Classical & Quantum Regimes of Harmonic Generation in FELs

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Outline

1. Introduction
2. Classical Harmonic Generation
3. Quantum Harmonic Generation
4. Summary

1. Introduction

In usual classical FEL theory, photon recoil momentum is neglected and electron-light momentum exchange is continuous.

When classical momentum spread ($\gamma_R mc\rho$) << one-photon recoil momentum($\hbar k$)

i.e. $\bar{\rho} << 1$ where $\bar{\rho} = \dfrac{mc\gamma_R \rho}{\hbar k}$

then FEL dynamics changes dramatically,
as now the **discrete** nature of the electron-radiation momentum exchange becomes significant.

Quantum FEL regime has several interesting and potentially useful features
e.g. extremely narrow linewidth
Quantum entanglement between recoiling electrons and radiation

What are fundamental differences between harmonic generation in a classical FEL and a quantum FEL ?

2. Classical Harmonic Generation - recap

Classical planar wiggler FEL equations are :

$$\frac{d\theta_j}{d\bar{z}} = \bar{p}_j$$

$$\frac{d\bar{p}_j}{d\bar{z}} = -\sum_h F_h(\xi)\left(A_h e^{ih\theta_j} + c.c.\right) \qquad h = 1,3,5,\ldots$$

$$\frac{dA_h}{d\bar{z}} = F_h(\xi)\left\langle e^{-ih\theta}\right\rangle + ih\,\delta A_h$$

where

$$\theta_j = (k_w + k)z - \omega t_j \qquad \bar{p}_j = \frac{\gamma_j - \gamma_R}{\rho\gamma_R}$$

$$|\mathrm{A}_\mathrm{h}|^2 = \frac{\varepsilon_0 |E_h|^2}{\rho n \gamma_R mc^2} \qquad \bar{z} = \frac{z}{L_g} = \frac{4\pi\rho z}{\lambda_w} \qquad \rho = \frac{1}{\gamma_R}\left(\frac{a_w \omega_p}{4ck_w}\right)^{2/3}$$

$$\xi = \frac{a_w^2}{2\left(1 + a_w^2\right)} \qquad F_h(\xi) = (-1)^{\frac{h-1}{2}}\left(J_{\frac{h-1}{2}}(h\xi) - J_{\frac{h+1}{2}}(h\xi)\right)$$

$$\delta = \frac{\gamma_0 - \gamma_R}{\rho\gamma_R}$$ Detuning parameter

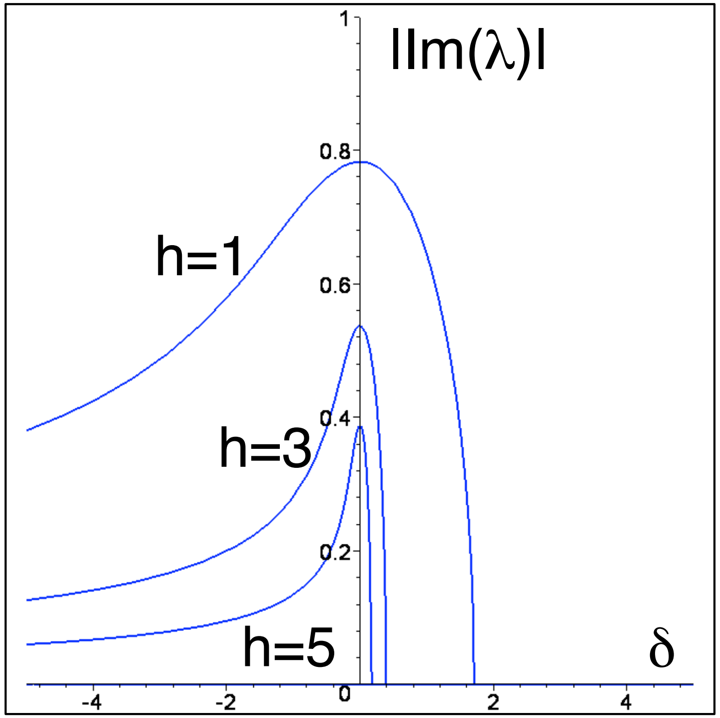
2. Classical Harmonic Generation – Linear Theory

Performing a linear analysis of these equations and looking for solutions $\propto \exp(i\lambda\bar{z})$ one obtains the dispersion relation.

$$\lambda^3 - h\delta\lambda^2 + hF_h{}^2 = 0$$

So looking for roots with Im(λ)<0 we find regions of instability/growth:

e.g. a_w=1



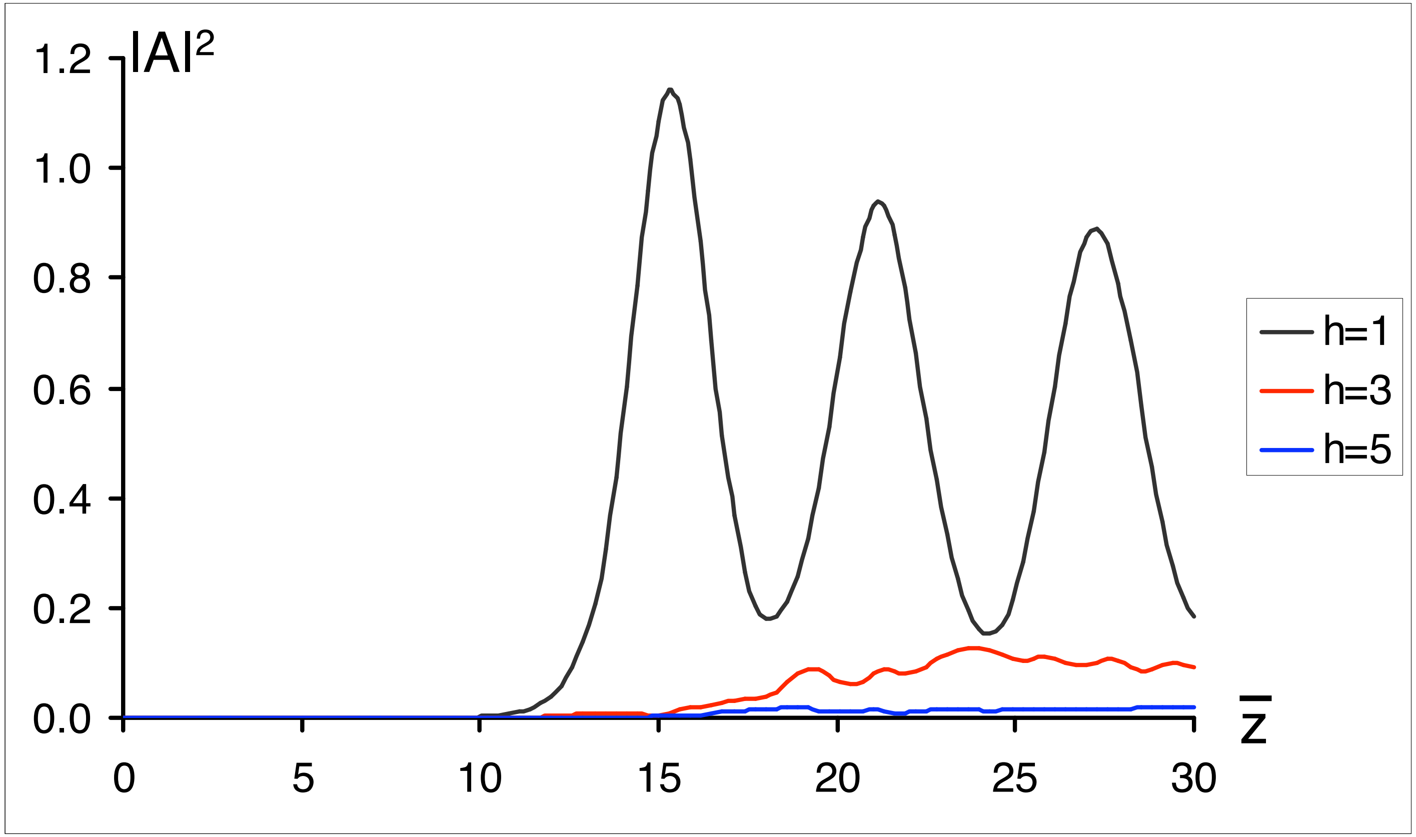
Maximum growth at δ=0
for fundamental and harmonics

Growth rate decreases
as h increases

Regions of growth for harmonics lie within that for fundamental

2. Classical Harmonic Generation – Nonlinear Regime

Solving classical equations for h=1,3,5 numerically for δ=0 :



Fundamental and harmonics evolve simultaneously

Fundamental dominates interaction unless disrupted

Peak intensity **decreases** with harmonic number

3. Quantum Harmonic Generation – Model

Similar procedure as described in previous talks i.e.

Electron dynamical equations

$$\frac{d\theta_j}{d\bar{z}} = \bar{p}_j$$

$$\frac{d\bar{p}_j}{d\bar{z}} = -\sum_h F_h(\xi)\left(A_h e^{ih\theta_j} + c.c.\right)$$

Single electron Hamiltonian

$$H_j = \frac{p_j^{\,2}}{2} - i\sum_h \frac{F_h}{h}\left(A_h e^{ih\theta_j} - \text{c.c.}\right)$$

Maxwell-Schrodinger equations for electron wavefunction Ψ and classical field A

$$i\frac{\partial\Psi(\theta,\bar{z})}{\partial\bar{z}} = -\frac{1}{2\bar{\rho}}\frac{\partial^2\Psi}{\partial\theta^2} - i\bar{\rho}\sum_h \frac{F_h}{h}\left(A_h e^{ih\theta_j} - \text{c.c.}\right)\Psi$$

$$\frac{dA_h}{d\bar{z}} = F_h\int_0^{2\pi}|\Psi|^2 e^{-ih\theta}\,d\theta + ih\delta A_h$$

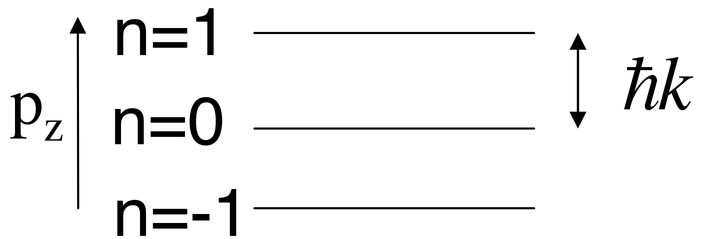
3. Quantum Harmonic Generation – Model

Assuming electron wavefunction is periodic in θ :

$$\Psi(\theta,\bar{z}) \propto \sum_{n=-\infty}^{\infty} c_n(\bar{z}) e^{in\theta}$$

$|c_n|^2$ = Probability of electron having momentum n(ħk)

Only <u>discrete</u> values of momentum are possible : $p_z = n\,(\hbar k)$, $n = 0, \pm 1, ..$



M-S equations in terms of momentum amplitudes

$$\frac{dc_n}{d\bar{z}} = -i\frac{n^2}{2\bar{\rho}}c_n - \bar{\rho}\sum_h \frac{F_h(\xi)}{h}\left(A_h c_{n-h} - A^*_h c_{n+h}\right)$$

$$\frac{dA_h}{d\bar{z}} = F_h \sum_{n=-\infty}^{\infty} c_n c^*_{n-h} + ih\,\delta A_h$$

3. Quantum Harmonic Generation – Linear Theory

Performing a linear analysis of quantum equations and looking for solutions $\propto \exp(i\lambda\bar{z})$ one obtains the dispersion relation.

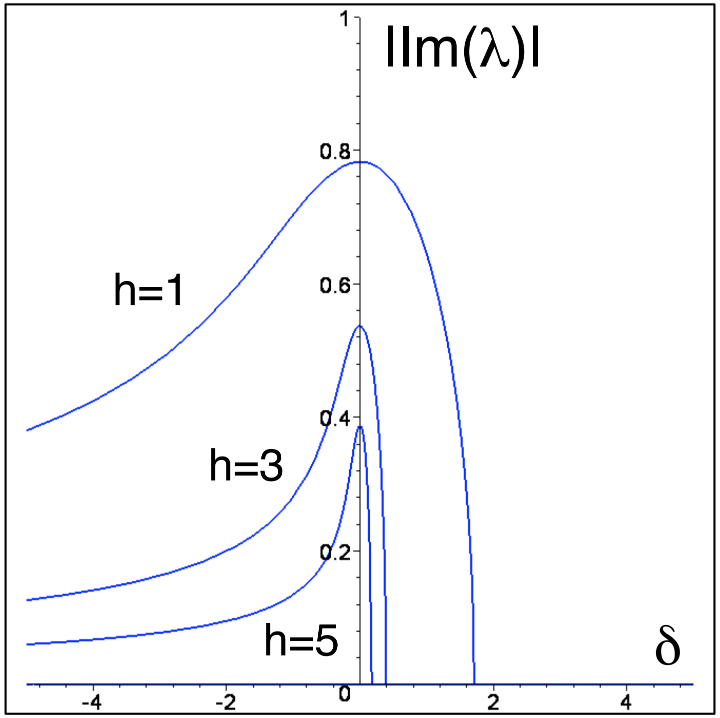
$$(\lambda - h\delta)\left(\lambda^2 - \frac{h^4}{4\bar{\rho}^2}\right) + hF_h^{\ 2} = 0$$

quantum term significant when $\bar{\rho}<1$

So looking for roots with $\mathrm{Im}(\lambda)<0$ we find regions of instability/growth:

For $\bar{\rho}>>1$ we obtain result from classical model

e.g. $\rho=10$, $a_w=1$



3. Quantum Harmonic Generation – Linear Theory

$$(\lambda - h\delta)\left(\lambda^2 - \frac{h^4}{4\bar{\rho}^2}\right) + hF_h^{\ 2} = 0$$

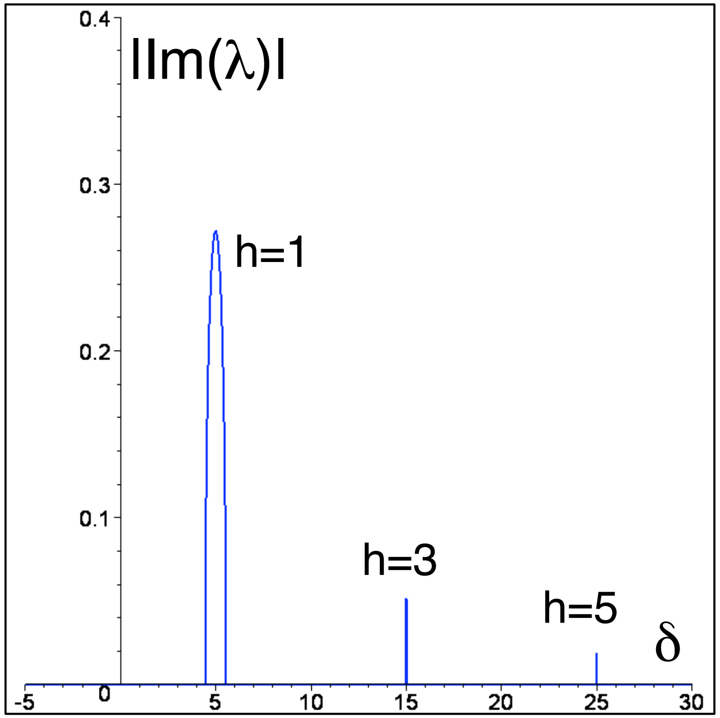
quantum term
significant when $\bar{\rho}<1$

For $\bar{\rho}<1$ (quantum regime) regions of instability change substantially

e.g. $\bar{\rho}=0.1$, $a_w=1$

Regions of instability are now well separated in frequency, centred on

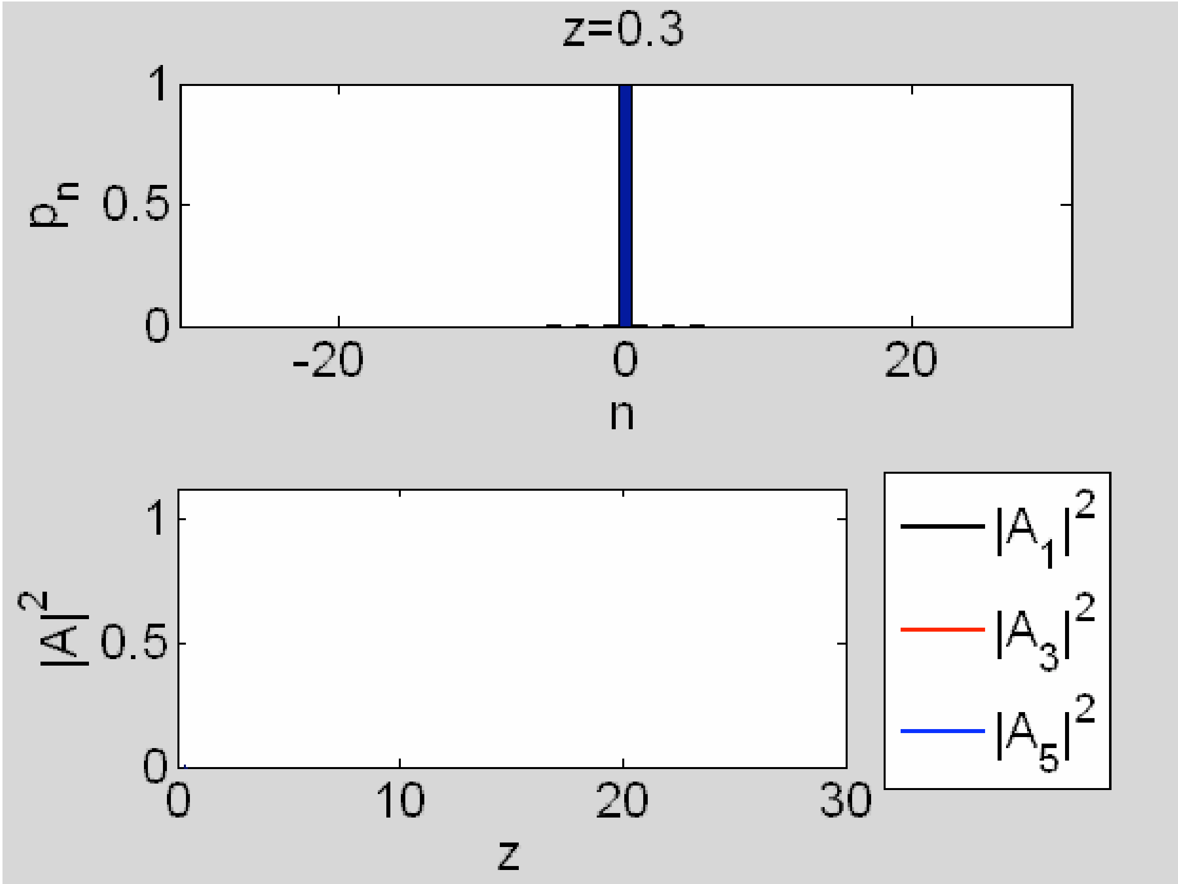
$$\delta = \frac{h}{2\bar{\rho}} = 5, 15, 25$$



3. Quantum Harmonic Generation – Nonlinear Regime

Classical case

$\bar{\rho} = 5, \; \delta = 0$



$p_n = |c_n|^2$

Field evolves as from solution of classical equations.

All harmonics evolve simultaneously – fundamental dominates

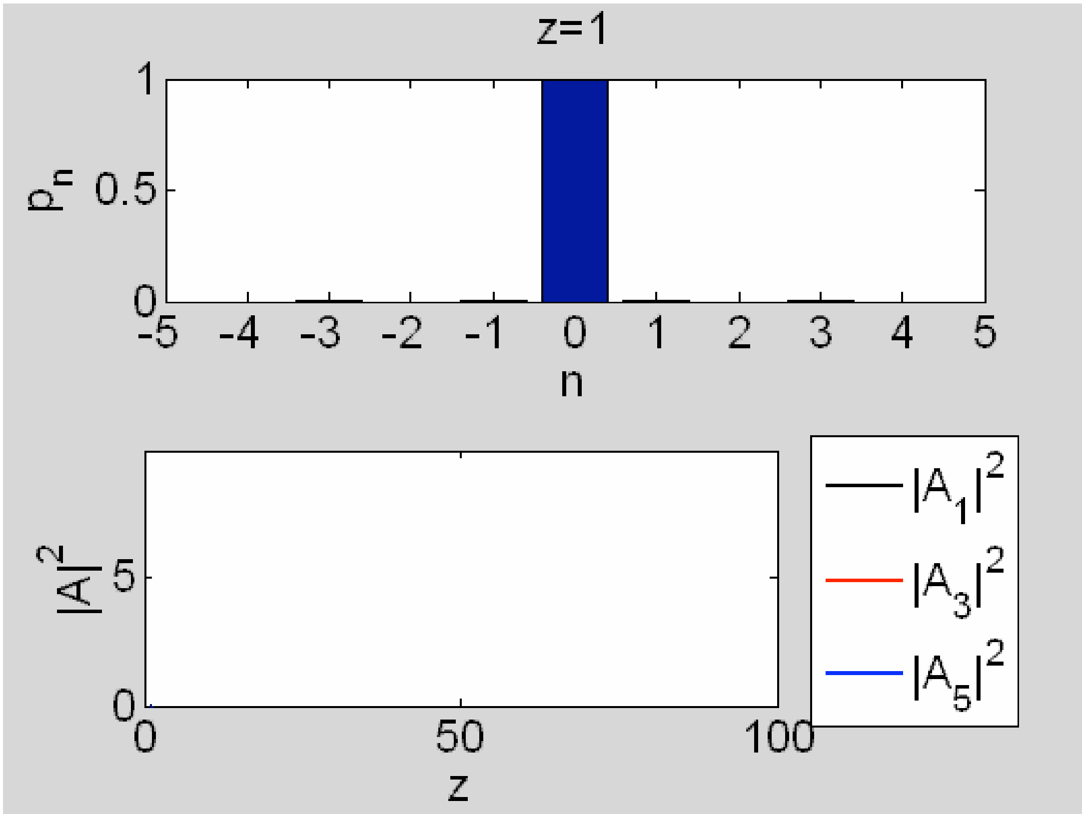
Many momentum states participate.

3. Quantum Harmonic Generation – Nonlinear Regime

In quantum case, each harmonic can be excited **independently** as regions of instability are separated in frequency.

$$\bar{\rho} = 0.1,$$

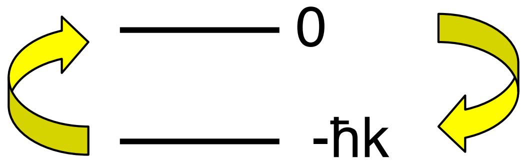
$$\delta = \frac{1}{2\bar{\rho}} = 5$$



$p_n = |c_n|^2$

Fundamental evolves as sech^2 pulse, but h=3,5 are not amplified

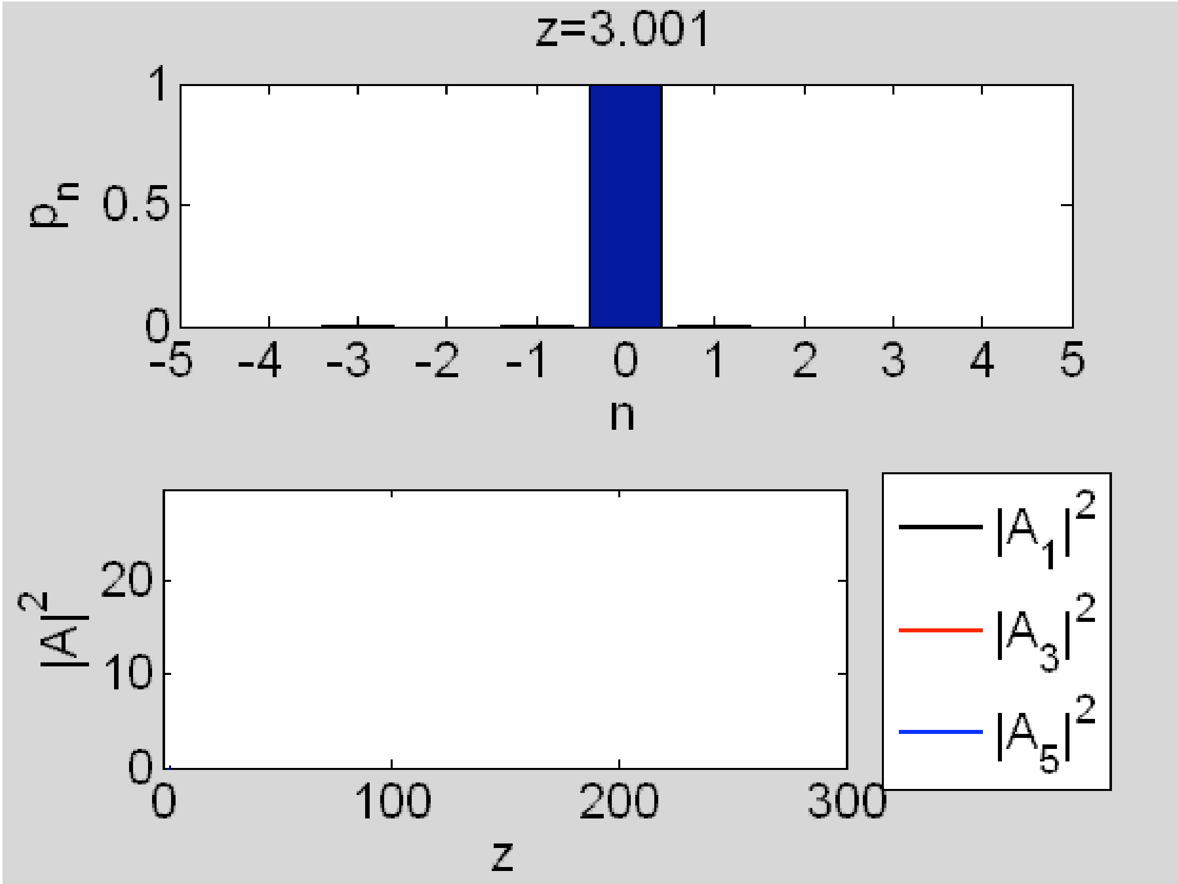
Only momentum states n=0 and -1 participate, emitting and reabsorbing photon with momentum ħk



3. Quantum Harmonic Generation – Nonlinear Regime

$$\overline{\rho} = 0.1,$$

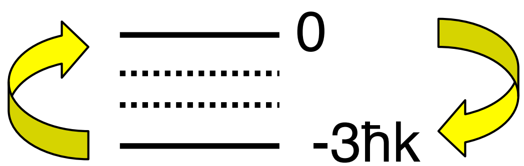
$$\delta = \frac{3}{2\overline{\rho}} = 15$$



$p_n = |c_n|^2$

3rd harmonic now evolves as sech² pulse, but h=1,5 are not amplified

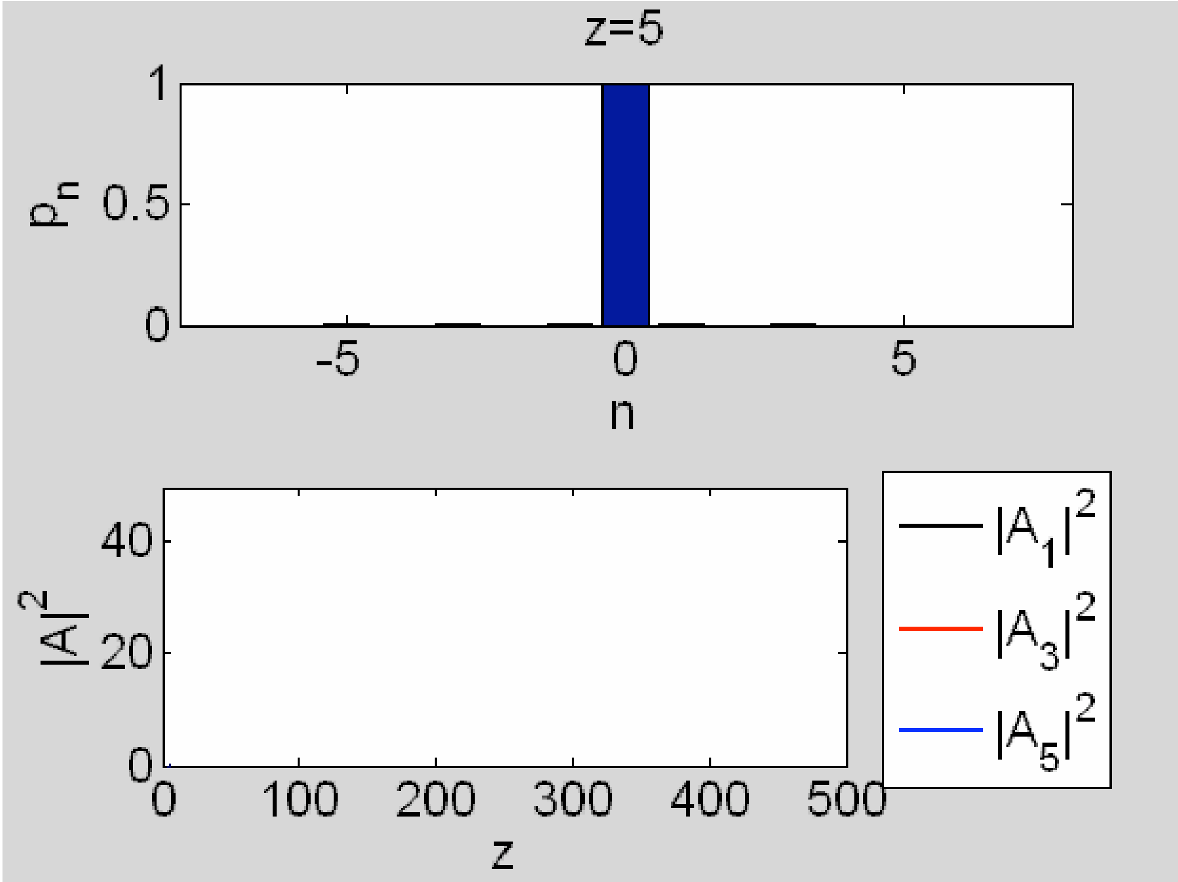
Only momentum states n=0 and -3 participate, emitting and reabsorbing photon with momentum 3ħk



3. Quantum Harmonic Generation – Nonlinear Regime

$$\overline{\rho} = 0.1,$$

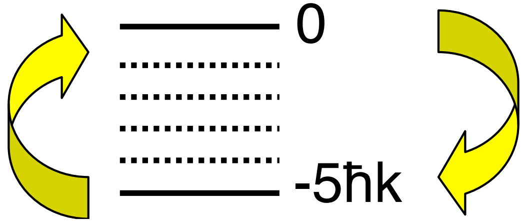
$$\delta = \frac{5}{2\overline{\rho}} = 25$$



$p_n = |c_n|^2$

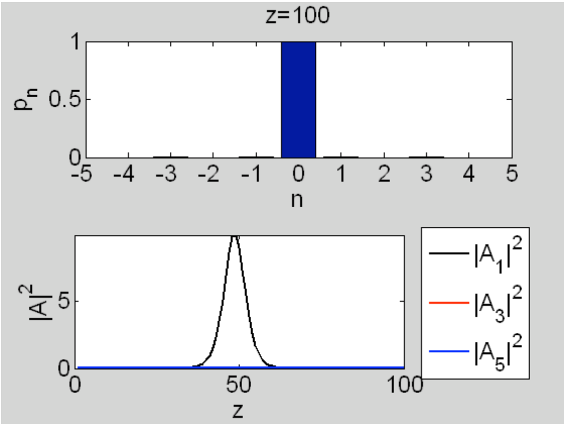
5^{th} harmonic evolves as $sech^2$ pulse, but h=1,3 are not amplified

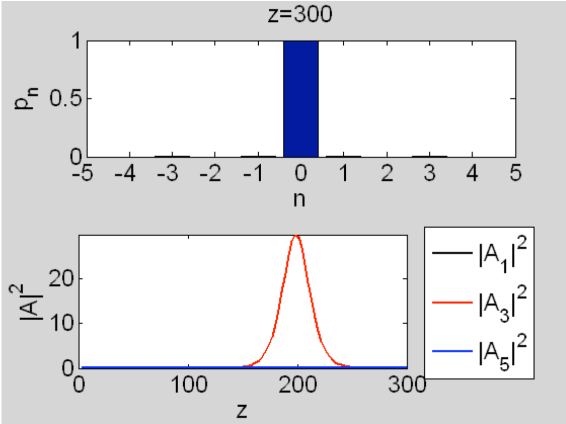
Only momentum states n=0 and -5 participate,
emitting and reabsorbing photon with momentum $5\hbar k$

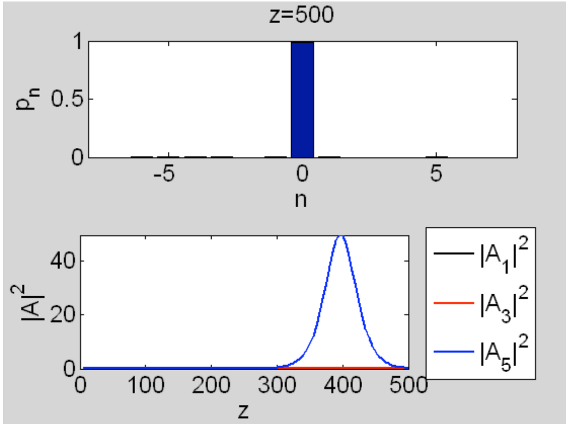


3. Quantum Harmonic Generation – Nonlinear Regime

Note that in quantum regime, although growth rate decreases with harmonic number, peak intensity <u>increases</u> with harmonic number







δ=5 : Fundamental

$|A_1|^2_{max} = 10$

δ=15 : 3rd harmonic

$|A_3|^2_{max} = 30$

δ=25 : 5th harmonic

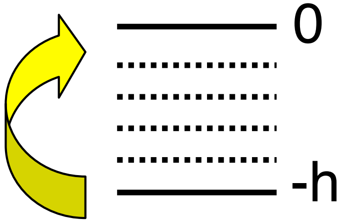
$|A_5|^2_{max} = 50$

Peak intensity increases linearly with harmonic number, h

Why?

3. Quantum Harmonic Generation – Nonlinear Regime

In quantum regime, amplification of harmonic number h involves only momentum states n=0 and n=-h.



Effectively a 2-level system

Quantum FEL equations reduce to Maxwell-Bloch equations
- analogous to an unstable pendulum (mentioned previously)

$$\frac{dS}{d\bar{z}} = i\frac{h^2}{2\bar{\rho}}S + \rho\frac{F_h}{h}A_h D$$

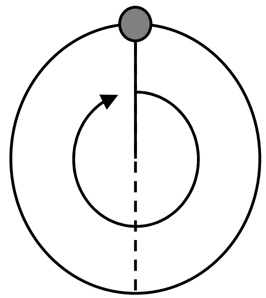
$$\frac{dD}{d\bar{z}} = -\frac{2\bar{\rho}}{h}F_h\left(A_h S^* + \text{c.c.}\right)$$

$$\frac{dA_h}{d\bar{z}} = F_h S + ih\delta A$$

where

$$S = c_{-h}{}^{*}c_0$$

$$D = \left|c_0\right|^2 - \left|c_{-h}\right|^2$$



Can show that peak $|A_h|^2$ occurs when $|c_{-h}|^2=1$

Value of peak is $\left|A_h\right|^2 = \dfrac{h}{\bar{\rho}}$

i.e. linear increase in peak intensity with harmonic number

i.e. in quantum regime every electron emits a photon of momentum h(ħk) coherently

4. Summary

Quantum regime of FEL evolution can be very different from the usual classical case

Also the case for harmonic generation in a planar wiggler FEL :
In classical regime, harmonics excited simultaneously with fundamental, and fundamental dominates – growth rate and peak intensity decrease with h.

In quantum regime, can choose detuning to selectively amplify harmonics.
Growth rate decreases with h, but peak intensity increases linearly with h due to increased momentum of photons being emitted coherently by electrons.

Next steps?

Currently simplest possible model (ideal cold beam, 1D steady-state etc.)

Harmonics will be more sensitive to energy spread
- can increase in harmonic intensity with h survive ?
- will require e.g. Wigner description