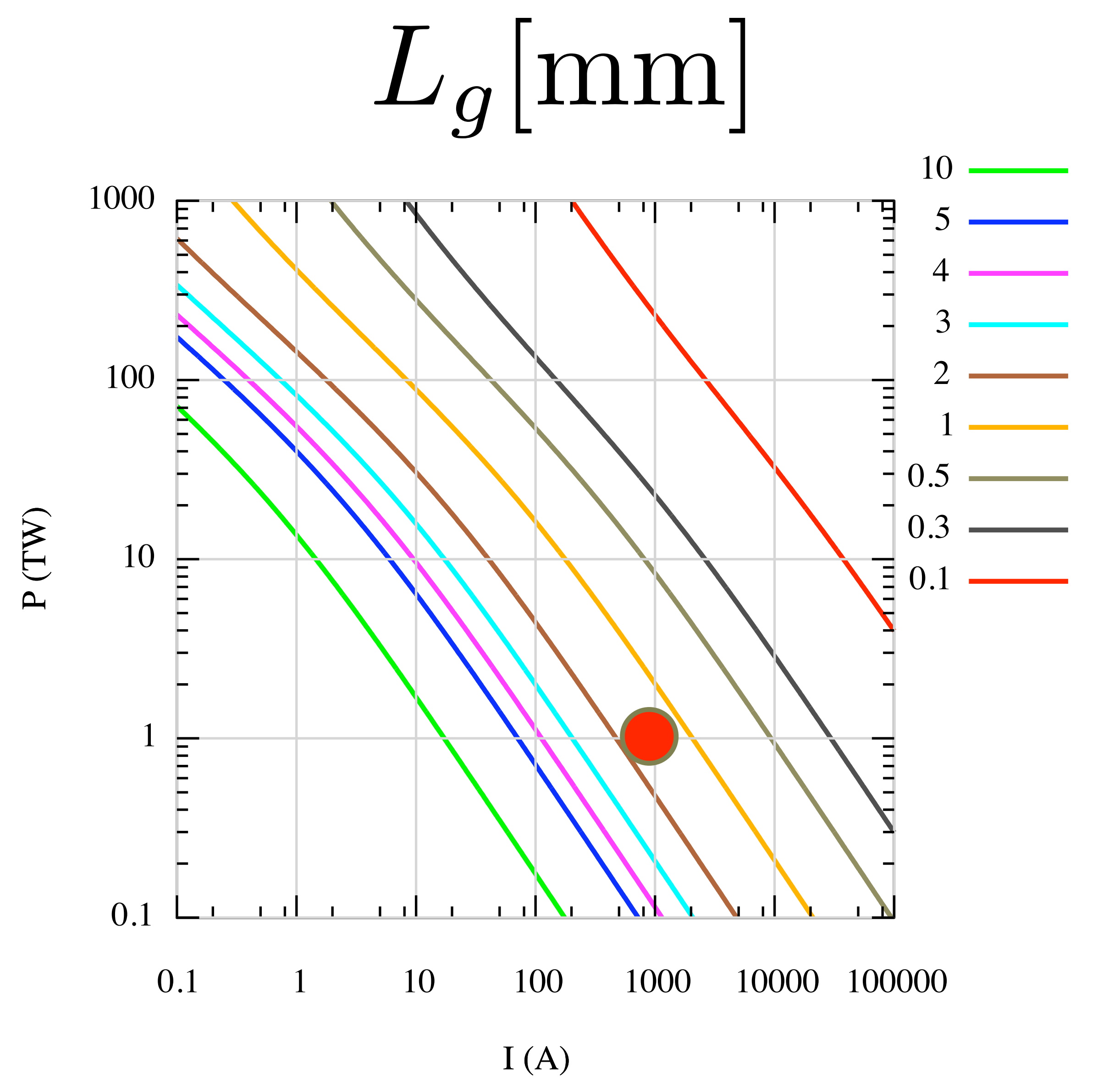


Energy output stability

Laser wiggler with Nd:glass

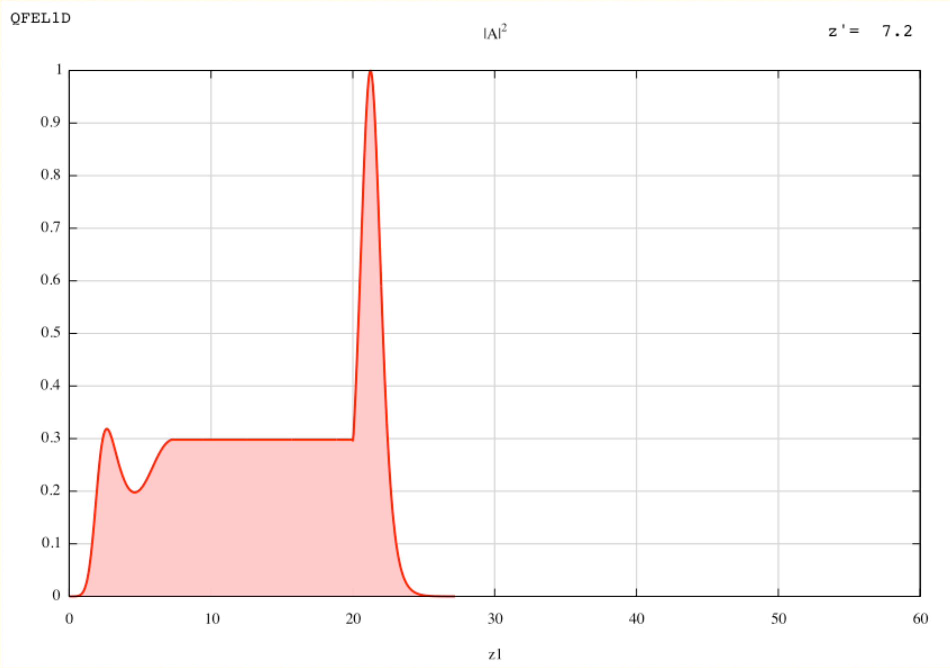
$$\lambda_L = 1\mu m$$
$$\gamma_r = 36$$
$$\sigma = 10\mu m$$
$$R = 20\mu m$$



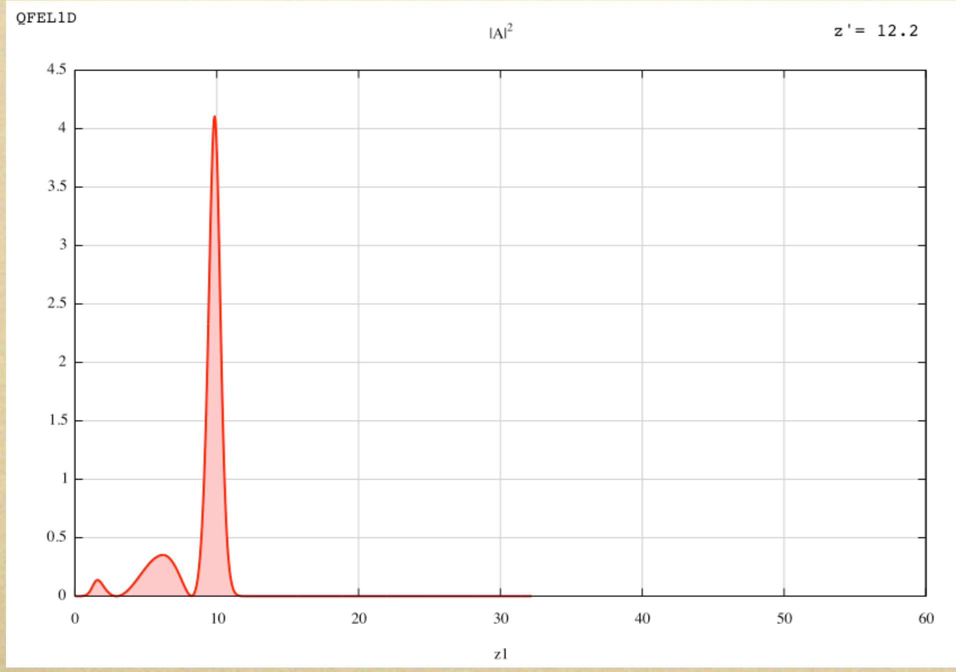


Operation mode zoo

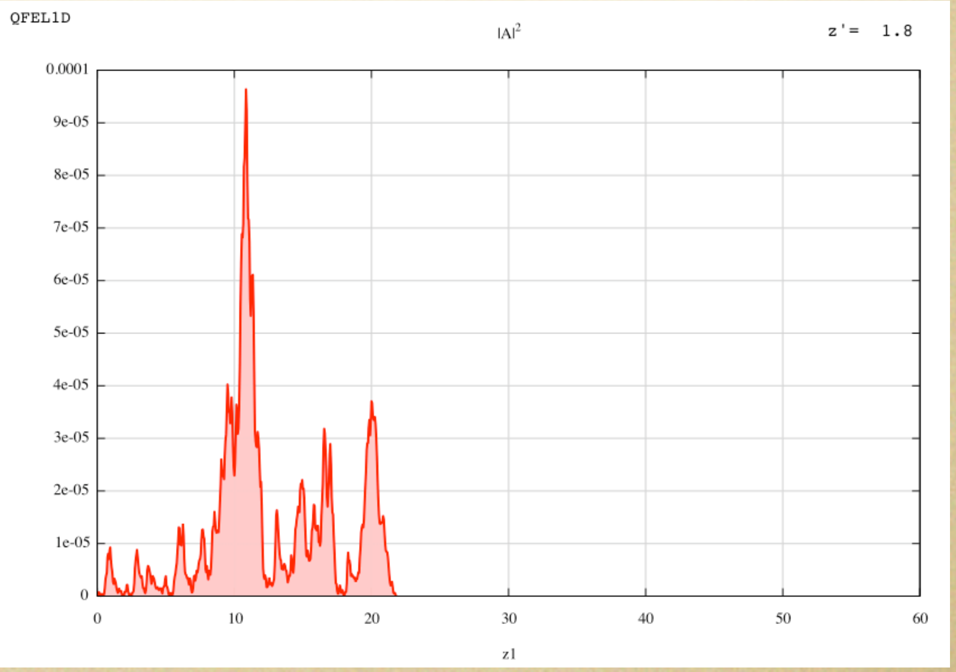
Resonant Coherent Seed



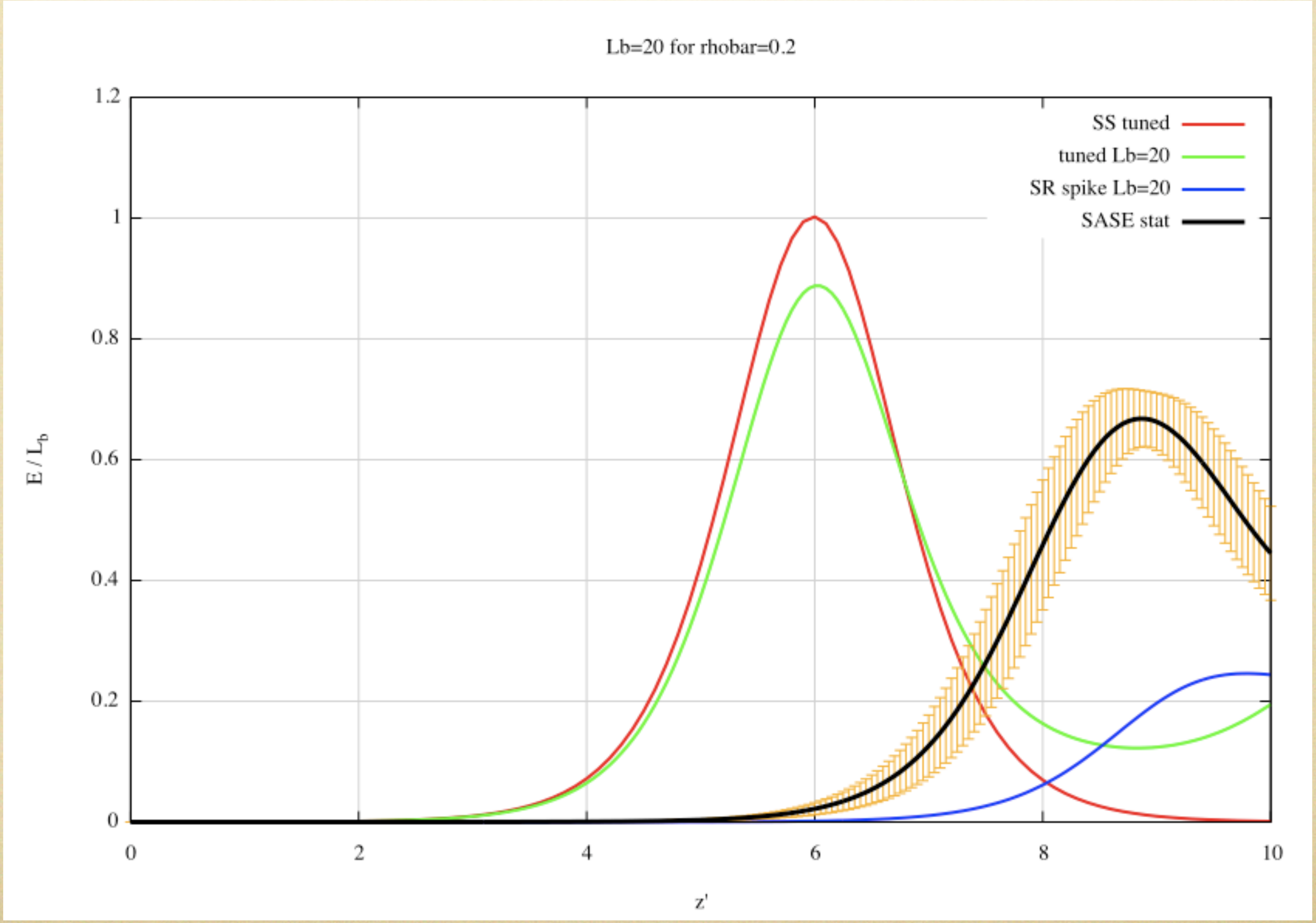
Detuned Coherent Seed



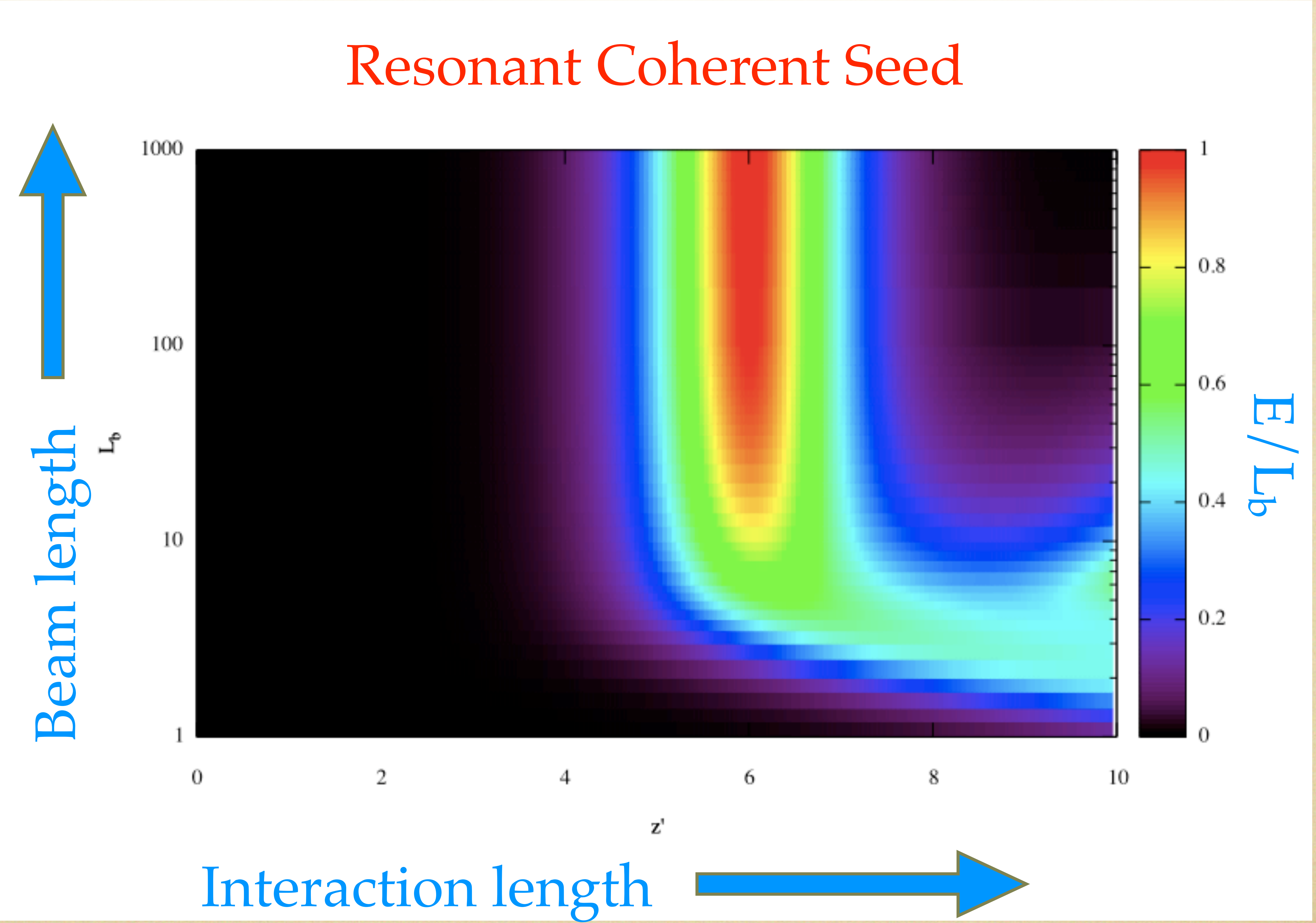
SASE



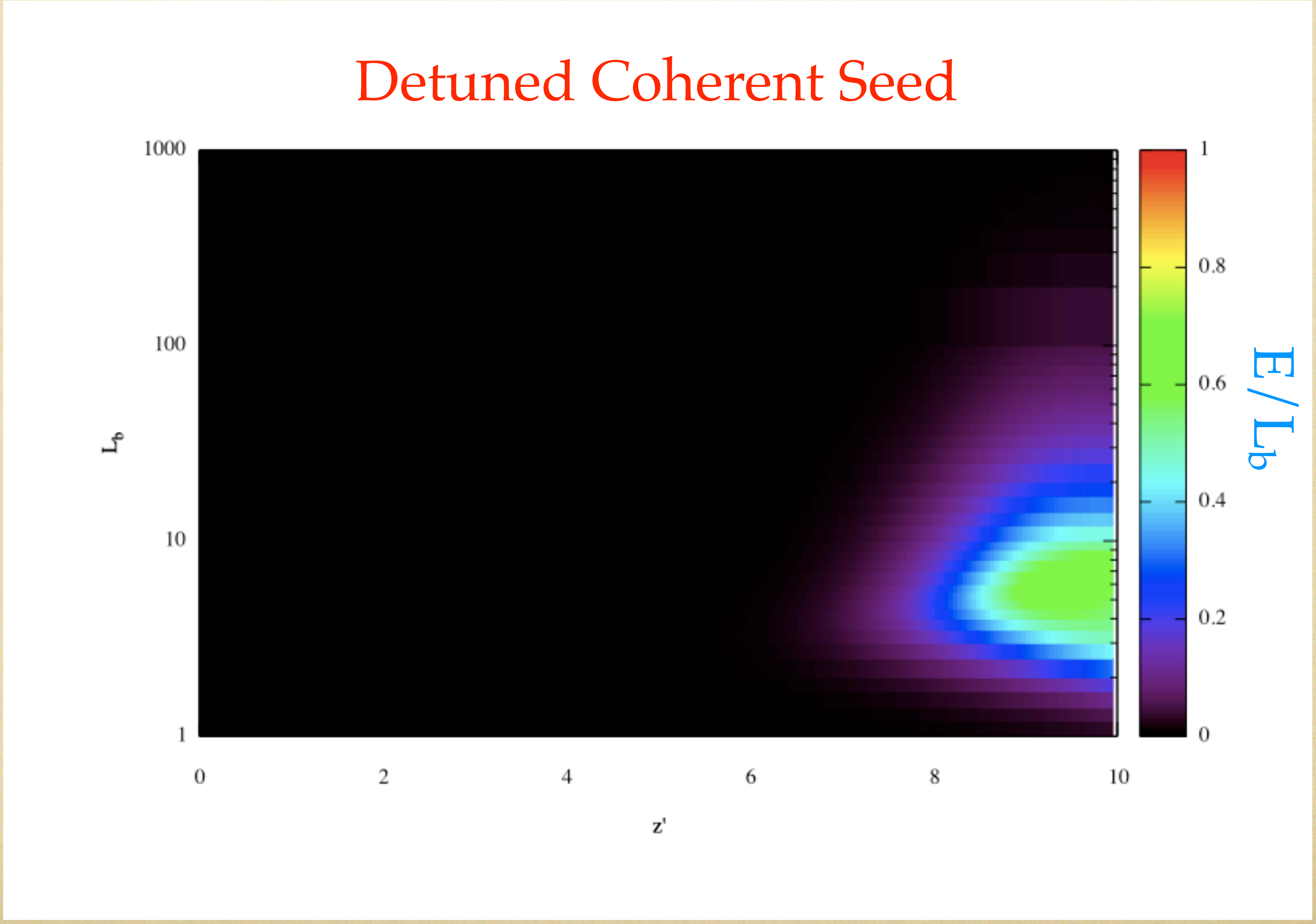
Radiation Energy



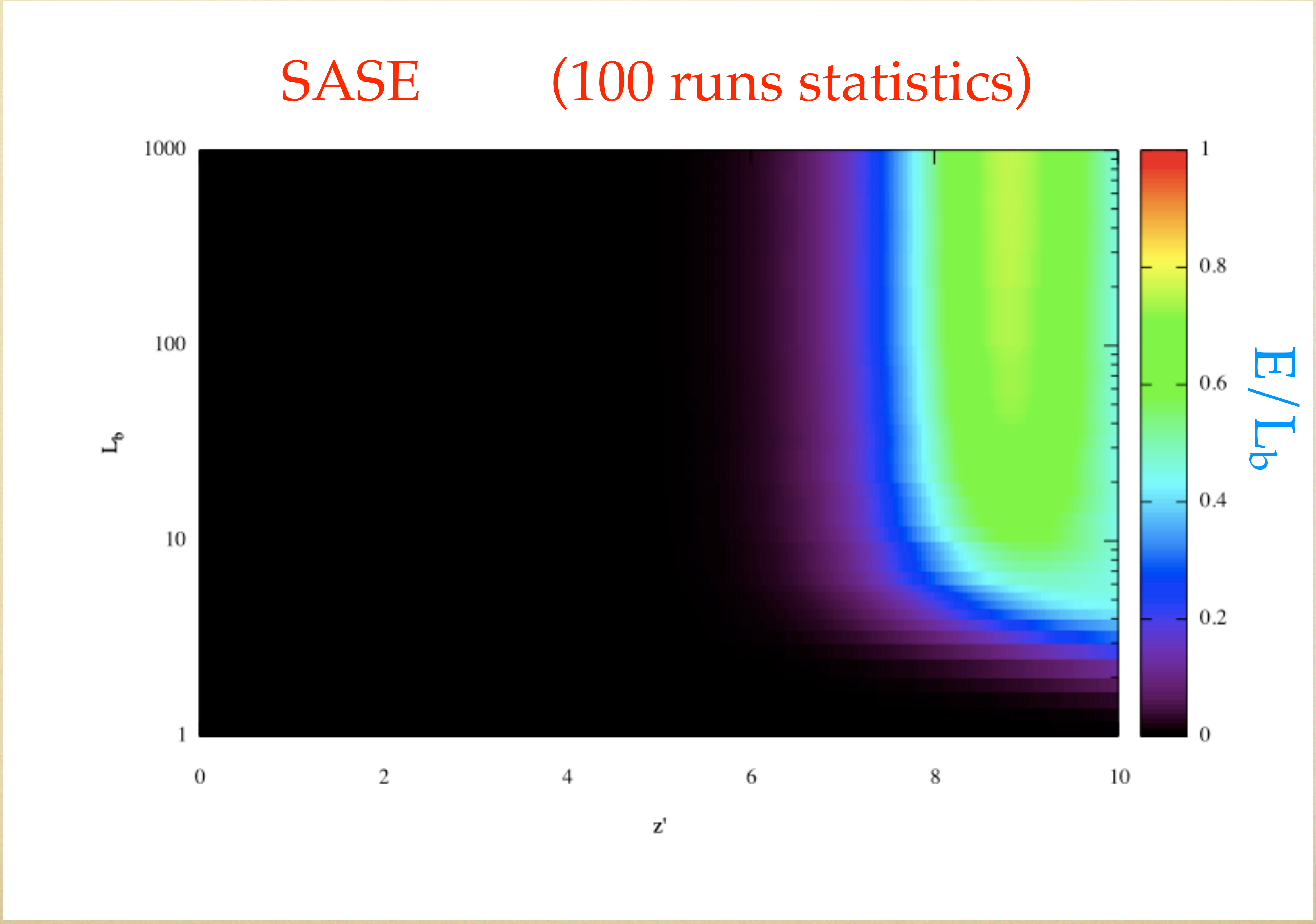
Radiation Energy



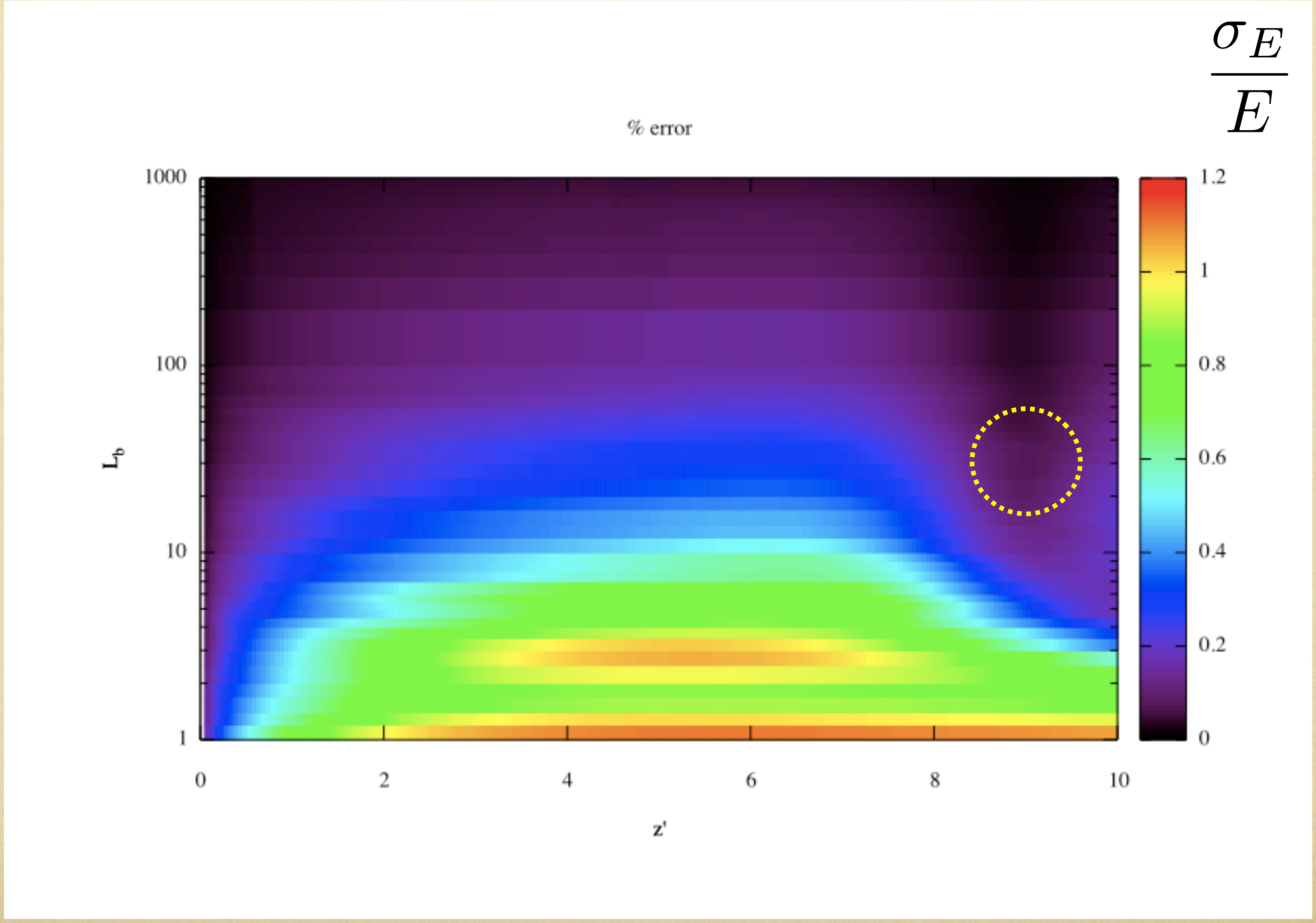
Radiation Energy



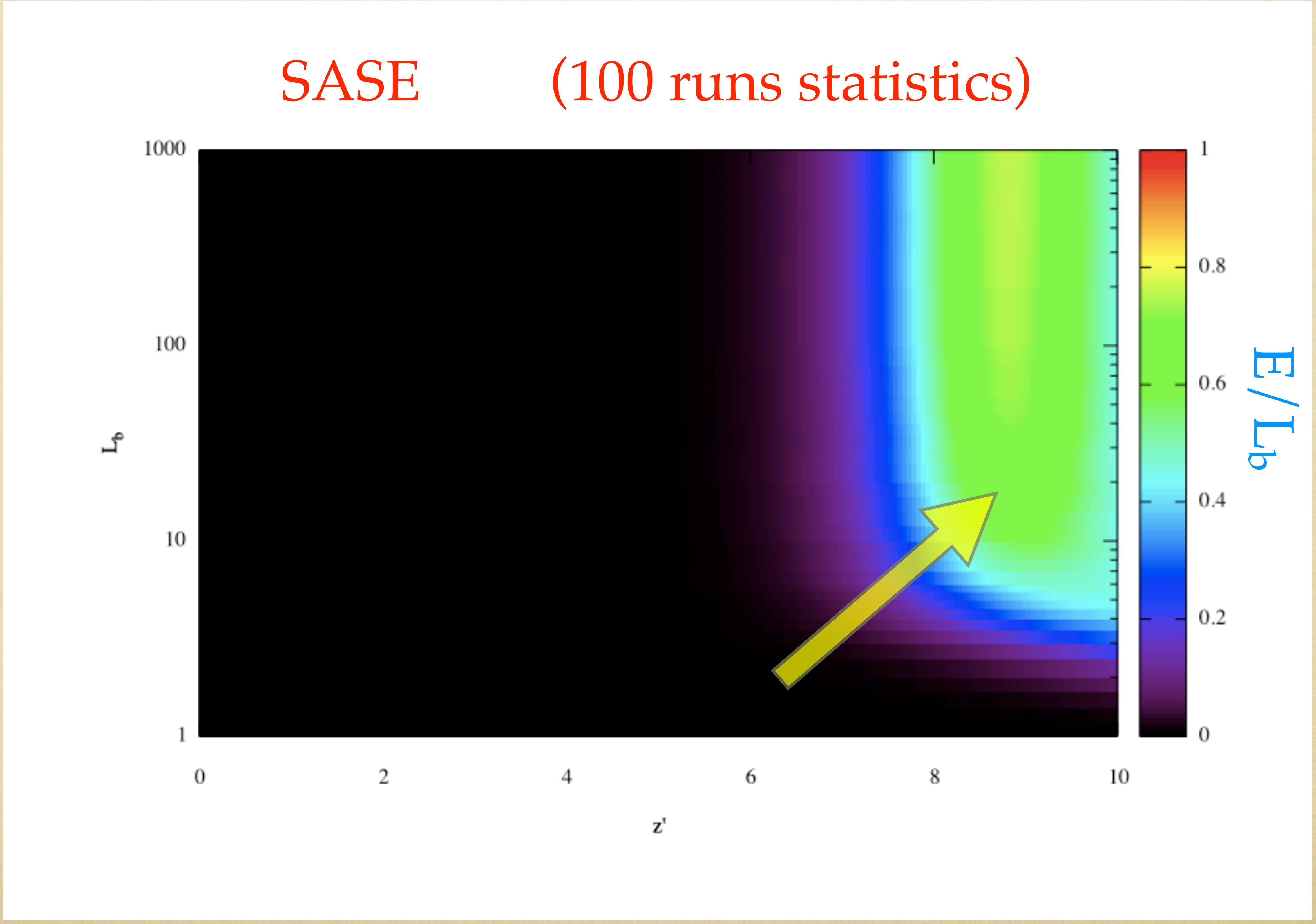
Radiation Energy



Relative energy fluctuations



Radiation Energy



Adding the transverse dynamics

Model equations in 3D

$$\frac{\partial w_s}{\partial \hat{z}} = -\left[\frac{s}{\hat{\rho}} + \hat{\delta} - \frac{\hat{b}^2}{4\hat{a}} p_\perp^2 + \frac{\xi}{2\rho\sqrt{\bar{\rho}}}(1 - |g|^2)\right] \frac{\partial w_s}{\partial \theta}$$

$$+(g^* \hat{A} e^{i\theta} + c.c.) \left(w_{s+1/2} - w_{s-1/2}\right)$$

$$-\hat{b}\, \mathbf{p}_\perp \cdot \nabla_\perp w_s$$

$$\frac{\partial \hat{A}}{\partial \hat{z}} + \frac{\partial \hat{A}}{\partial \hat{z}_1} = g \sum_m \int d^2\bar{\mathbf{p}}_\perp \int_{-\pi}^{+\pi} d\theta e^{-i\theta} w_{m+\frac{1}{2}} \quad + i\hat{a}\nabla_\perp^2 \hat{A}$$

quantum scaling

Parameters of the model

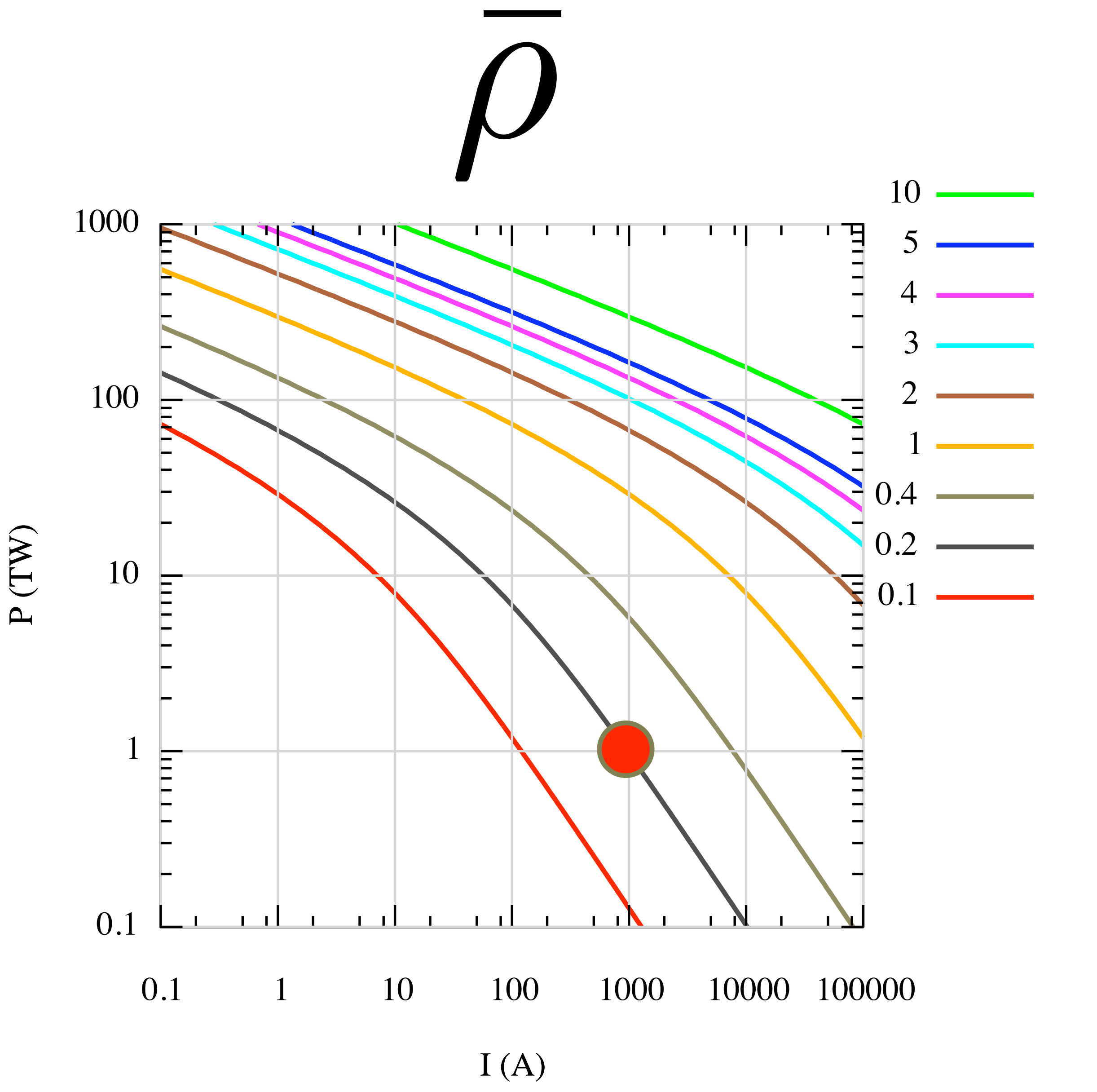
$$\bar{\rho} = \rho \, \frac{mc\gamma_r}{\hbar k} = \rho \, \gamma_r \frac{\lambda_r}{\lambda_c}$$

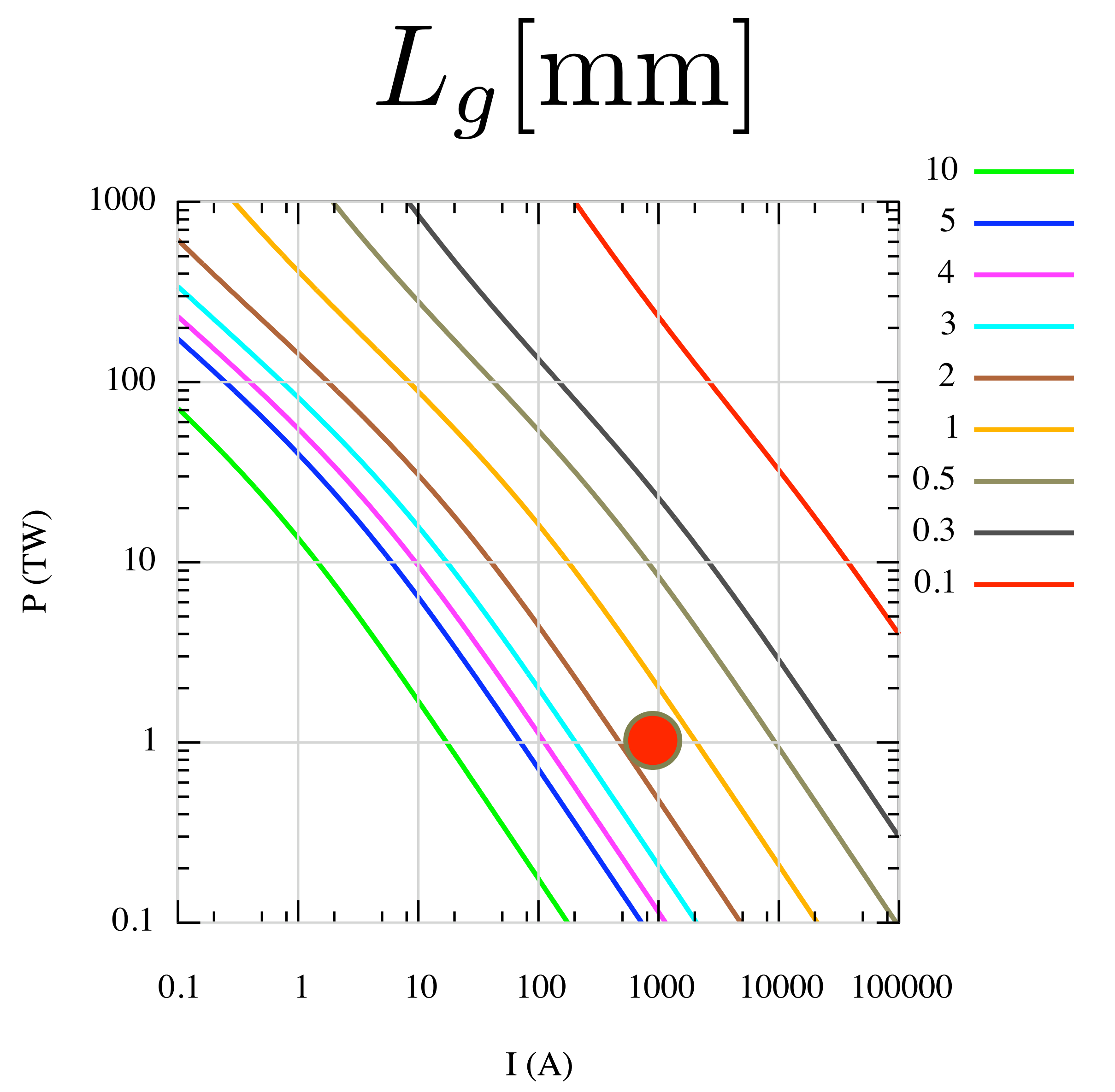
$$b = \frac{L_g}{\beta^*} \qquad a = \frac{L_g}{Z_r} \qquad X = \frac{4\pi\epsilon_n}{\gamma\lambda_r} = \frac{b}{a} = \frac{Z_r}{\beta^*}$$

$$L_g = \frac{\lambda_L}{8\pi\rho} \qquad \beta^* = \frac{\sigma^2\gamma_r}{\epsilon_n} \qquad Z_r = \frac{4\pi\sigma^2}{\lambda_r}$$

Laser wiggler with Nd:glass

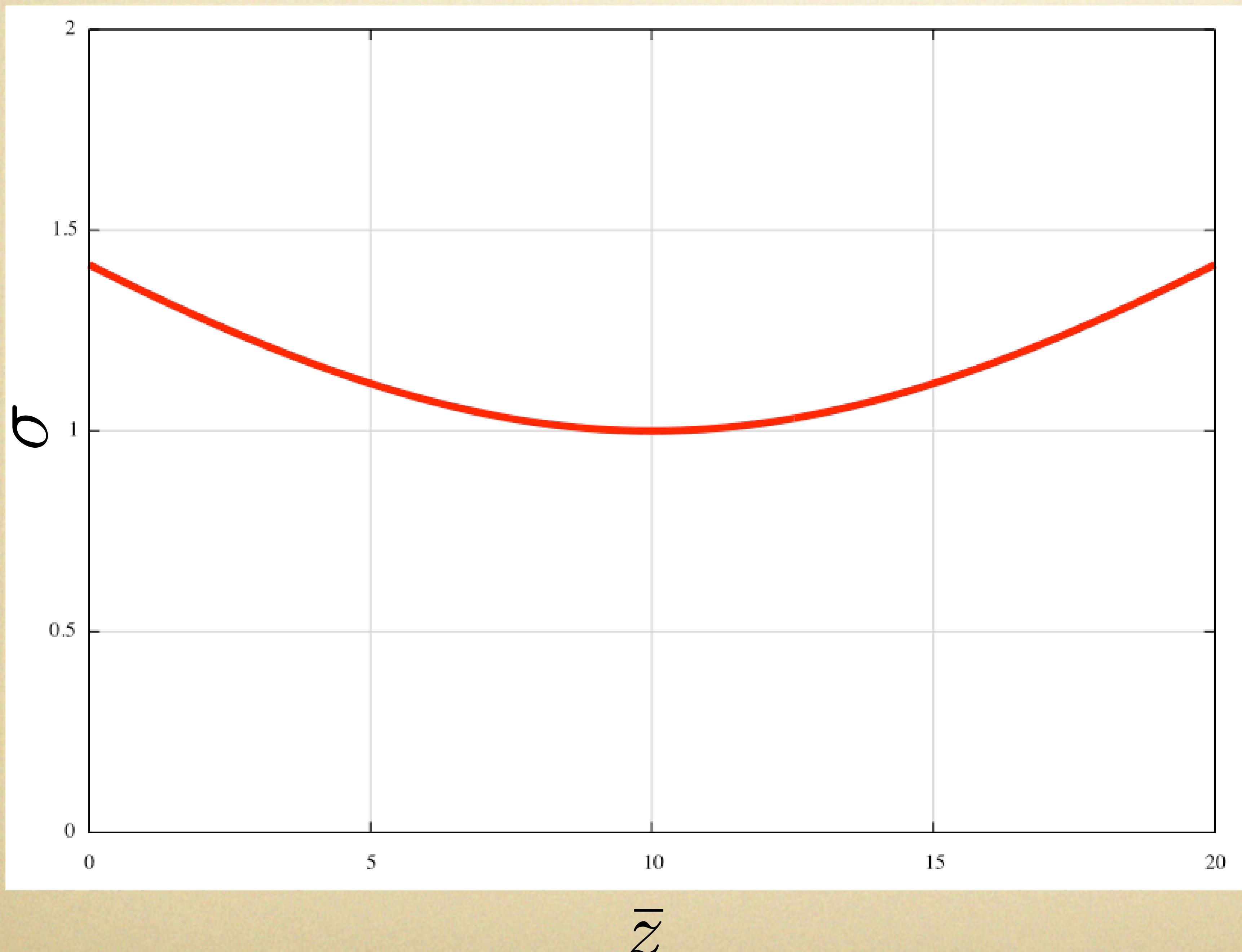
$$\lambda_L = 1\mu m$$
$$\gamma_r = 36$$
$$\sigma = 10\mu m$$
$$R = 20\mu m$$





Beam envelope evolution

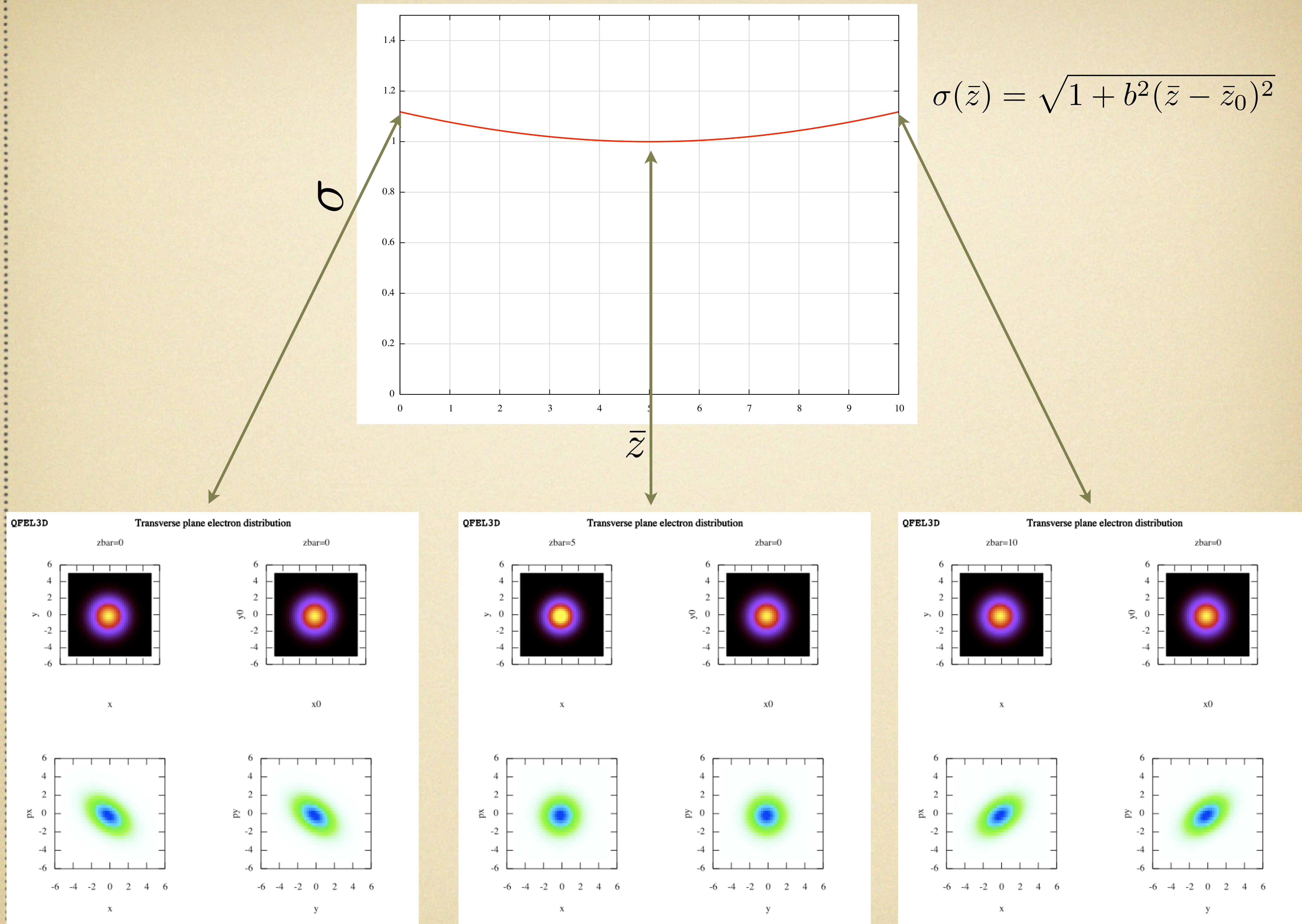
$$\sigma(\bar{z}) = \sqrt{1 + b^2(\bar{z} - \bar{z}_0)^2}$$



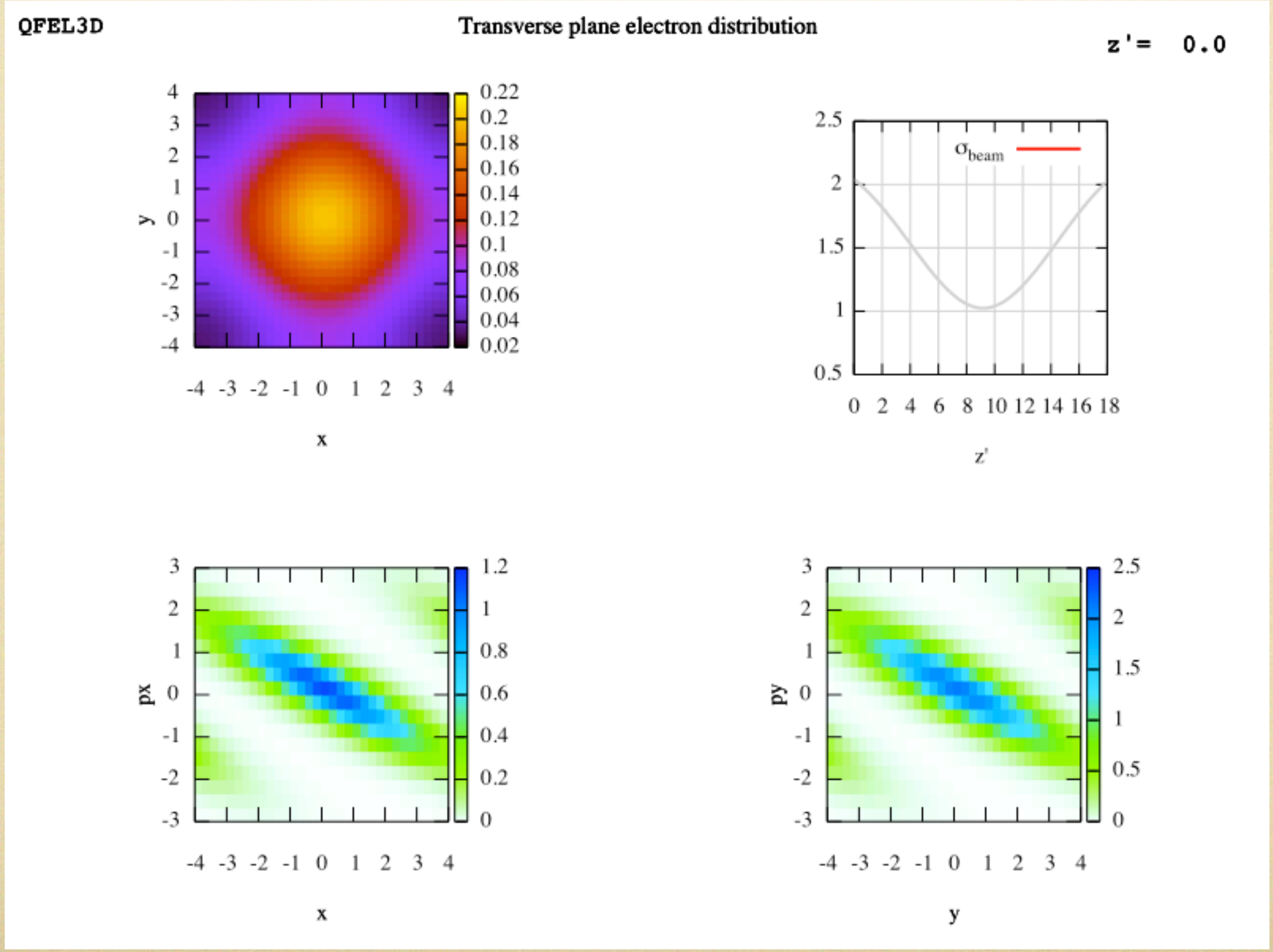
$$b = 0.1$$

$$\beta^* = 10L_g$$

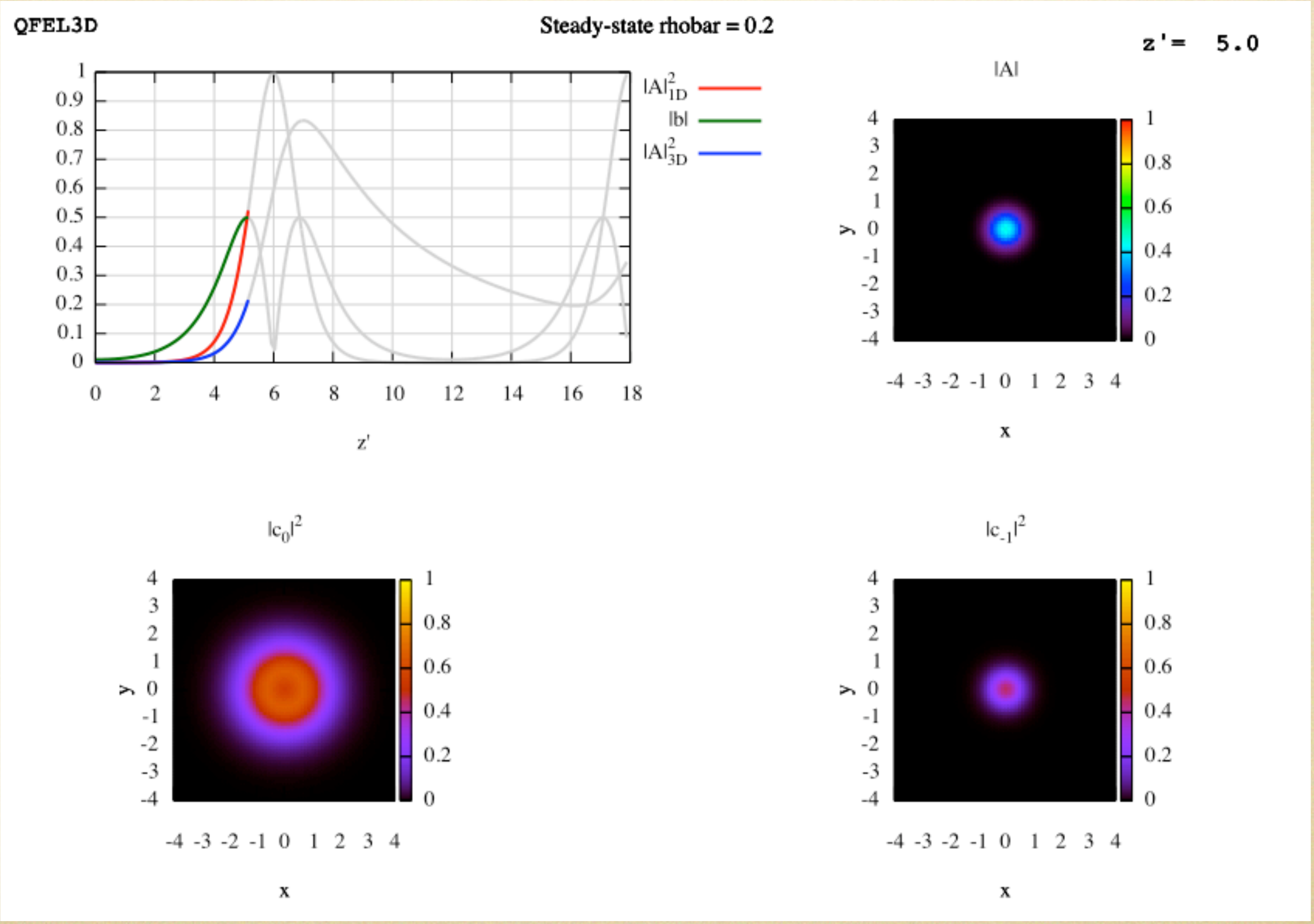
$$\sigma(\bar{z}) = \sqrt{1 + b^2(\bar{z} - \bar{z}_0)^2}$$



Beam transport

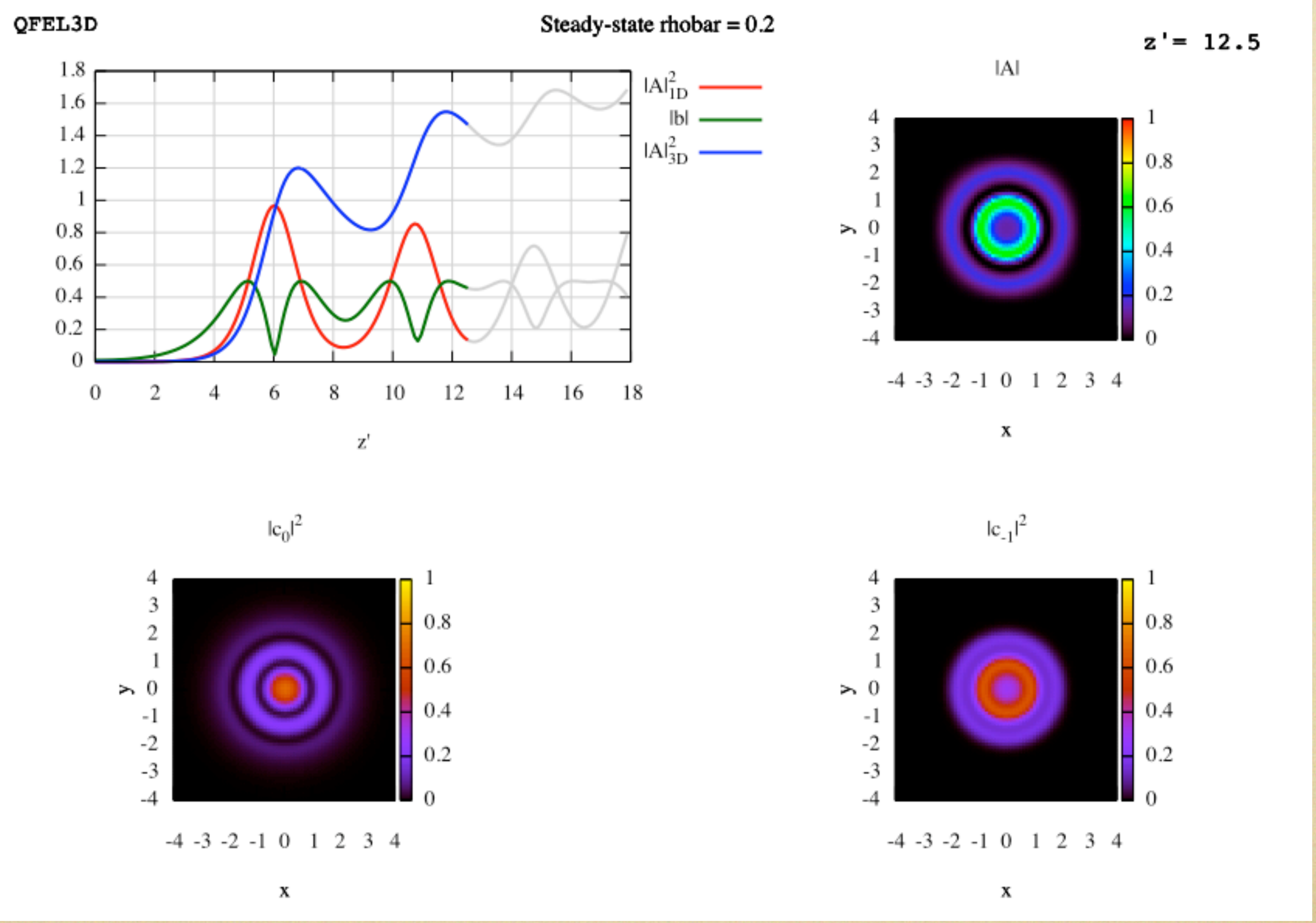


No emitt No diffr

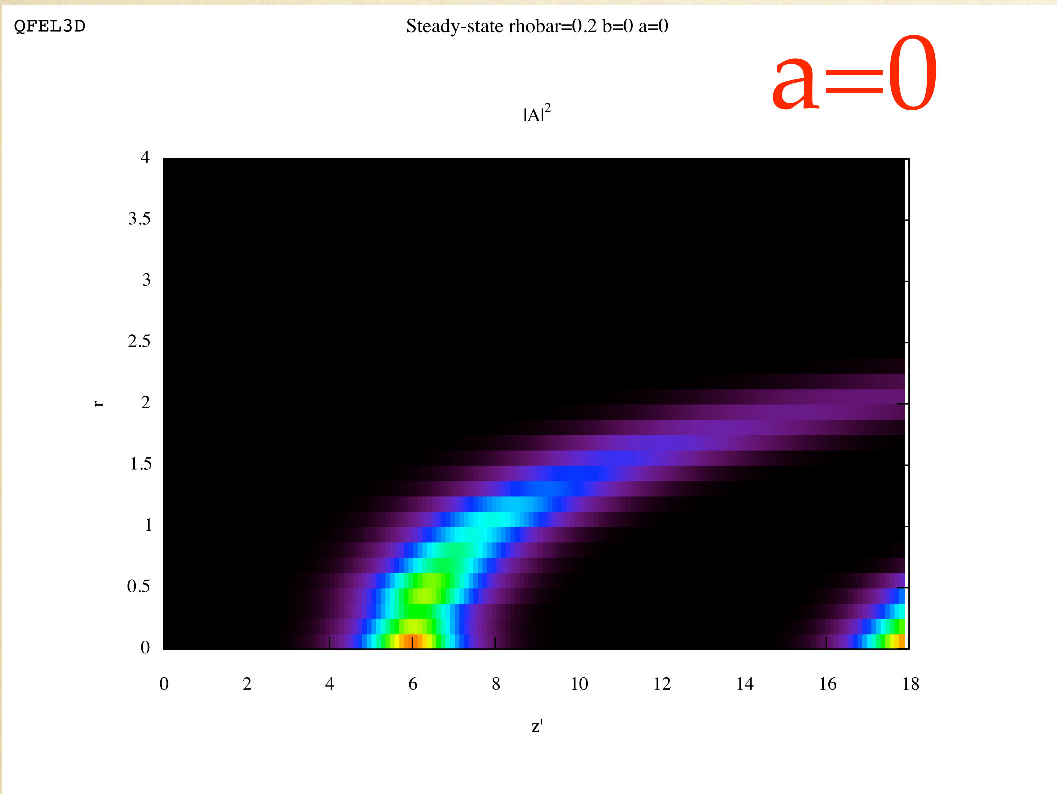


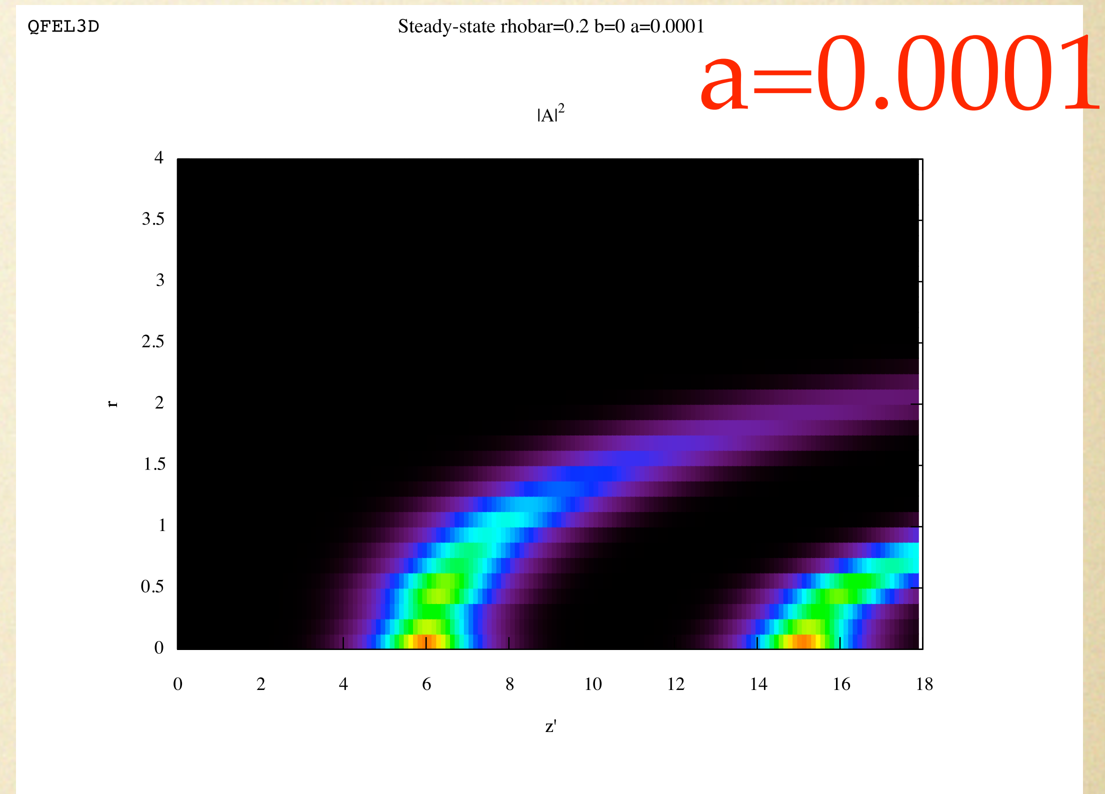
Diffraction only

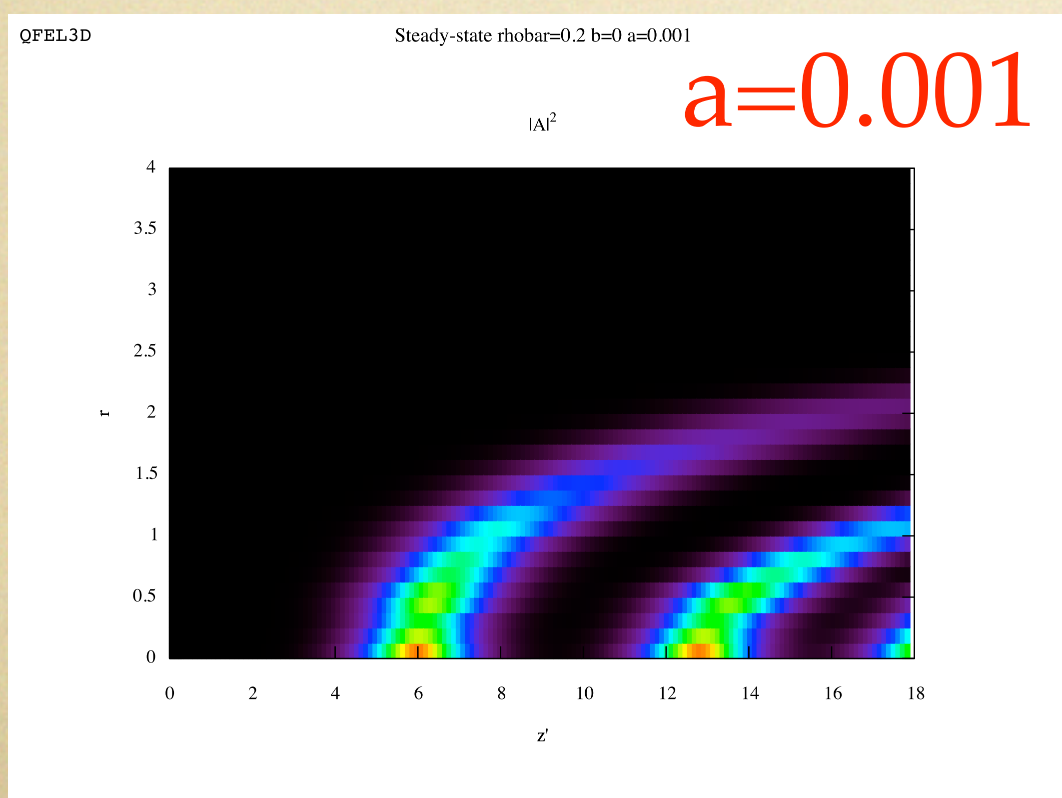
a=0.01

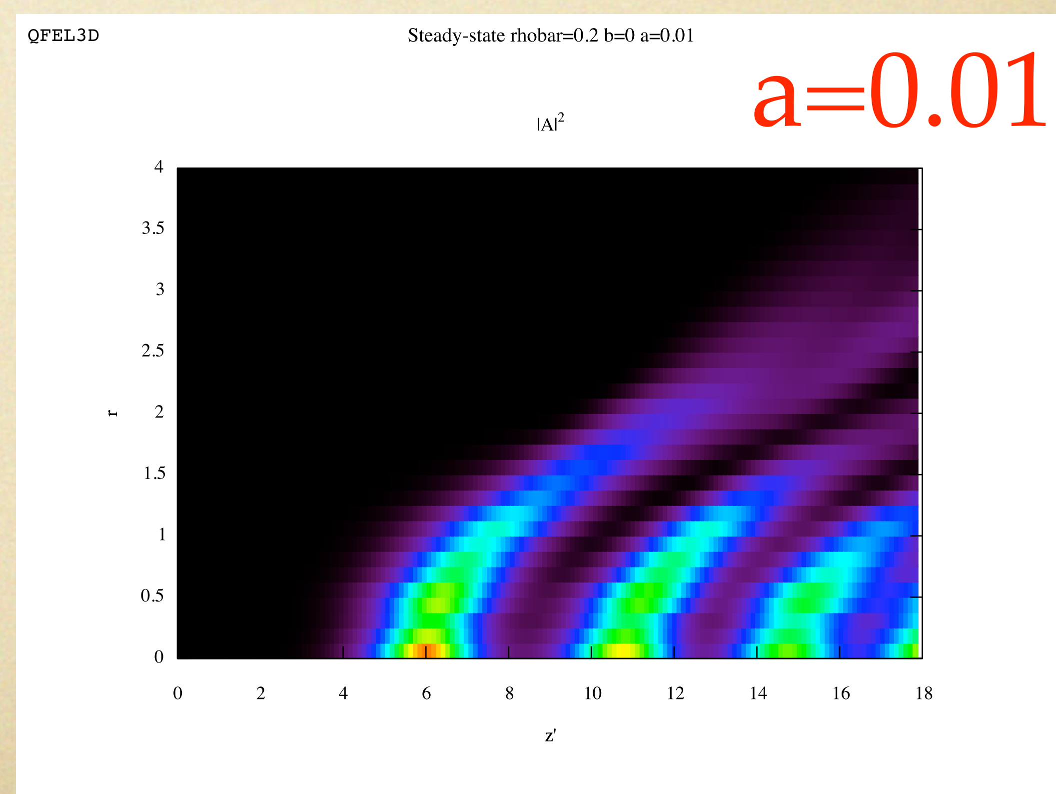


Diffraction only



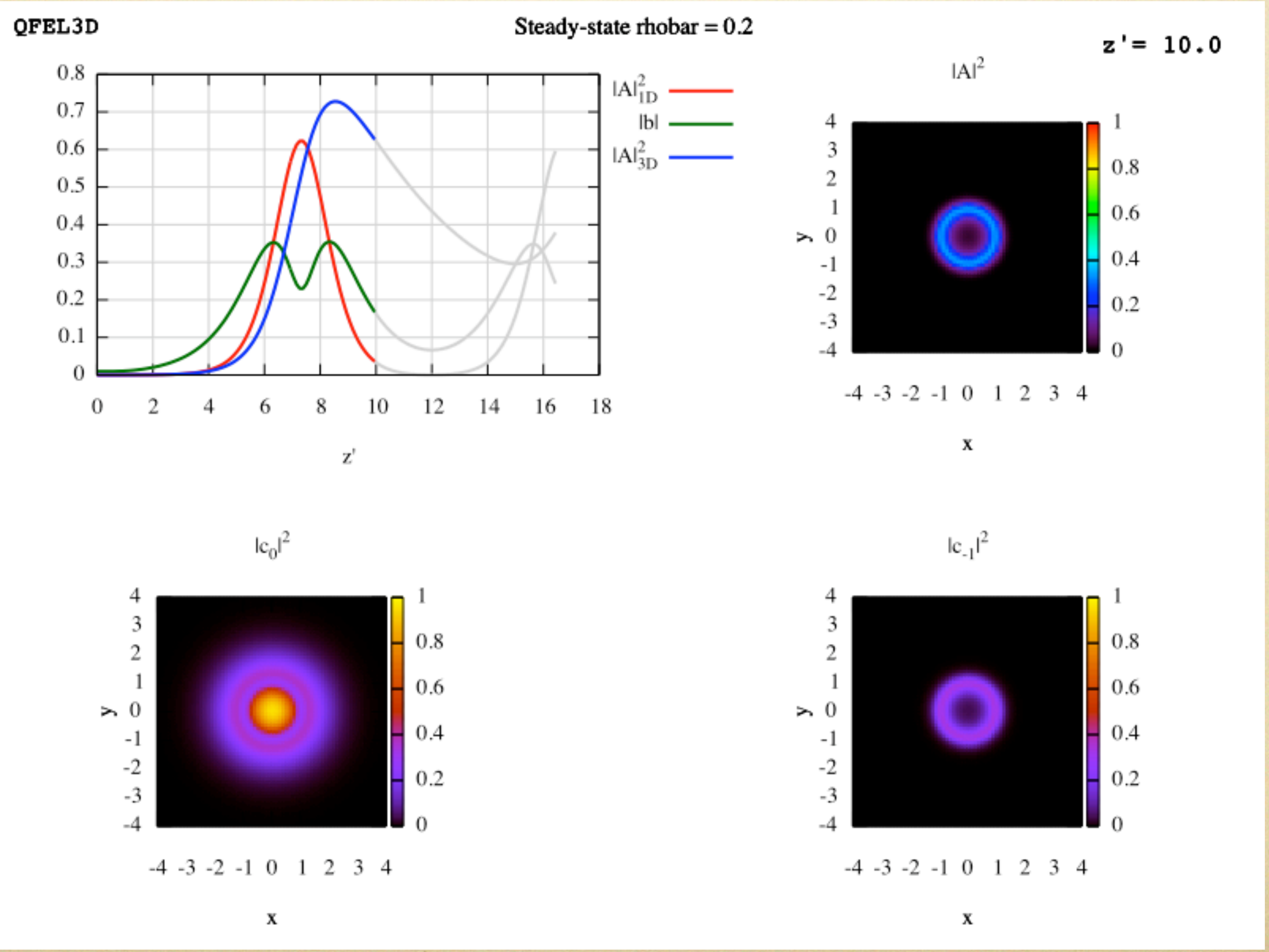






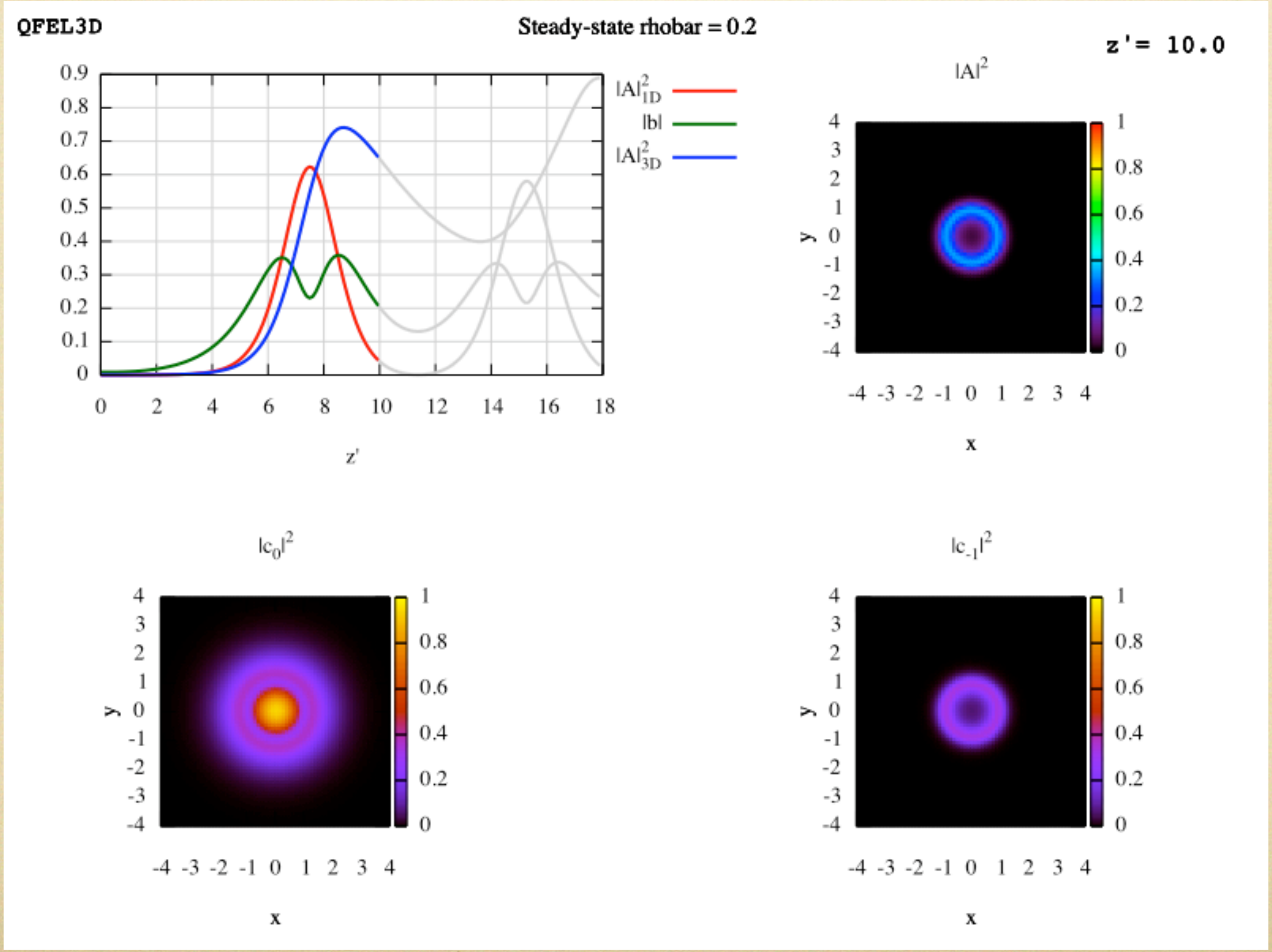
St.St.(emitt + diffr) $\frac{\hat{b}^2}{4\hat{a}} = 0.25$

X=100 b=0.01 a=0.0001



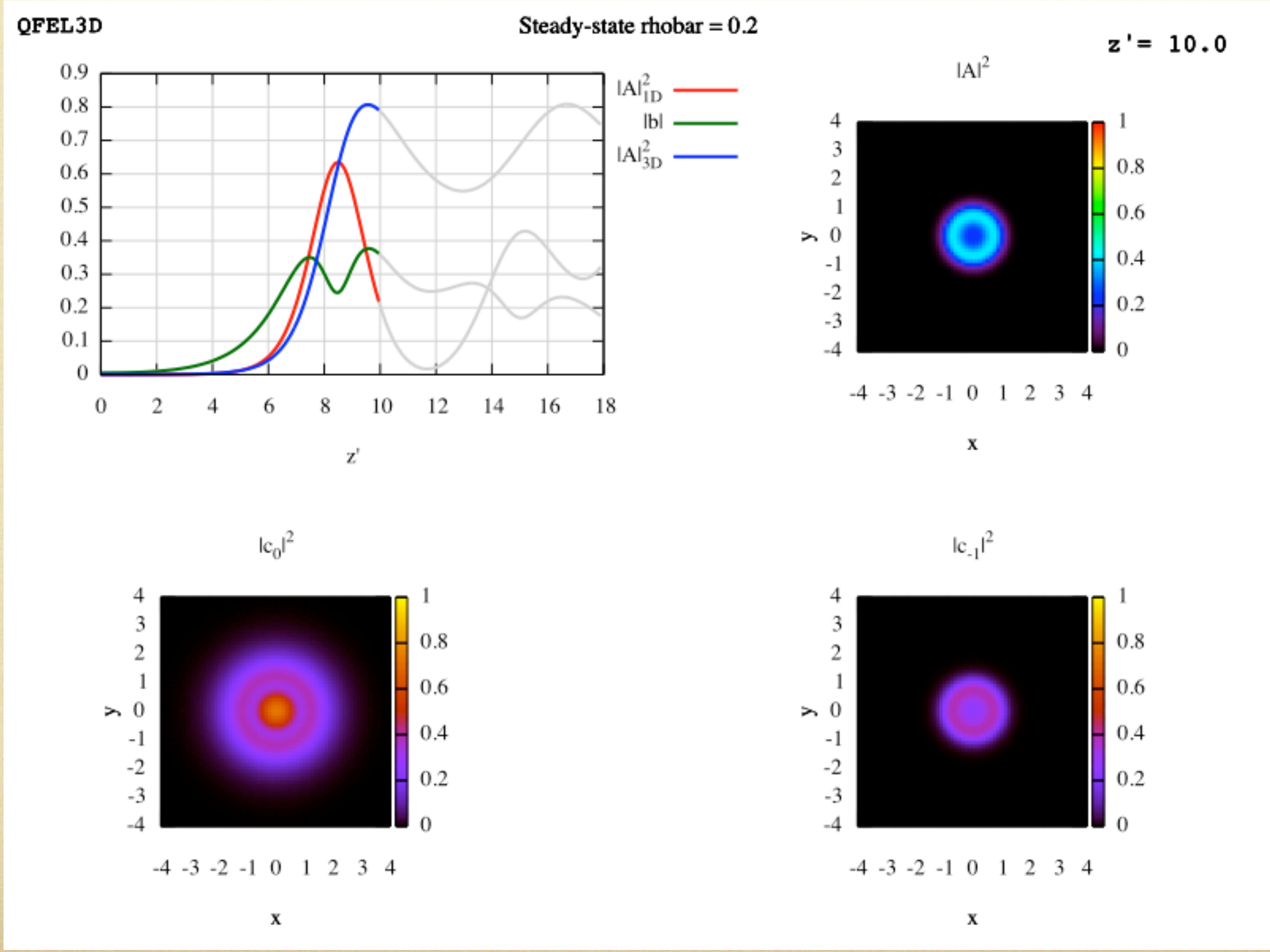
$$\frac{\hat{b}^2}{4\hat{a}} = 0.25$$

X=50 b=0.02 a=0.0004



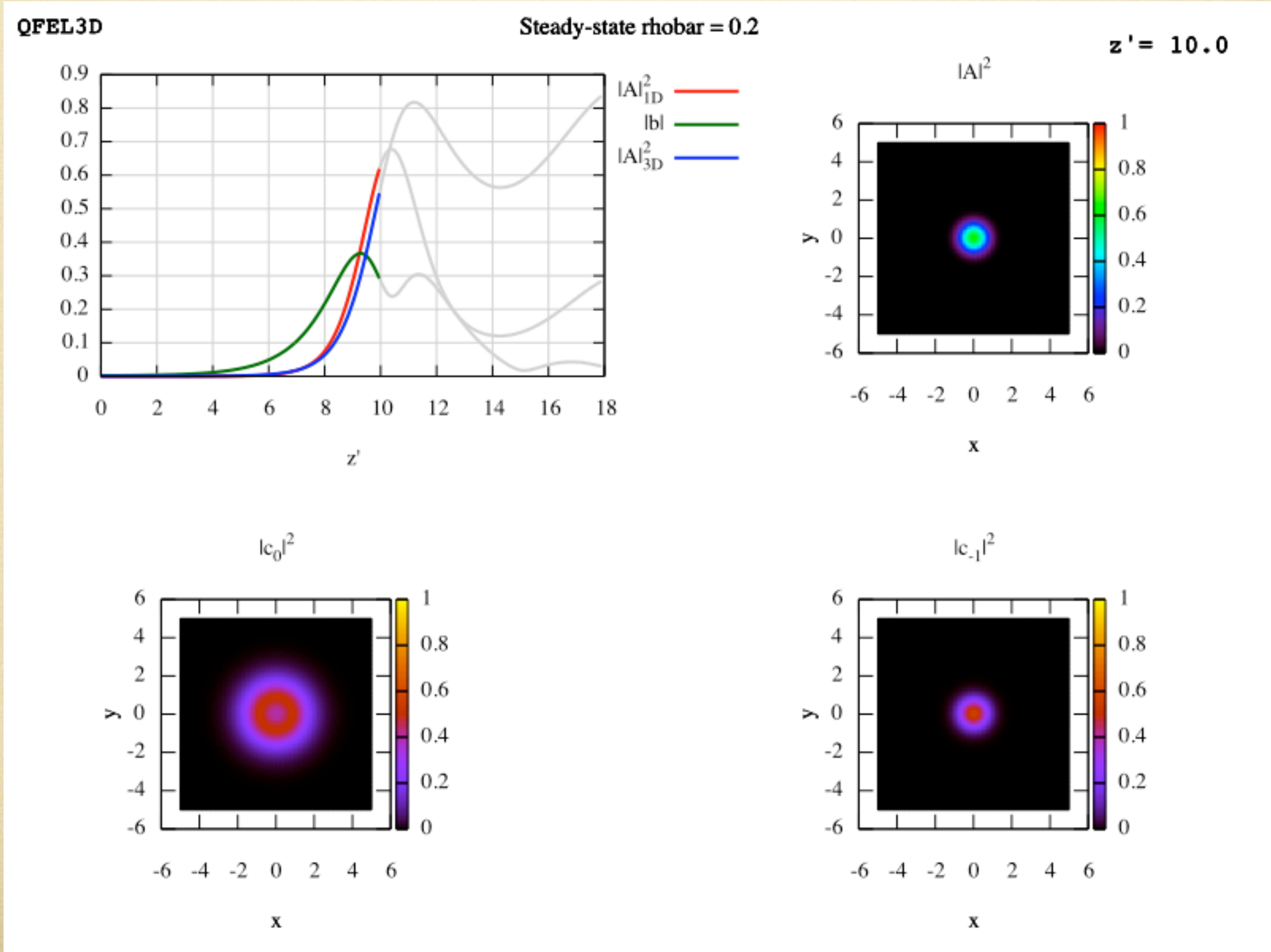
$$\frac{\hat{b}^2}{4\hat{a}} = 0.25$$

X=20 b=0.05 a=0.0025

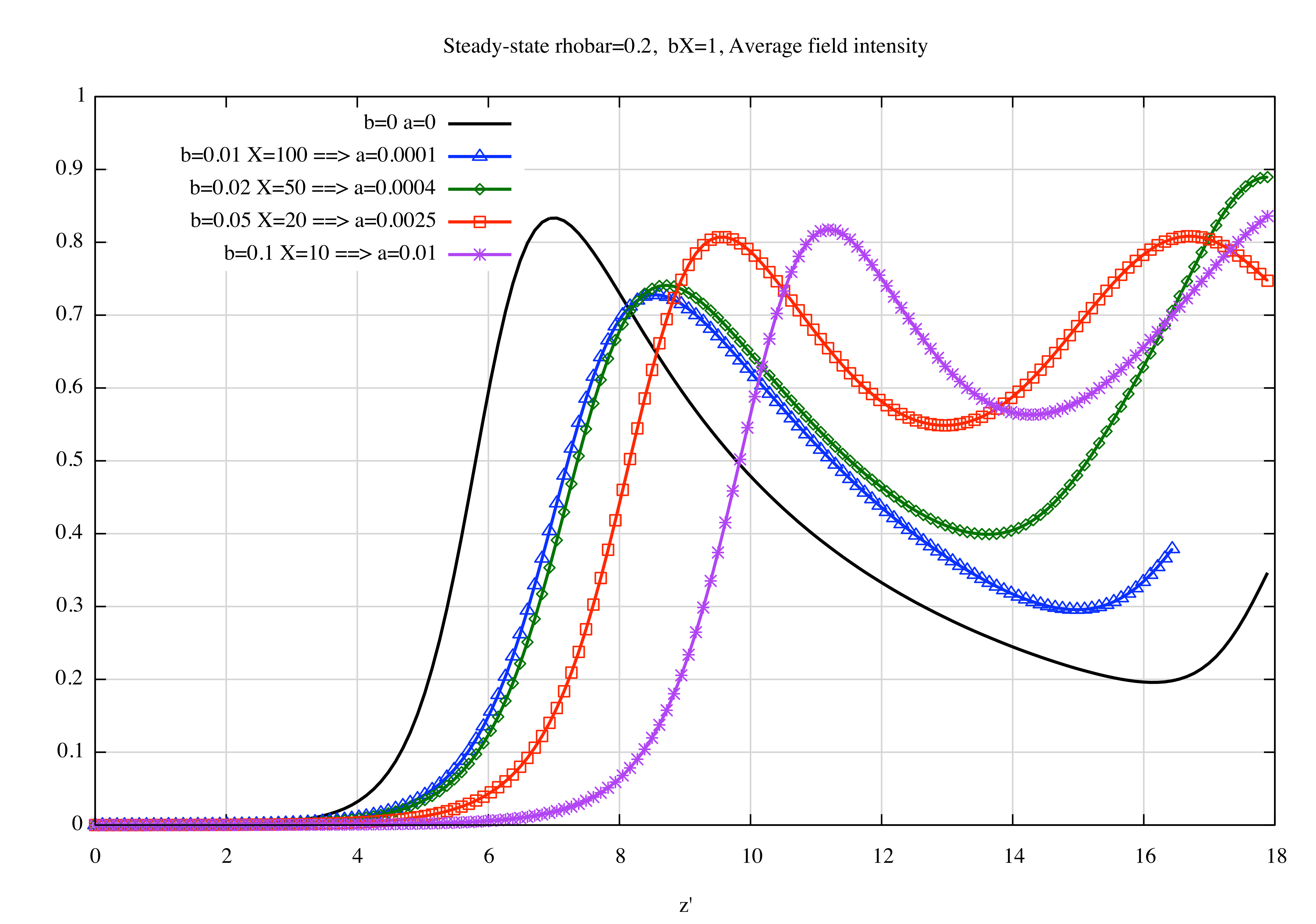


$$\frac{\hat{b}^2}{4\hat{a}} = 0.25$$

X=10 b=0.1 a=0.01



St.St.(emitt + diffr) $\frac{\hat{b}^2}{4\hat{a}} = 0.25$



Full 3D model eqns

$$\frac{\partial w_s}{\partial \hat{z}} = -\left[\frac{s}{\hat{\rho}} + \hat{\delta} - \hat{X}\, p_\perp^2 + \hat{\xi}(1 - |g|^2)\right] \frac{\partial w_s}{\partial \theta}$$

$$+(g^* \hat{A} e^{i\theta} + c.c.) \left(w_{s+1/2} - w_{s-1/2}\right)$$

$$-\hat{b}\, \mathbf{p}_\perp \cdot \nabla_\perp w_s$$

$$\frac{\partial \hat{A}}{\partial \hat{z}} + \frac{\partial \hat{A}}{\partial \hat{z}_1} = g \sum_m \int d^2\bar{\mathbf{p}}_\perp \int_{-\pi}^{+\pi} d\theta e^{-i\theta} w_{m+\frac{1}{2}} \quad + i\hat{a}\nabla_\perp^2 \hat{A}$$

quantum scaling

Scaling

classical scaling   quantum scaling

$$\bar{\rho} \quad \longrightarrow \quad \hat{\rho} = \sqrt{\bar{\rho}}\bar{\rho} = \bar{\rho}^{3/2}$$

$$\bar{z} \quad \longrightarrow \quad \hat{z} = \sqrt{\bar{\rho}}\bar{z}$$

$$\bar{z}_1 \quad \longrightarrow \quad \hat{z}_1 = \sqrt{\bar{\rho}}\bar{z}_1$$

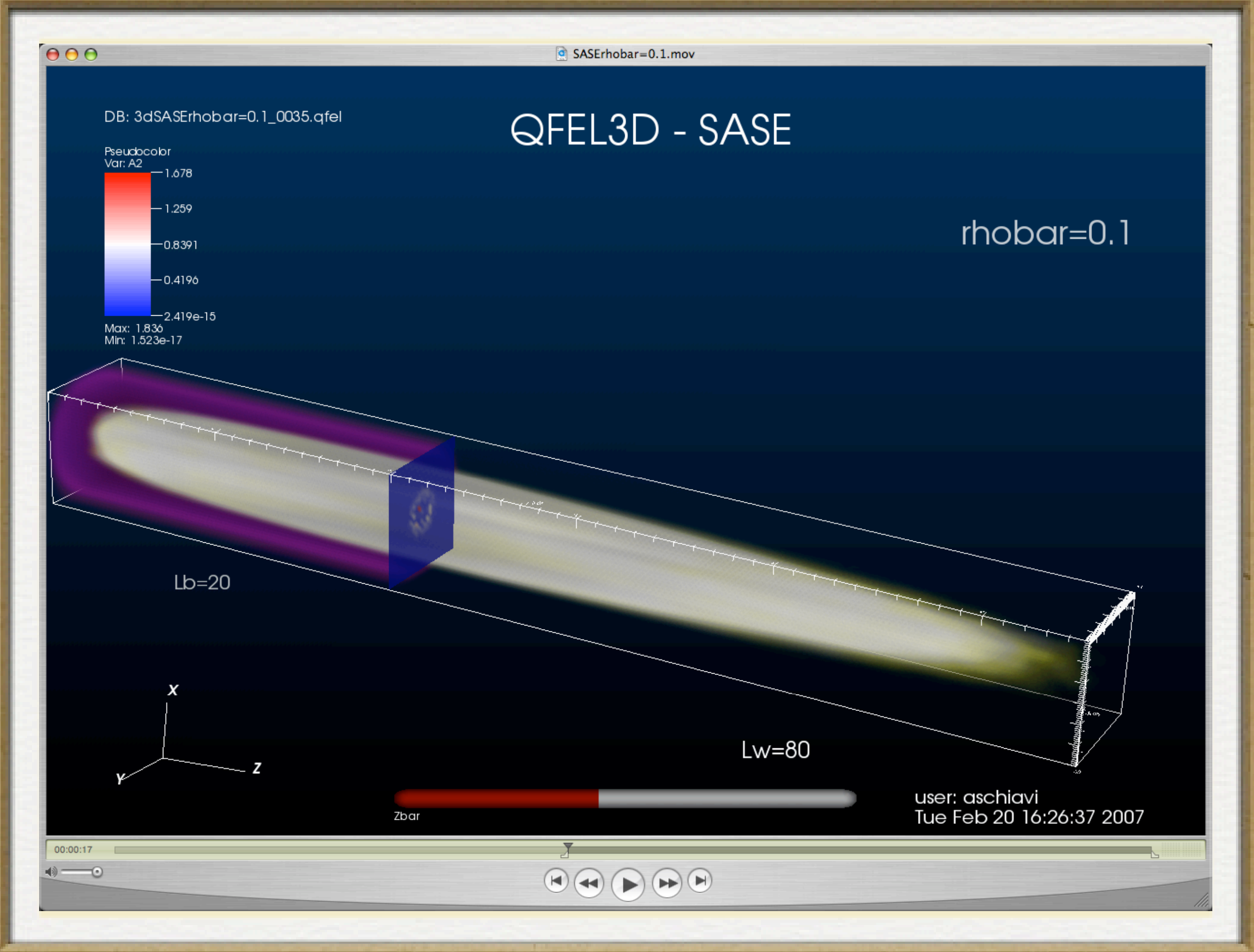
$$A \quad \longrightarrow \quad \hat{A} = \sqrt{\bar{\rho}}A$$

$$\bar{\delta} \quad \longrightarrow \quad \hat{\delta} = \bar{\delta}/\sqrt{\bar{\rho}}$$

$$a \quad \longrightarrow \quad \hat{a} = a/\sqrt{\bar{\rho}}$$

$$b \quad \longrightarrow \quad \hat{b} = b/\sqrt{\bar{\rho}}$$

unchanged: $X, \xi, g, x_\perp, p_\perp, \ldots$



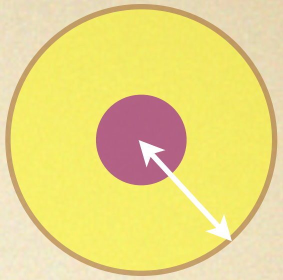
The full monty

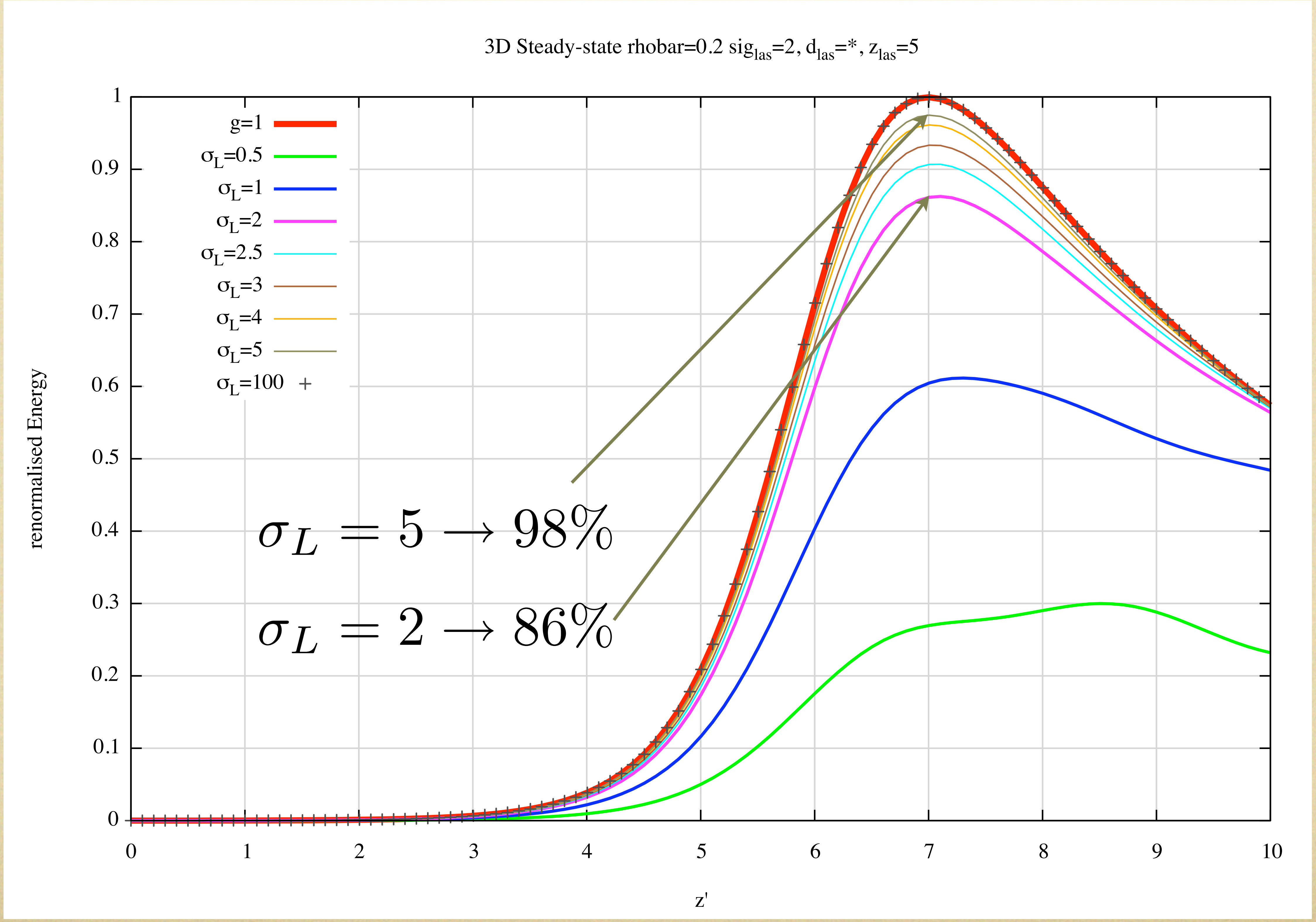
Laser wiggler profile

$$g(r,\bar{z}) = \frac{1}{1 - i(\bar{z} - \bar{z}_0)/\bar{Z}_L} \exp(-\frac{r^2}{4\sigma_L^2[1 - i(\bar{z} - \bar{z}_0)/\bar{Z}_L]})$$

3D steady state

LASER FOCAL SPOT



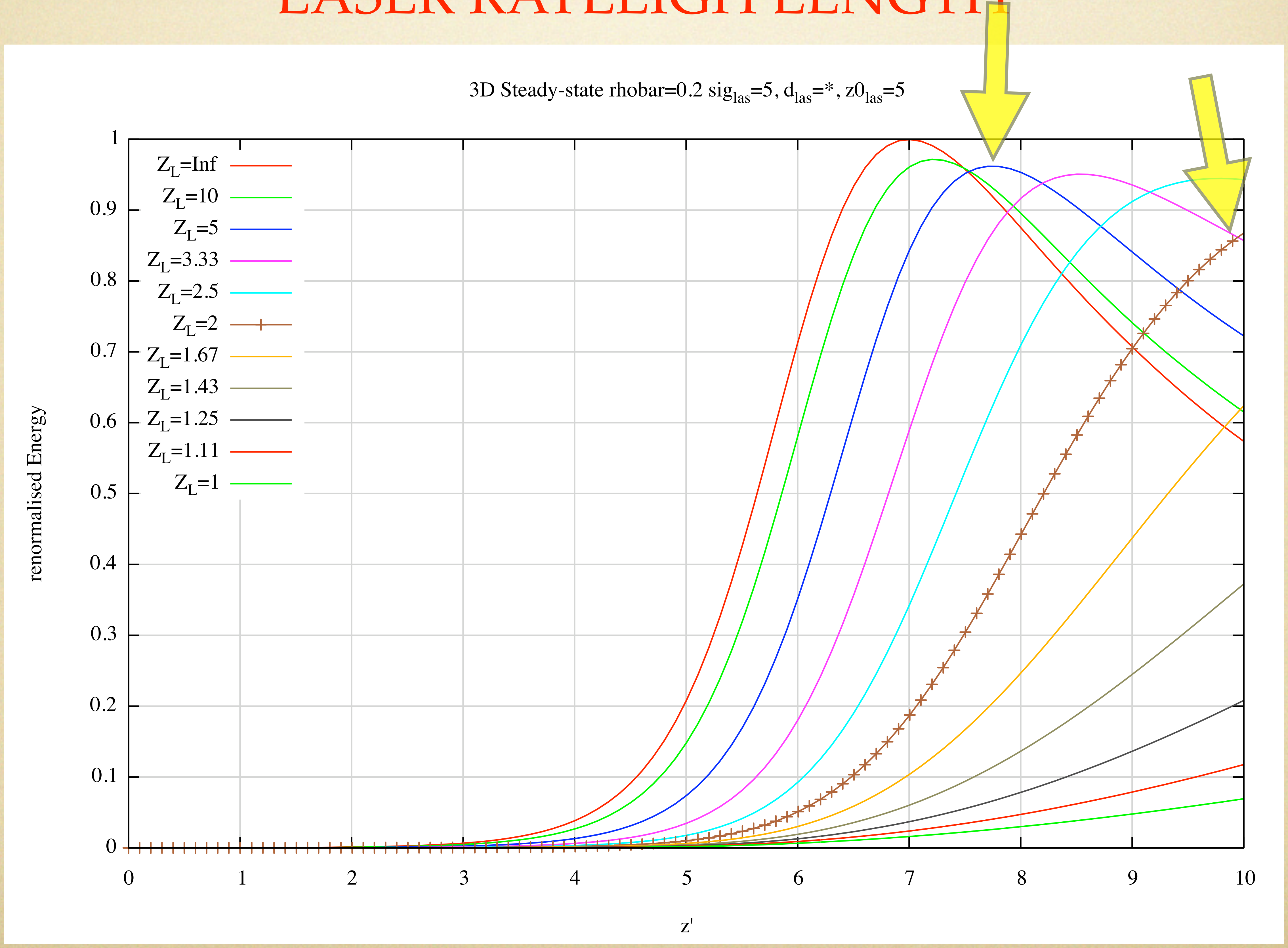


gaussian beams

3D steady state

$\sigma_L = 5$

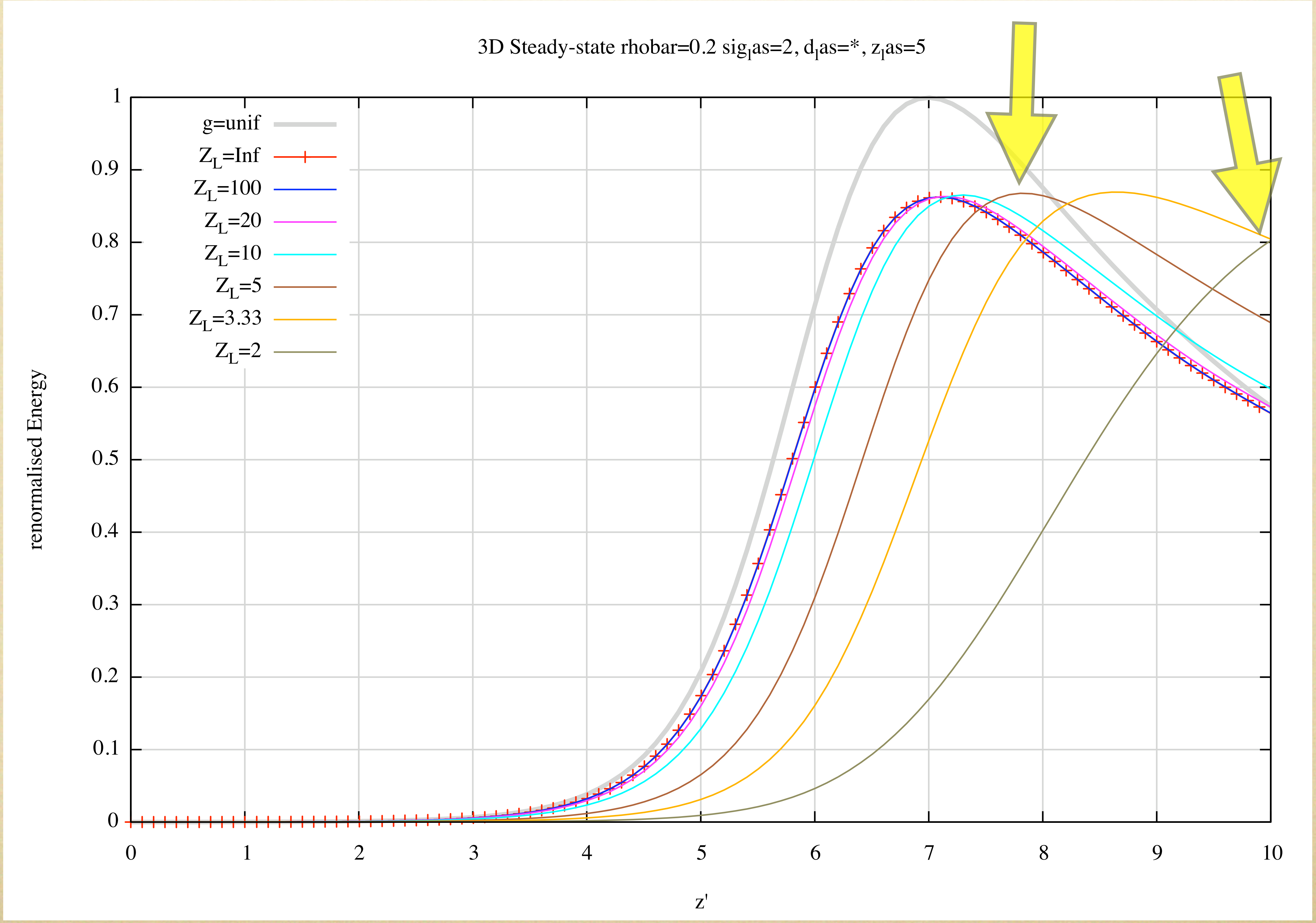
LASER RAYLEIGH LENGTH



3D steady state

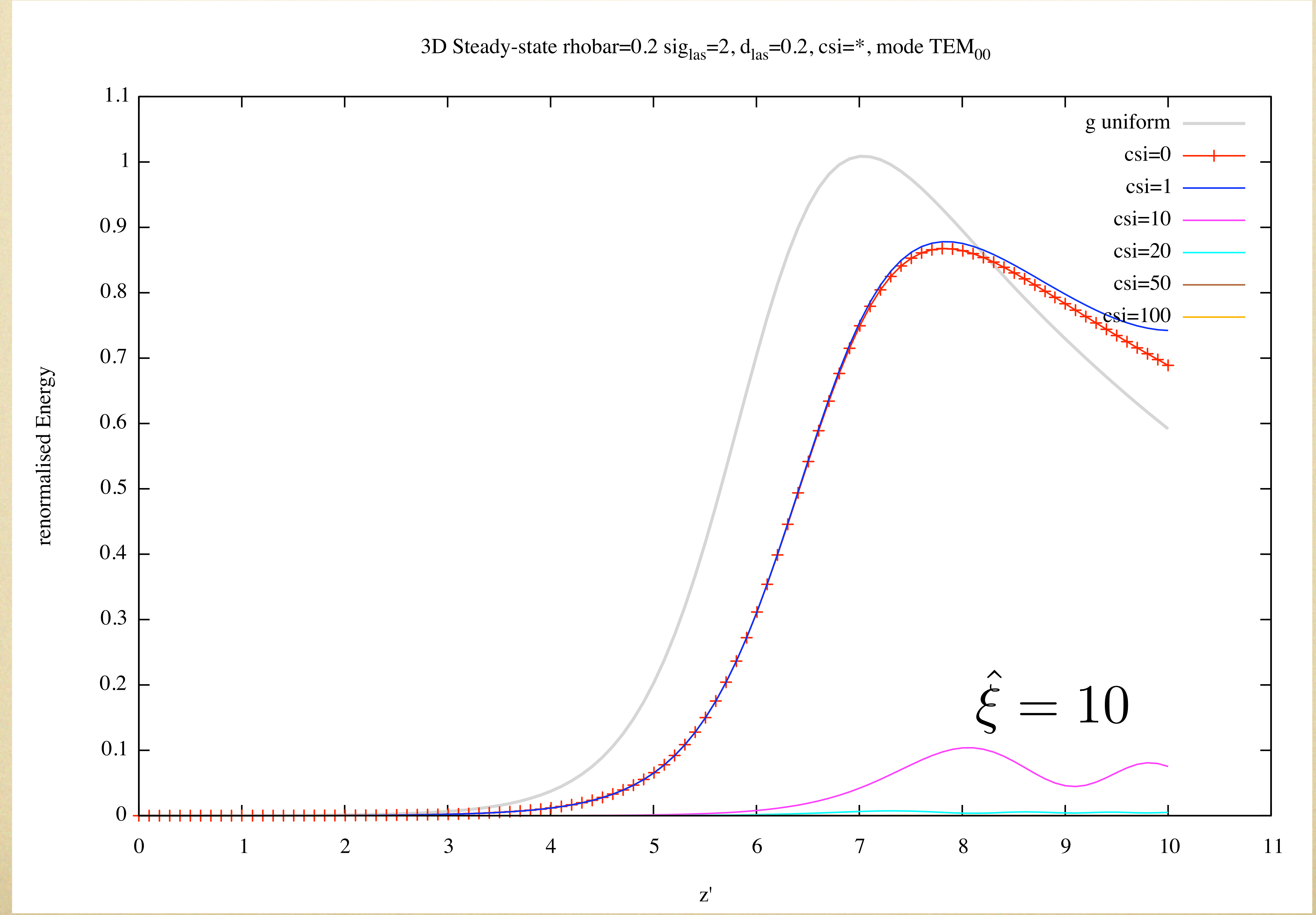
$\sigma_L = 2$

LASER RAYLEIGH LENGTH



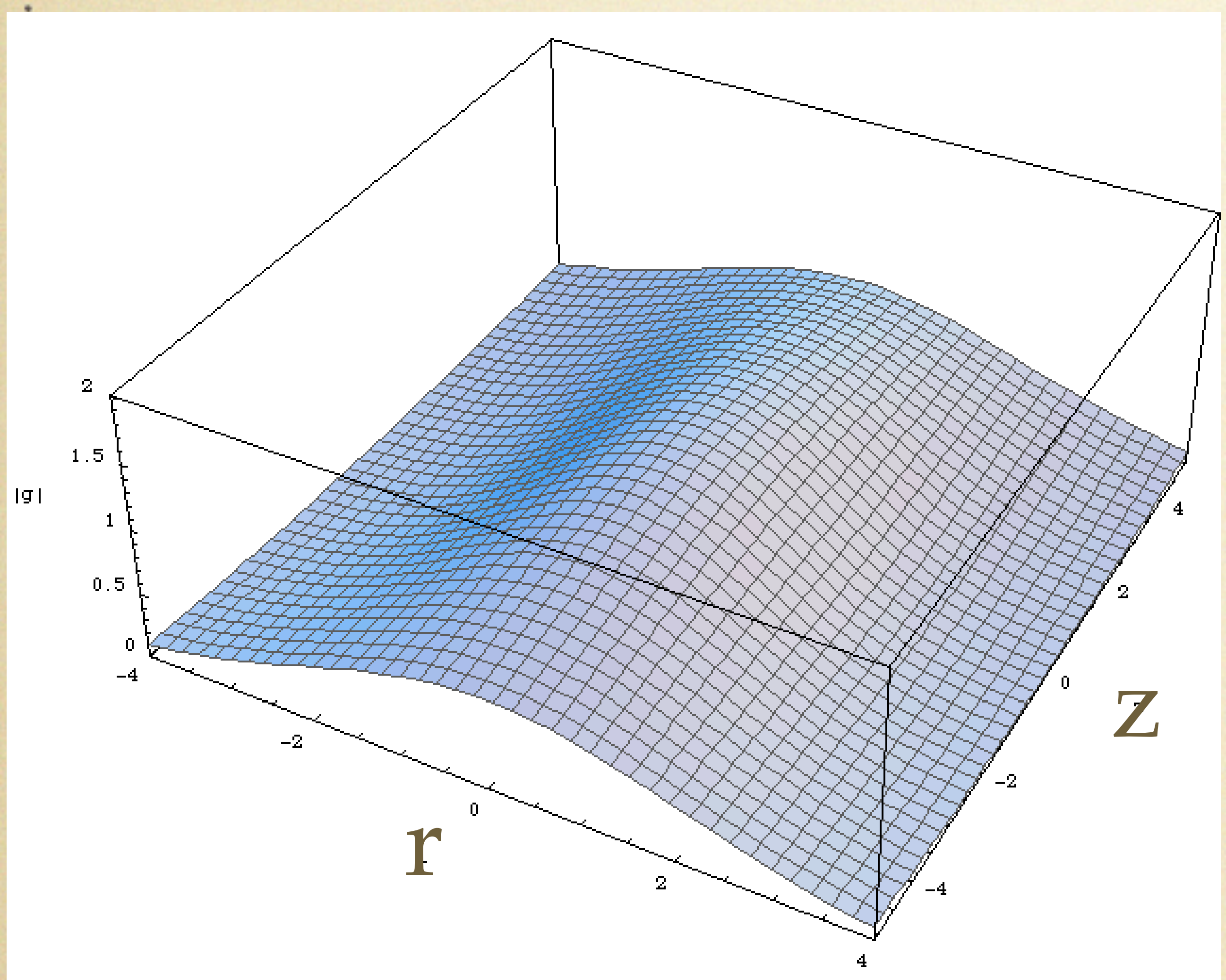
csi hat gaussian

$$\frac{\partial w_s}{\partial \hat{z}} = -\left[\frac{s}{\hat{\rho}} + \hat{\delta} - \hat{X}\, p_\perp^2 + \hat{\xi}(1 - |g|^2)\right]\frac{\partial w_s}{\partial \theta} \dots$$

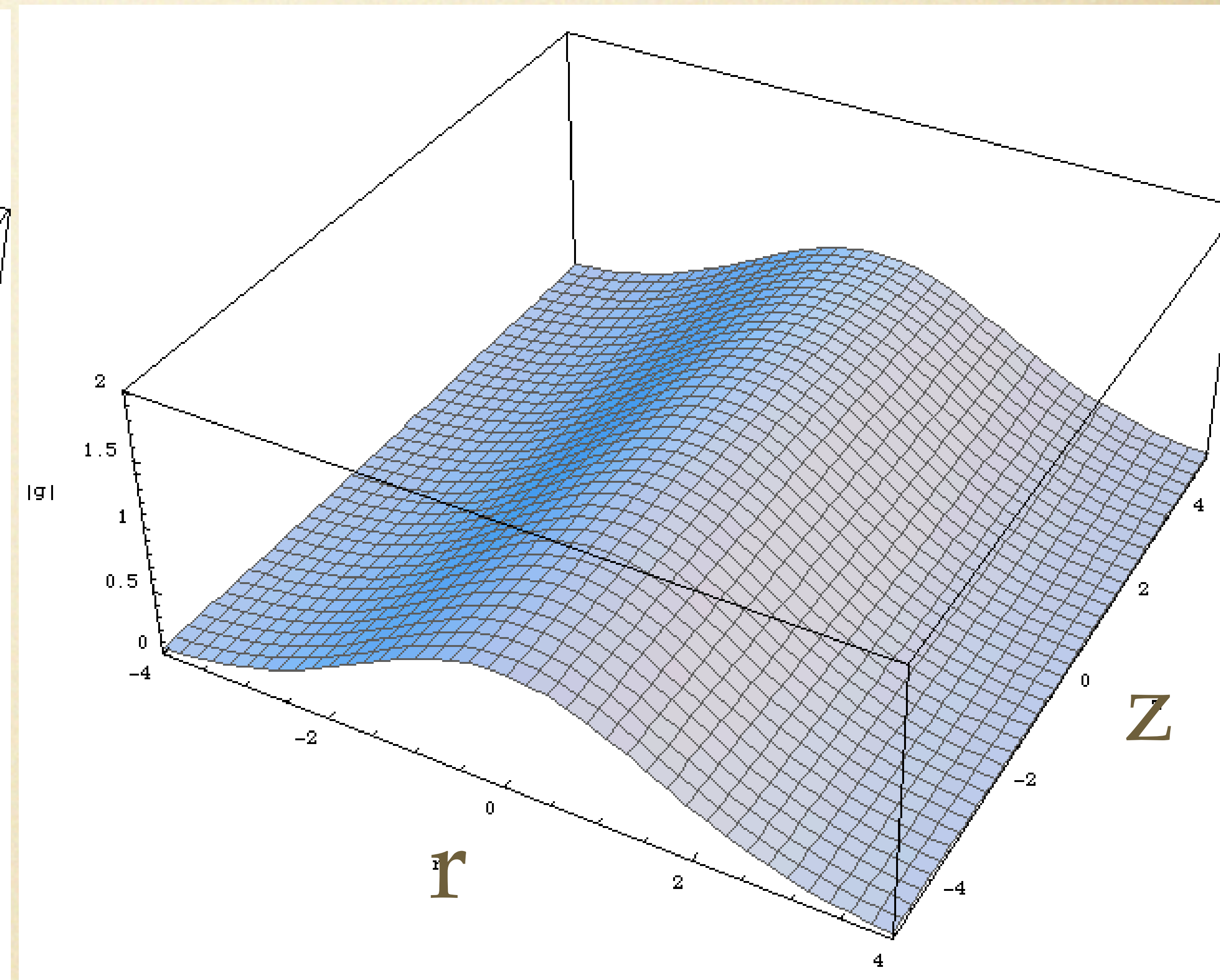


Gaussian mode TEM_{00}

$Z_L=5$



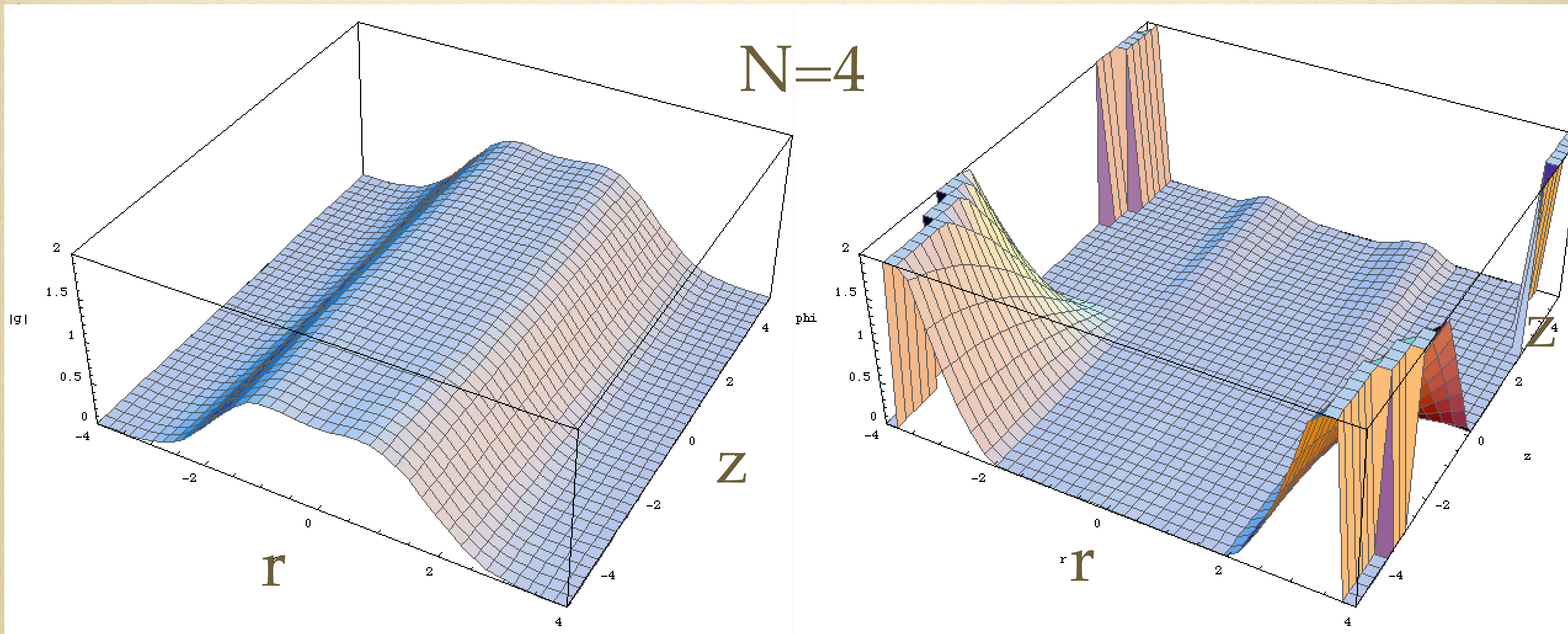
$Z_L=10$



$$g(r,\bar{z}) = \frac{1}{1-i(\bar{z}-\bar{z}_0)/\bar{Z}_L}\exp\left(-\frac{r^2}{4\sigma_L^2[1-i(\bar{z}-\bar{z}_0)/\bar{Z}_L]}\right)$$

Flattened Gaussian Beam

$$g_N(r,0) = exp[-(N+1)\left(\frac{r}{w_0}\right)^2] \sum_{n=0}^{N} c_n^{(N)} L_n \left[\frac{2(N+1)r^2}{w_0^2}\right]$$



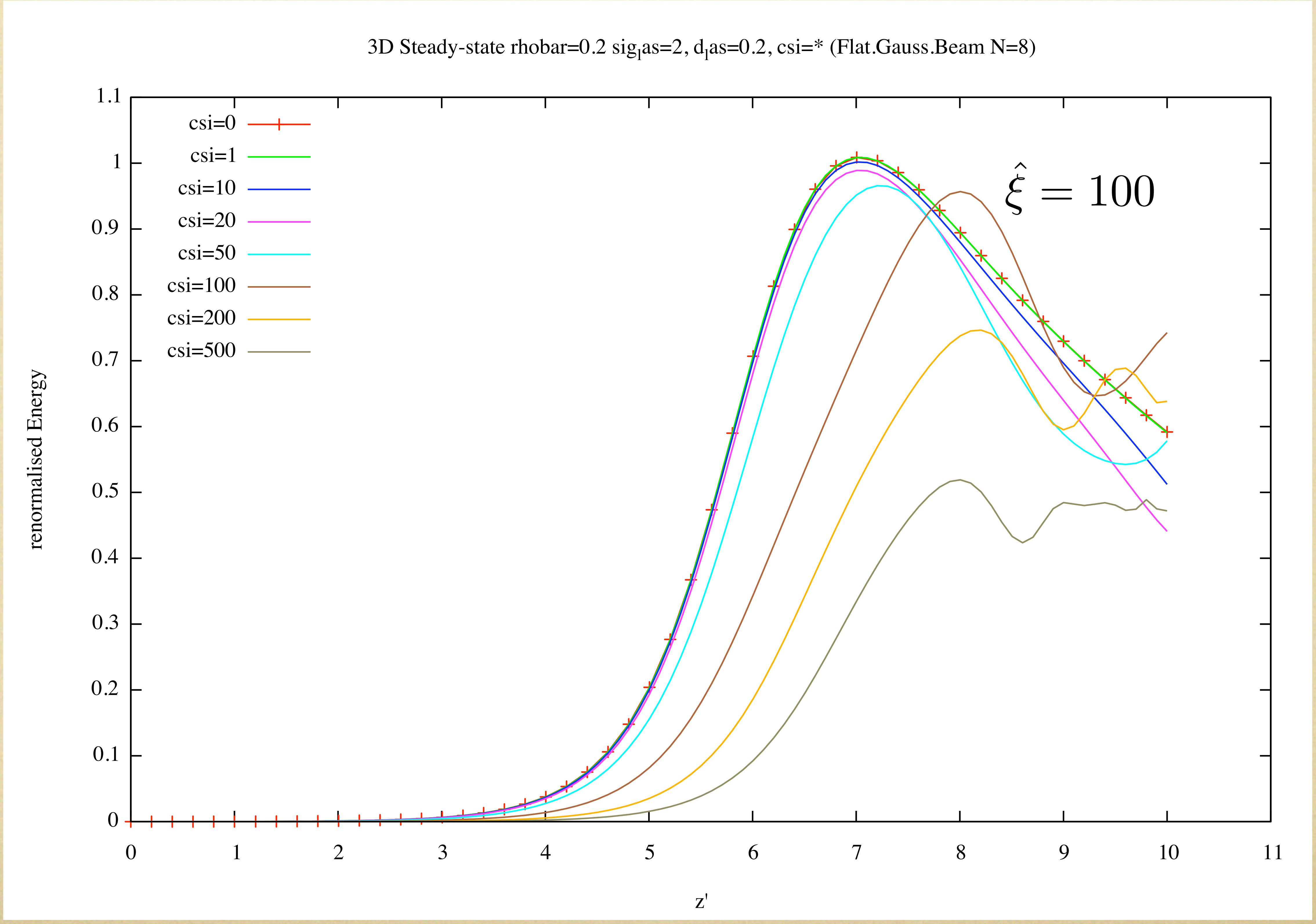
Abs(g)

Arg(g)

V. Bagini, R. Borghi, F. Gori et al, J. Opt. Soc. Am. A 13 (1996) 1385

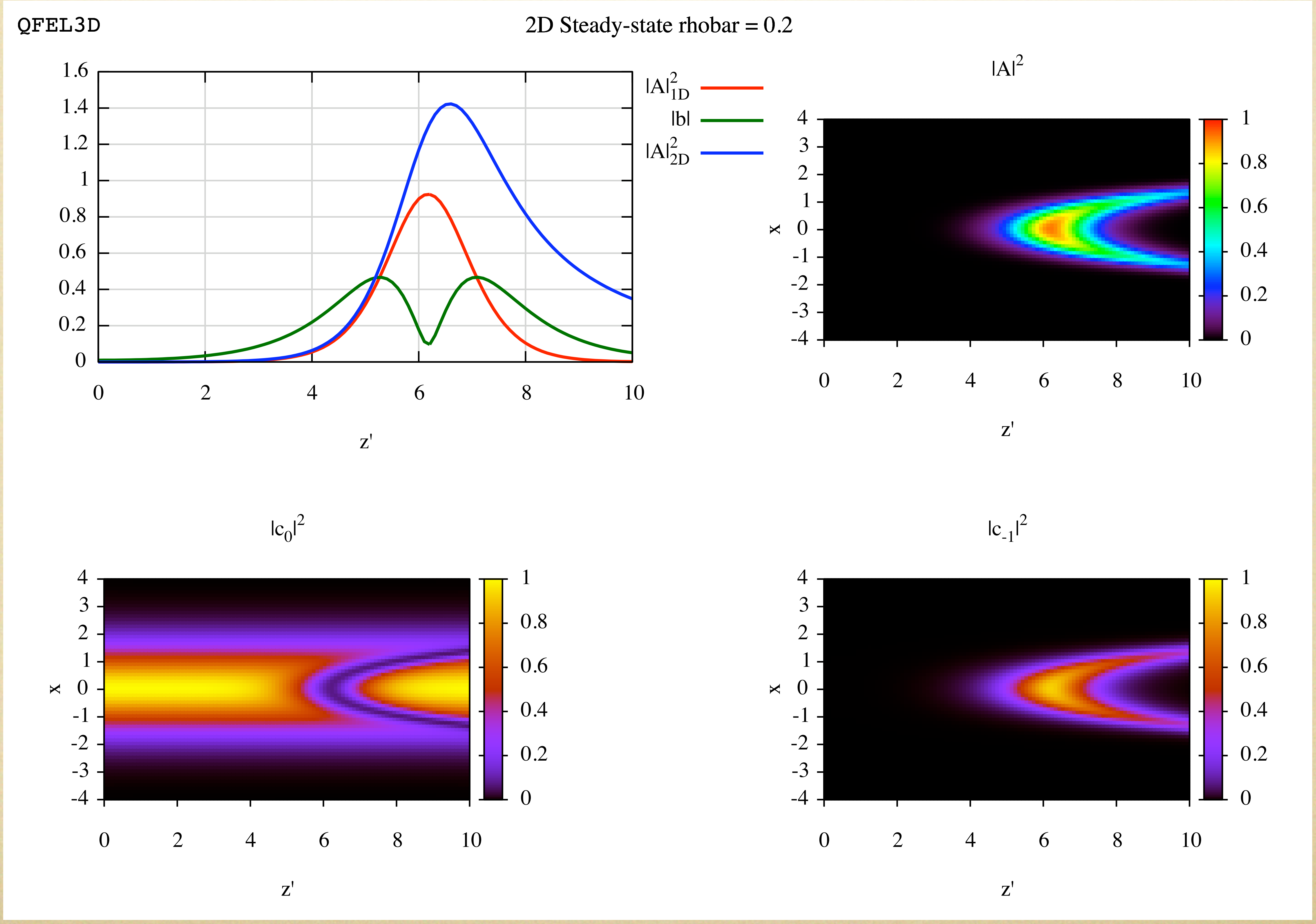
csi hat with FGB

$$\frac{\partial w_s}{\partial \hat{z}} = -\left[\frac{s}{\hat{\rho}} + \hat{\delta} - \hat{X}\, p_\perp^2 + \hat{\xi}(1 - |g|^2)\right]\frac{\partial w_s}{\partial \theta} \ldots$$



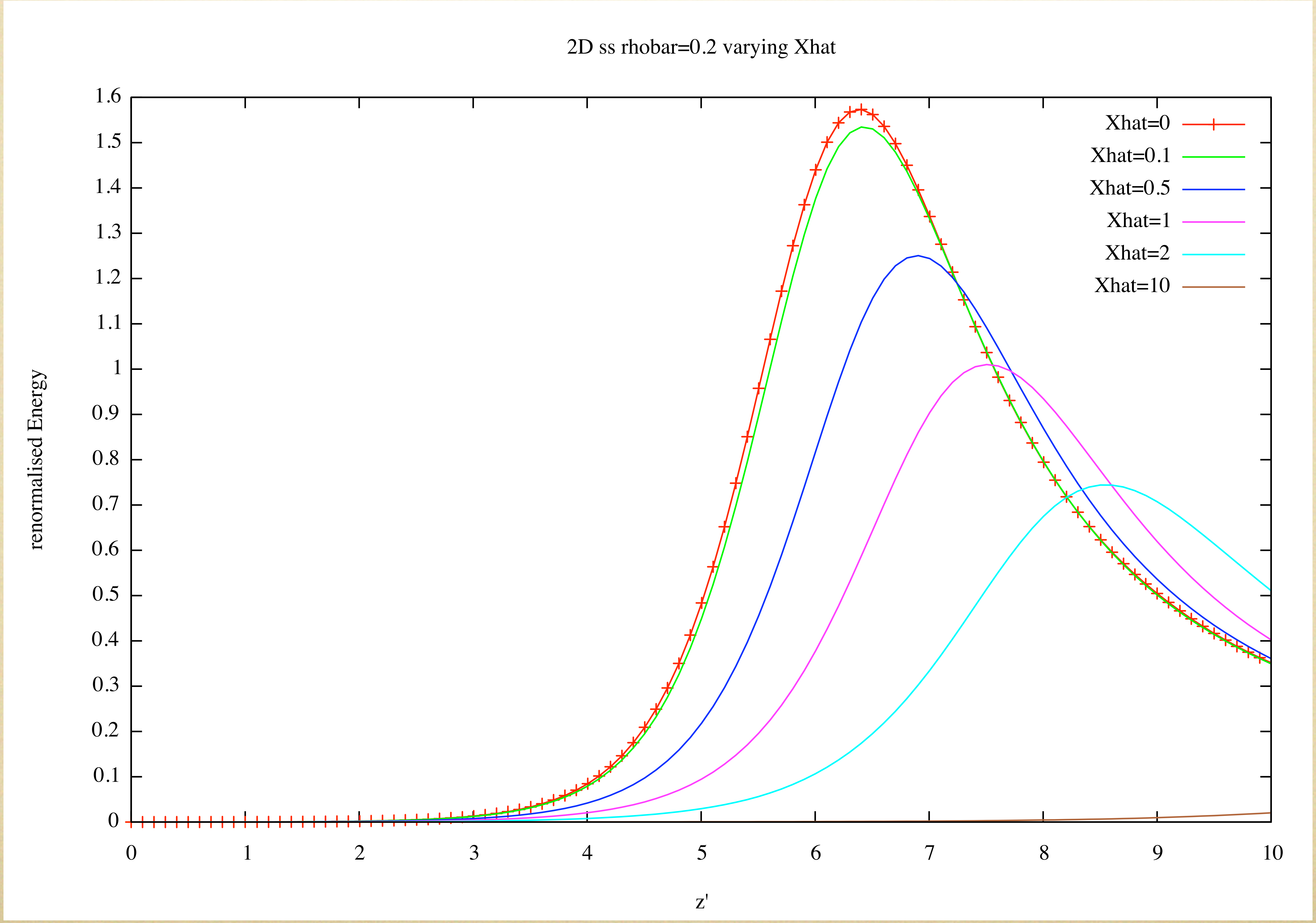
2D Xhat = 0.25

$$\frac{\partial w_s}{\partial \hat{z}} = -\left[\frac{s}{\hat{\rho}} + \hat{\delta} - \hat{X}\, p_\perp^2 + \hat{\xi}(1 - |g|^2)\right]\frac{\partial w_s}{\partial \theta} \ldots$$



2D Xhat steady state

$$\frac{\partial w_s}{\partial \hat{z}} = -\left[\frac{s}{\hat{\rho}} + \hat{\delta} - \hat{X}\, p_\perp^2 + \hat{\xi}(1 - |g|^2)\right] \frac{\partial w_s}{\partial \theta} \ldots$$



Concluding remarks

- the Wigner function model matches perfectly with the Schroedinger model in the classical and quantum regime in 1D
- inclusion of transverse dynamics and beam transport
- energy output stable for $L_b>20\ L_c$
- runs with propagation (SASE) confirm Steady-state sims
- laser profile resonance detuning (Gauss. beams->Flat beams)
- emittance resonance detuning (keep X hat < 5)