

3D WIGNER FUNCTION MODEL FOR A QUANTUM FREE ELECTRON LASER

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Outline

1. From the classical to the quantum FEL description
2. Schrödinger vs. Wigner models in 1D
3. From 1D to 3D quantum FEL model with a laser wiggler
4. 3D Wigner model
5. Discussion and results

CLASSICAL FEL 1D MODEL

$$\frac{\partial^2 \theta_j}{\partial \bar{z}^2} = -(A e^{i\theta_j} + c.c.)$$

$$\frac{\partial A}{\partial \bar{z}} + \frac{\partial A}{\partial z_1} = \frac{1}{N} \sum_{j=1}^N e^{-i\theta_j}$$

R.Bonifacio, C.Pellegrini, L.Narducci,
Opt. Comm. (**1984**) (steady-state)

$$\bar{z} = z / L_g$$

$$L_g = \lambda_w / 4\pi\rho$$

gain length

$$z_1 = (z - v_r t) / L_c$$

$$L_c = \lambda_r / 4\pi\rho$$

cooperation length

$$\theta_j = (k + k_w)z - \omega t_j;$$

$$\rho |A|^2 = \frac{P_{\text{rad}}}{P_{\text{beam}}}$$

$$\rho \propto \frac{1}{\gamma_R} \left(\frac{a_w \lambda_w}{\sigma} \right)^{2/3} I^{1/3}$$

FEL parameter

HIGH-GAIN REGIME

- exponential growth of intensity and bunching
- start up from noise
- saturation at $A \sim 1$ ($P_{\text{rad}} \sim \rho P_{\text{beam}}$) after several L_g

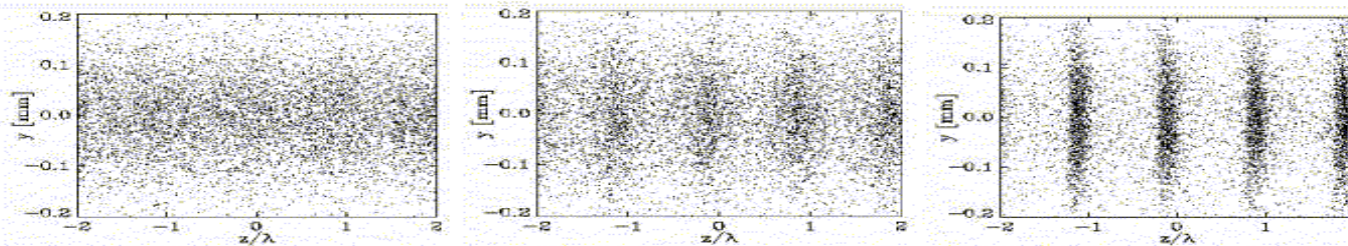
$b \sim 0$



$b \sim 0.8$

bunching:

$$b = \frac{1}{N} \sum_{j=1}^N e^{-i\theta_j}$$



wiggler length (several L_g)

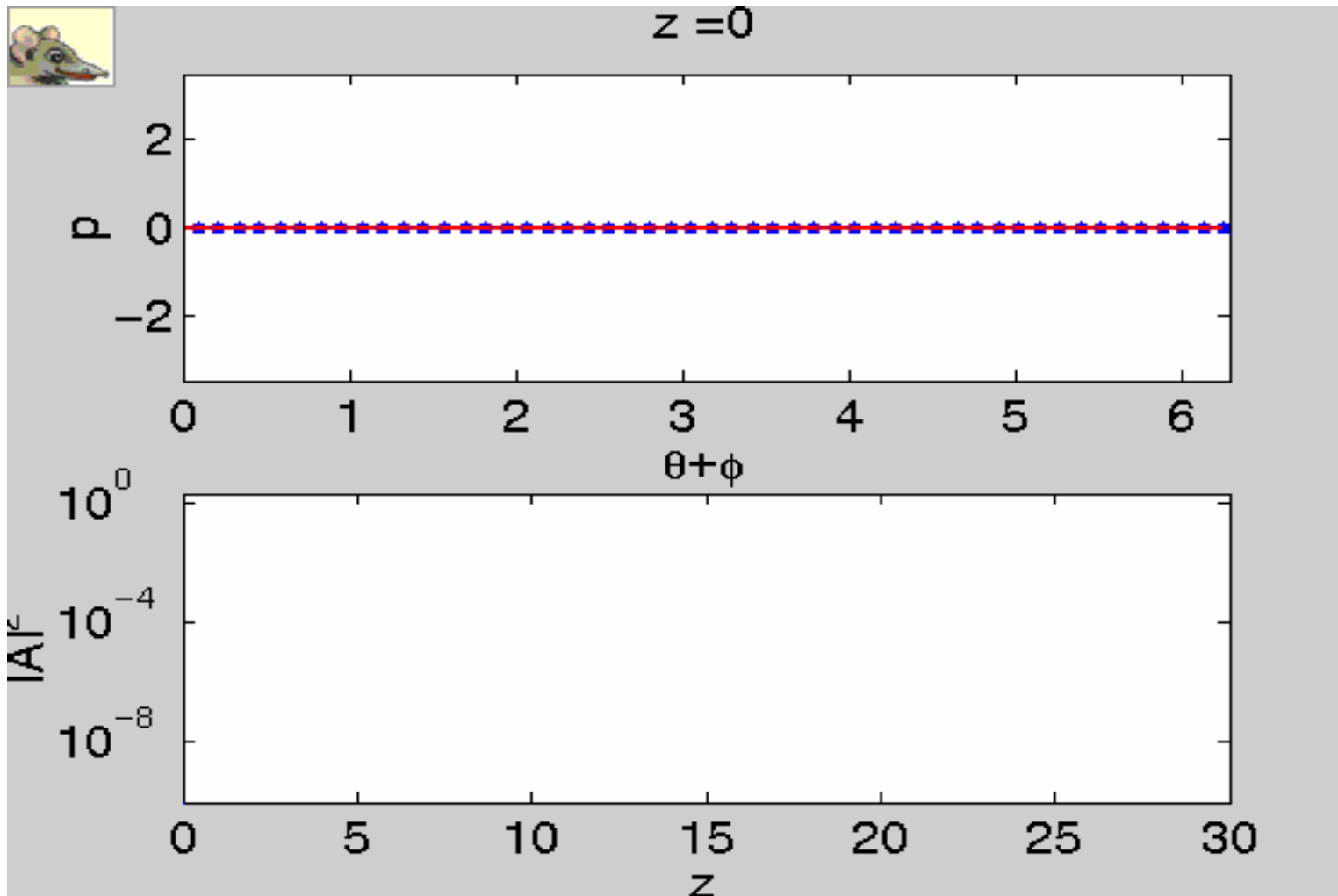
neglecting SLIPPAGE:

$$\cancel{\frac{\partial A}{\partial z_1}}$$

(**STEADY-STATE** regime)

electrons behave as coupled pendula
in a self-consistent potential:

$$V(\theta + \phi) = 2 |A| \cos(\theta + \phi)$$



$$p = \frac{\gamma - \gamma_r}{\rho \gamma_r}$$

Simulation
by G.R.M Robb

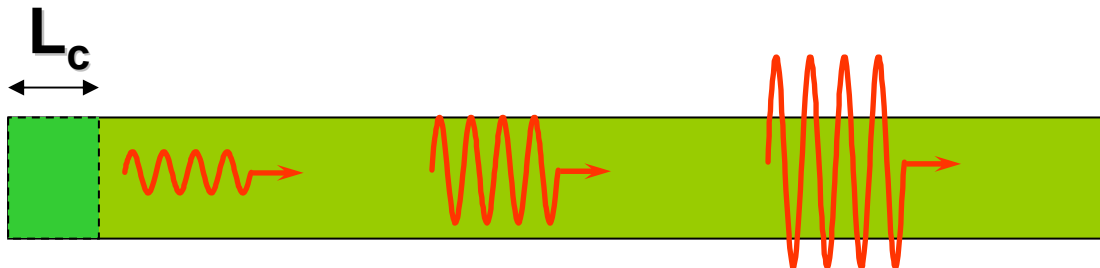
SLIPPAGE EFFECT: SUPERRADIANCE

R. Bonifacio, B.W. McNeil, P. Pierini PRA (1989)

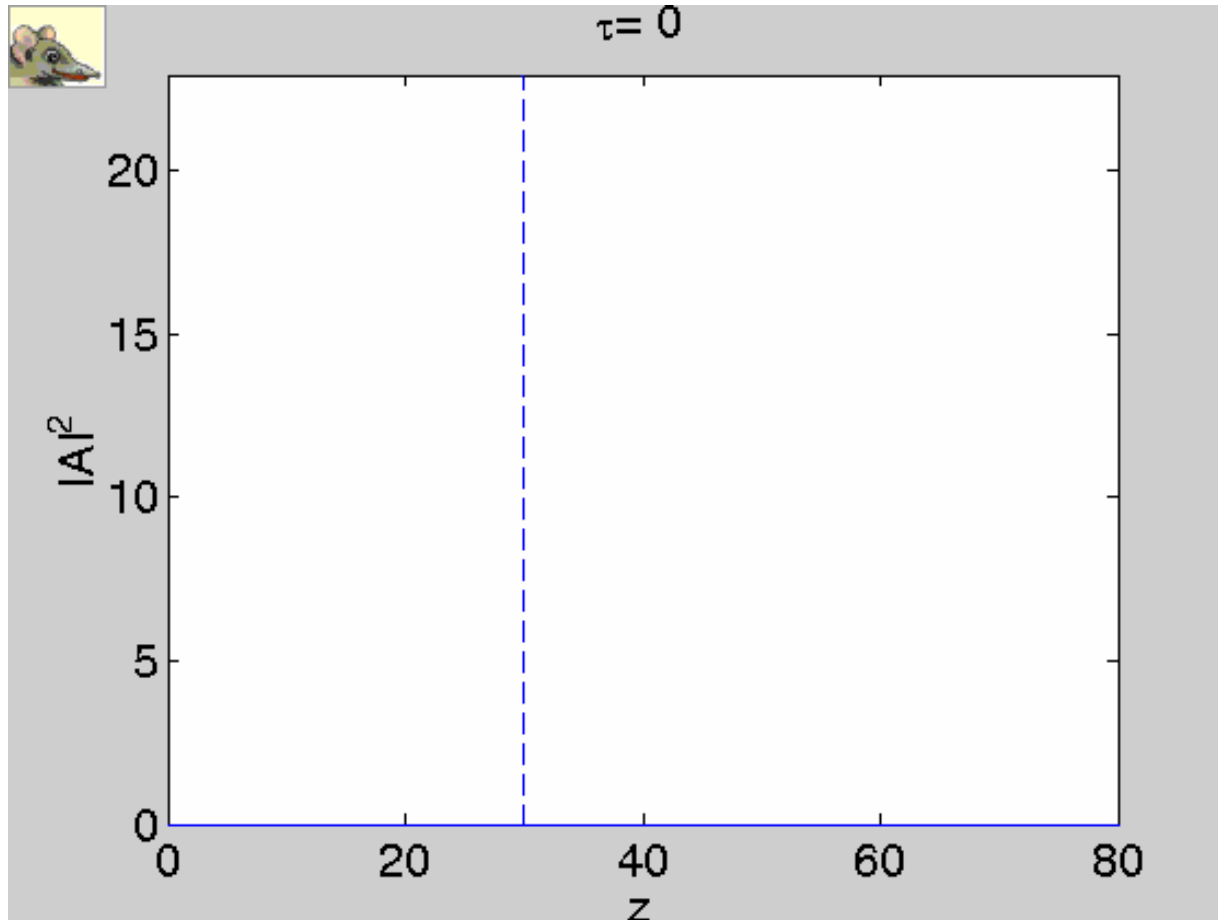
Radiation propagates faster than electrons and advance them by a slippage length $(\lambda_r/\lambda_w)L_w = \lambda_r N_w$

Electrons in a cooperation length $L_c = \lambda_r/4\pi\rho$ radiate a **SUPERRADIANT (SR) PULSE**.

if $L_b \gg L_c$ the **SR** pulse gets amplified as it propagates forward through the beam with **no saturation**.



STEADY-STATE + SUPERRADIANT SPIKE ('second instability') from a resonant coherent seed



$$L_b = 30 L_c$$

$$0 < z < 50 L_g$$

$$\delta = 0$$

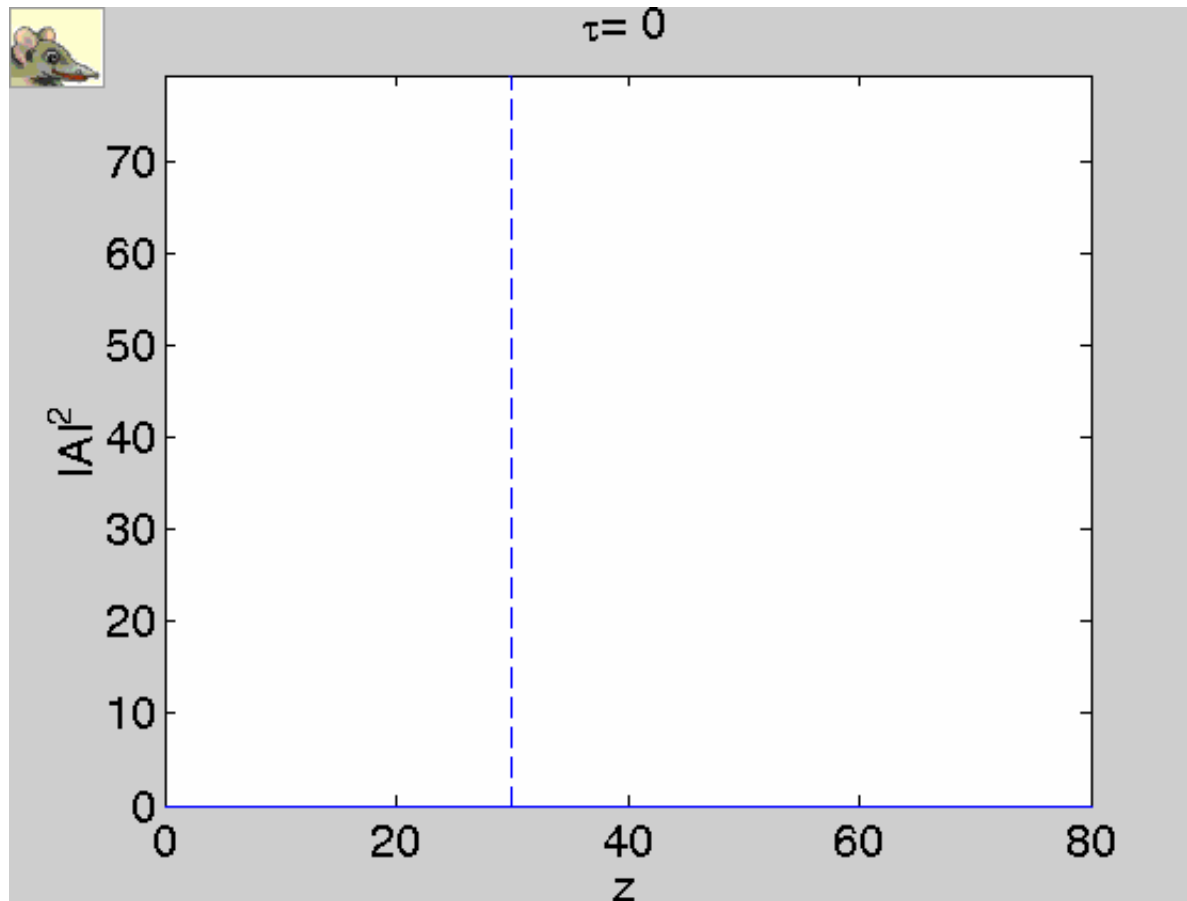
$$\left(z_1 = \frac{z - vt}{L_c} \right)$$

simulation
by G.R.M Robb

SUPERRADIANT SPIKE in a detuned case



SELF-SIMILAR SOLUTION



$$L_b = 30 L_c$$

$$0 < z < 50 L_g$$

$$\delta = 2 \text{ (out of resonance)}$$

simulation
by G.R.M Robb

$$\left(z_1 = \frac{z - vt}{L_c} \right)$$

SELF-SIMILAR SUPERRADIANT SOLUTION

particular solution of the
FEL eqs:

$$|A|_{\text{peak}}^2 \propto z_1^2 \Rightarrow P \propto N^2$$

$$\text{width} \propto z_1^{-1/2} \propto 1/\sqrt{N}$$

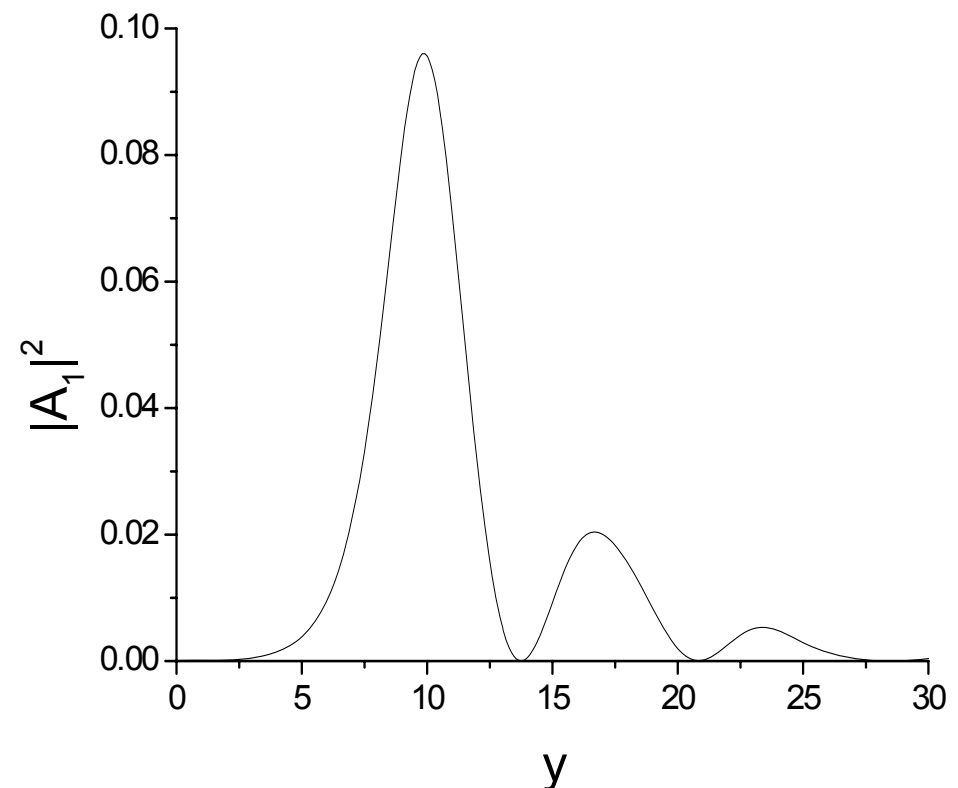
$$A(\bar{z}, z_1) = z_1 A_1(y)$$

$$\theta_j(\bar{z}, z_1) = \theta_{1j}(y)$$

$$y = \sqrt{z_1}(\bar{z} - z_1)$$

$$\frac{d^2 \theta_{1j}}{dy^2} = -(A_1 e^{i\theta_{1j}} + \text{c.c.})$$

$$\frac{y}{2} \frac{dA_1}{dy} + A_1 = \frac{1}{N} \sum_{j=1}^N e^{-i\theta_{1j}}$$



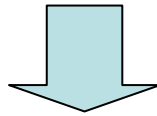
SASE mode for FELs

Ingredients of Self Amplified Spontaneous Emission (SASE)

- i) Start up from noise
- ii) Propagation effects (slippage)
- iii) Superradiance instability

R.Bonifacio, L. De Salvo, P.Pierini,
N.Piovella, C. Pellegrini, PRL (1994)

each cooperation length in the e-beam radiates a SR spike
which is amplified when it propagates forward on the beam



SASE mode operation to generate 'coherent' X-ray
radiation (LCLS, Desy, etc..)

QUANTUM-SASE REGIME

- In a **QUANTUM REGIME** an FEL behaves as a TWO-LEVEL system
- electrons emit coherent photons as in a **LASER**
- in the SASE mode the spectrum is **intrinsically narrow** ('quantum purification')
- the transition between the **classical** and the **quantum** regime depends on a **single** parameter:

$$\bar{\rho} = \left(\frac{mc \gamma_r}{\hbar k} \right) \rho$$

QUANTUM EFFECTS IN FELs

CLASSICAL LIMIT OF FEL:

electron recoil: $mc (\delta\gamma) \gg \hbar k$: photon recoil

since in the classical regime $(\delta\gamma / \gamma_r)_{\max} \approx \rho \Rightarrow \bar{\rho} \gg 1$ $\bar{\rho} = \left(\frac{mc \gamma_r}{\hbar k} \right) \rho$

many recoils implies many photons, hence..
classically, each electron emits many photons

$$\frac{\langle N \rangle_{\text{photon}}}{N} = \bar{\rho} |A|^2 \gg 1 \quad (\text{since } A \sim 1)$$

the **QUANTUM REGIME** of **FEL** occurs when:

$$mc (\delta\gamma) \leq \hbar k \quad \longrightarrow \quad \bar{\rho} < 1$$

each electron emits **only** a single photon!



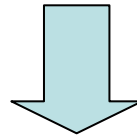
COHERENCE

Quantum FEL behaves like a **two-level system**
(i.e. a '**laser**')

QUANTUM FEL MODEL

Procedure :

Describe N particle system as a **Quantum Mechanical** ensemble

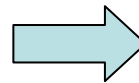


Write a **Schrödinger equation** for macroscopic wavefunction $\Psi(\theta, \bar{z})$

or equivalently ..

the equation for the **Wigner function** (quantum distribution) $W(\theta, p, \bar{z})$

Include **slippage** (i.e. propagation)
using a **multiple-scaling** approach



$\Psi(\theta, \bar{z}, z_1)$
or
 $W(\theta, p, \bar{z}, z_1)$

Canonical Quantization

$$H = \frac{p^2}{2\rho} - i\bar{\rho}(Ae^{i\theta} - c.c.)$$

$$\begin{aligned}\dot{\theta} &= \frac{p}{\rho} = \frac{\partial H}{\partial p} \\ \dot{p} &= -\bar{\rho}(Ae^{i\theta} + c.c.) = -\frac{\partial H}{\partial \theta}\end{aligned}$$

$$\left[p = \frac{mc(\gamma - \gamma_0)}{\hbar k} \right]$$

$$[\hat{\theta}, \hat{p}] = i \quad \longrightarrow \quad p \rightarrow \hat{p} = -i \frac{\partial}{\partial \theta} \quad \longrightarrow \quad i \frac{\partial \Psi}{\partial \bar{z}} = \hat{H} \Psi$$

$$i \frac{\partial \Psi}{\partial \bar{z}} = -\frac{1}{2\bar{\rho}} \frac{\partial^2 \Psi}{\partial \theta^2} - i\bar{\rho}(A(\bar{z})e^{i\theta} - c.c.)\Psi$$

$$\frac{dA}{d\bar{z}} = \int_0^{2\pi} d\theta |\Psi(\theta, \bar{z})|^2 e^{-i\theta} + i\delta A$$

**G. Preparata,
PRA (1988)**

QUANTUM FEL PROPAGATION MODEL

θ describes the spatial evolution of Ψ on the scale of λ

z_1 describes the spatial evolution of A and Ψ on the scale of $L_c \gg \lambda$.

$$(z_1 = (z - vt) / L_c)$$

Using a **multiple-scale method** we derive the 1D quantum FEL model

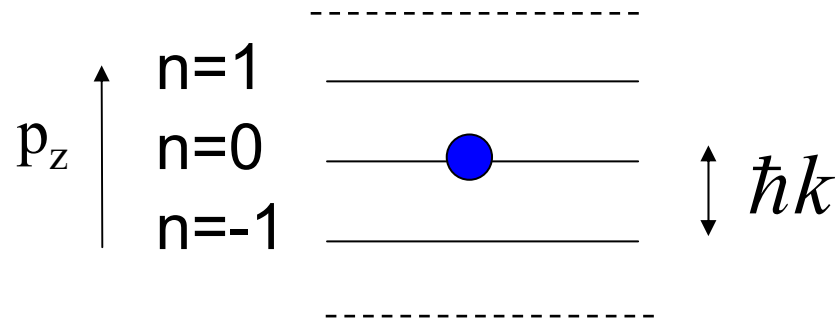
$$i \frac{\partial \Psi}{\partial \bar{z}} = - \frac{1}{2 \bar{\rho}} \frac{\partial^2 \Psi}{\partial \theta^2} - i \bar{\rho} [A(\bar{z}, z_1) e^{i\theta} - \text{c.c.}] \Psi$$
$$\frac{\partial A}{\partial \bar{z}} + \frac{\partial A}{\partial z_1} = \int_0^{2\pi} d\theta |\Psi(\theta, \bar{z}, z_1)|^2 e^{-i\theta} + i\delta A$$

Momentum representation:

$$\Psi(\theta, \bar{z}, z_1) = \sum_{n=-\infty}^{\infty} c_n(\bar{z}, z_1) e^{in\theta}$$

$|c_n|^2$ = probability to find an electron with $p_z = n(\hbar k)$ at z and z_1

discrete values of momentum : $p_z \sim mc\gamma = n(\hbar k)$, $n=0, \pm 1, \dots$



$$\frac{\partial c_n}{\partial \bar{z}} = -\frac{in^2}{2\bar{\rho}} c_n - \bar{\rho} (A c_{n-1} - A^* c_{n+1})$$

$$\frac{\partial A}{\partial \bar{z}} + \frac{\partial A}{\partial z_1} = \sum_{n=-\infty}^{\infty} c_n c_{n-1}^* + i\delta A$$

STEADY-STATE regime

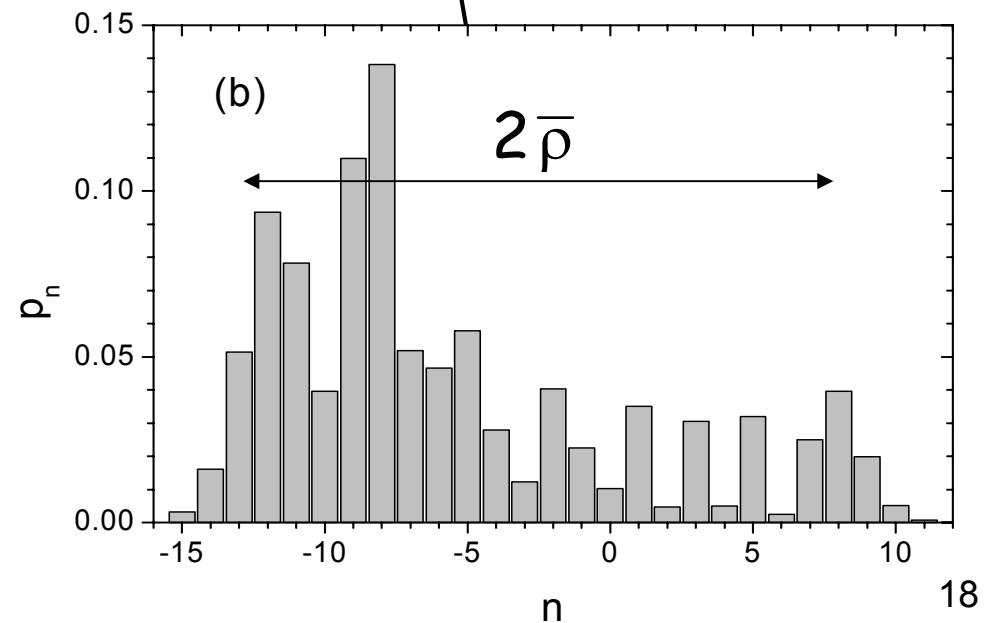
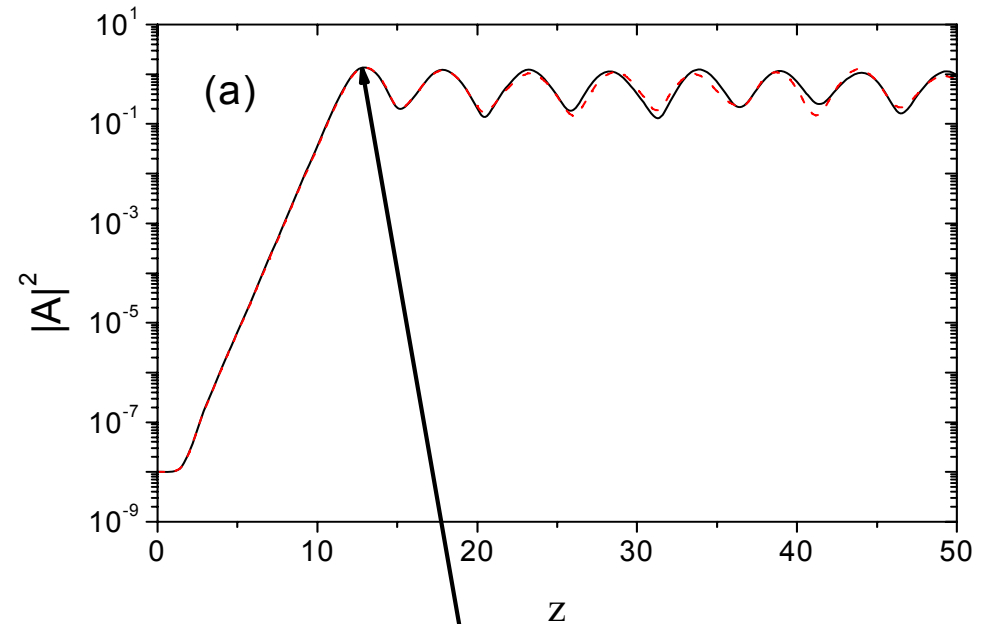
$$\left(\frac{\partial \mathbf{A}}{\partial \mathbf{z}_1} \right)$$

The classical limit
is recovered for

$$\bar{\rho} \gg 1$$

many momentum states
occupied,
both with $n > 0$ and $n < 0$

$\bar{\rho}=10, \delta=0$, no propagation

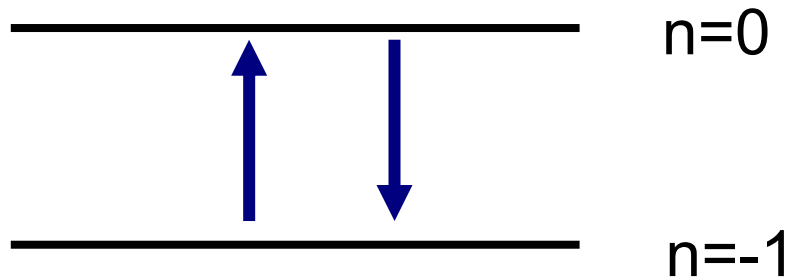


Quantum limit for $\bar{\rho} \leq 1$

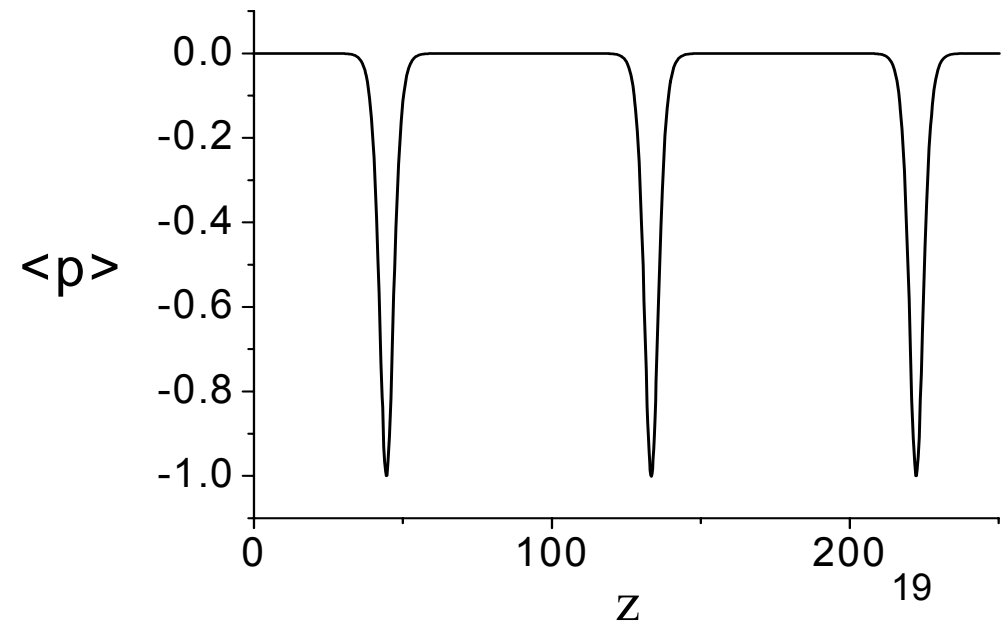
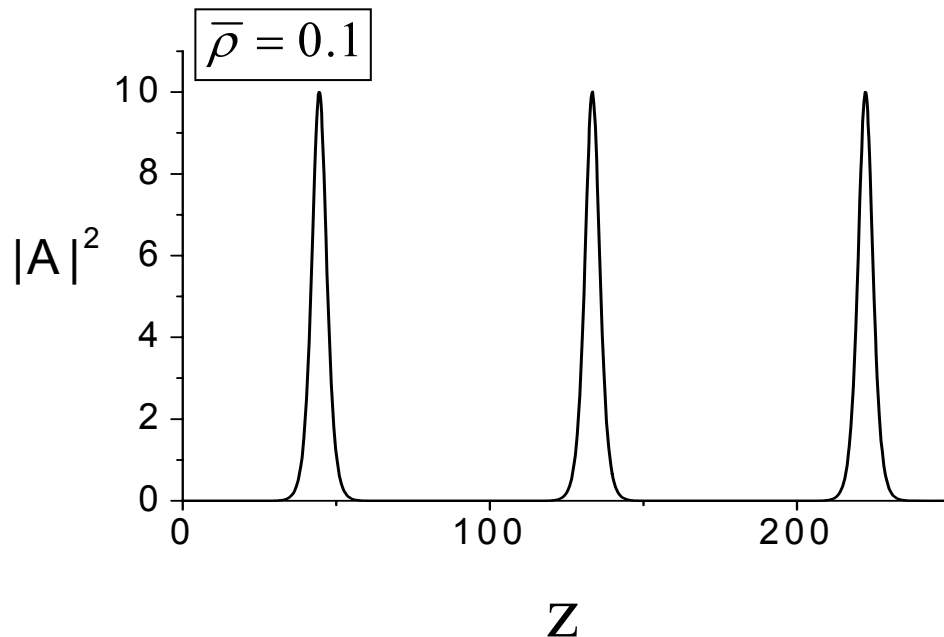
$$\left(\frac{\partial A}{\partial z_1} \right)$$

Only TWO momentum states : $p_z = 0$ and $p_z = -\hbar k$

$$\Psi(\theta, \bar{z}) \propto c_0(\bar{z}) + c_{-1}(\bar{z})e^{-i\theta}$$



Dynamics are those of a **2-level system** coupled to an optical field, as in a **LASER**



Schrödinger vs. Wigner model

Wigner function: quantum phase-space distribution:

$$\Psi(q, t) \rightarrow W(q, p, t) = \frac{1}{2\pi\hbar} \int_{-\infty}^{+\infty} dq' e^{-iq'p/\hbar} \Psi^*\left(q - \frac{q'}{2}, t\right) \Psi\left(q + \frac{q'}{2}, t\right)$$

$$\int_{-\infty}^{+\infty} dp W(q, p, t) = |\Psi(q, t)|^2 \quad \text{space distribution}$$

$$\int_{-\infty}^{+\infty} dq W(q, p, t) = |\tilde{\Psi}(p, t)|^2 \quad \text{momentum distribution}$$

$$\iint_{\mathbb{R}^2} dq dp W(q, p, t) = 1 \quad \text{normalization}$$

for a **mixed state**: $\psi(q) \rightarrow \hat{\rho}(t) \rightarrow W(q, p, t) = \frac{1}{2\pi\hbar} \int_{-\infty}^{+\infty} dq' e^{-iq'p/\hbar} \left\langle q + \frac{q'}{2} \left| \hat{\rho}(t) \right| q - \frac{q'}{2} \right\rangle$

time evolution of the system..

for a pure state :

$$i\hbar \frac{\partial \Psi}{\partial t} = \hat{H} \Psi \quad \Rightarrow \quad \frac{\partial W}{\partial t} = \dots$$

for a mixed state :

$$i\hbar \frac{d\hat{\rho}}{dt} = [\hat{H}, \hat{\rho}] \quad \Rightarrow \quad \frac{\partial W}{\partial t} = \dots$$

what about **quantum FEL**? $\Psi(\theta)$ is a **periodic** function ($0 < \theta < 2\pi$)

$[\hat{\theta}, \hat{p}] = i$ analogue to **phase** and **angular momentum** operators

$\rightarrow p = m$, $m = 0, \pm 1, \dots$

$\rightarrow W(\theta, p)$ with $p \in (-\infty, +\infty)$ is not well defined!



discrete Wigner function

since θ is an angular variable in $(0, 2\pi]$, then p is discrete ($p=n$),
 .. we need a discrete Wigner function !

$$W_n(\theta) = \frac{1}{\pi} \int_{-\pi/2}^{\pi/2} d\theta' e^{-2in\theta'} \langle \theta + \theta' | \hat{\rho} | \theta - \theta' \rangle = w_n(\theta) + \sum_{n'=-\infty}^{+\infty} \text{sinc}[(n - n' - 1/2)\pi] w_{n'+1/2}(\theta)$$

two 'extra' Wigner functions w_n and $w_{n+1/2}$, with **integer** and **half-integer** index:

$$w_s(\theta) = \frac{1}{2\pi} \int_{-\pi}^{\pi} d\theta' e^{-2is\theta'} \langle \theta + \theta' | \hat{\rho} | \theta - \theta' \rangle \quad (s=n \text{ or } s=n+1/2)$$

1D Wigner model for QFEL

N Piovella, MM Cola, L Volpe,
A Schiavi, R Gaiba, R Bonifacio
Optics Comm. (2007)

$$\frac{\partial w_s}{\partial \bar{z}} + \frac{s}{\bar{\rho}} \frac{\partial w_s}{\partial \theta} - \bar{\rho} (A e^{i\theta} + \text{c.c.}) \{w_{s+1/2} - w_{s-1/2}\} = 0 \quad (s=n \text{ or } s=n+1/2)$$

$$\frac{\partial A}{\partial \bar{z}} + \frac{\partial A}{\partial z_1} = \sum_{n=-\infty}^{+\infty} \int_{-\pi}^{\pi} d\theta e^{-i\theta} w_{n+1/2}(\theta, z_1, \bar{z}) + i\delta A$$

it is equivalent to the ' c_n ' equations for a pure state with:

$$\hat{\rho} = |\psi\rangle\langle\psi| \quad \Psi(\theta) = \sum_{n=-\infty}^{\infty} c_n e^{in\theta}$$

$$w_{n+\mu/2} = \frac{1}{2\pi} \sum_{n'=-\infty}^{+\infty} e^{i(2n'+\mu)\theta} c_{n+n'+\mu}^* c_{n-n'} \quad (\mu=0,1)$$

CLASSICAL LIMIT:

for $\bar{\rho} \gg 1$ $\bar{p} = \frac{s}{\bar{\rho}}$ becomes a continuous variable

$$\bar{\rho}(w_{s+1/2} - w_{s-1/2}) \rightarrow \bar{\rho} \left[w \left(\bar{p} + \frac{1}{2\bar{\rho}} \right) - w \left(\bar{p} - \frac{1}{2\bar{\rho}} \right) \right] \rightarrow \frac{\partial w}{\partial \bar{p}}$$

$$\frac{\partial w}{\partial \bar{z}} + \bar{p} \frac{\partial w}{\partial \theta} - (A e^{i\theta} + \text{c.c.}) \frac{\partial w}{\partial \bar{p}} = 0$$

$$\frac{\partial A}{\partial \bar{z}} + \frac{\partial A}{\partial z_1} = \int_{-\infty}^{+\infty} d\bar{p} \int_{-\pi}^{\pi} d\theta e^{-i\theta} w(\theta, \bar{p}, z_1, \bar{z}) + i\delta A$$

VLASOV
(or 'Liouville') equation

TO MAKE THE QUANTUM FEL MODEL **MORE REALISTIC**..
..it must include:

1. **initial energy spread** ($\rightarrow$ inhomogeneous broadening (see later..))
2. **3D dynamics** (emittance, focusing, optical guiding..)
3. **laser wiggler**

main points:

- A. Quantum effects relevant only **longitudinally** (photon recoil)
- B. Electrons in their transverse motion behave **classically**
- C. Emittance described by a **thermal** initial distribution..

$\Psi(\theta, x, y)$?? **NO**, because momentum spread is $\Delta p_x \sim \hbar/(\Delta x)$.

since $\Delta p_x = mc\gamma\Delta x'$, the normalized emittance is

$$\boxed{\varepsilon_n \approx \gamma \Delta x \Delta x' \approx \frac{\hbar}{mc} = \hat{\lambda}_{\text{Compton}}} \quad (\sim 10^{-7} \text{ mm}\cdot\text{mrad} !!)$$

SOLUTION:

Electron's 3D dynamics described by the Wigner function

$$W_n(\theta, x, y, p_x, p_y, z, z_1)$$

Radiation 3D evolution described by paraxial Maxwell eqs. for

$$A(x, y, z, z_1)$$

Laser wiggler described by a given (x, y, z) profile

$$a_w(x, y, z) = a_{w0} g(x, y, z)$$

From the exact quantum 3D **Hamiltonian** ..

from the Liouville – Von Neuman eq. for a mixed state :

$$i\hbar \frac{d\hat{\rho}}{dt} = [\hat{H}_{3D}, \hat{\rho}]$$

$$H_{3D} = \underbrace{\frac{p^2}{2\bar{\rho}} - i(g^* A e^{i\theta} - \text{h.c.})}_{\text{longitudinal motion}} + \underbrace{\alpha b \frac{p_{\perp}^2}{2} + \frac{\xi}{2\alpha \rho X} |g|^2}_{\text{transverse motion}} + \underbrace{\left[\frac{\xi}{2\rho} (1 - |g|^2) - \alpha^2 \frac{b^2}{4a} |\vec{p}_{\perp}|^2 \right]}_{\text{'mixed' (detuning)}} p$$

longitudinal motion

transverse motion

'mixed' (detuning)

$$[\theta, p] = i \quad [x, p_x] = [y, p_y] = i$$

$$\left(\vec{p}_{\perp} = \frac{m\gamma \vec{v}_{\perp}}{\hbar}, \vec{p}_{\perp} = -i\vec{\nabla}_{\perp} \right)$$

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(see poster by Mary Cola)



...to the equation for the **3D Wigner function**...

$$w_s(\theta, \vec{x}, \vec{p}_\perp) = \frac{1}{2\pi^3} \int d^2x' \int_{-\pi}^{+\pi} d\theta' e^{-2i(s\theta' + \vec{p}_\perp \cdot \vec{x}')} \langle \theta + \theta', \vec{x} + \vec{x}' | \hat{\rho} | \vec{x} - \vec{x}', \theta - \theta' \rangle$$

$(s = m, m + 1/2 \quad m = 0, \pm 1, \dots)$

... and to the **classical limit** for transverse motion

$$\alpha = \frac{\hat{\lambda}_{\text{Compton}}}{\varepsilon_n} \ll 1$$

We expand the exact equation for w_s in power of α
and keep the lowest terms

3D QUANTUM FEL MODEL

(Bonifacio, Piovella, Cola, Volpe, Schiavi, *submitted* 2007)

$$\begin{aligned} & \frac{\partial w_s}{\partial \bar{z}} + \left\{ \frac{s}{\bar{\rho}} + \delta + \frac{\xi}{2\rho} (1 - |g|^2) - \frac{b^2}{4a} |\mathbf{p}_\perp|^2 \right\} \frac{\partial w_s}{\partial \theta} + b(\vec{p}_\perp \cdot \vec{\nabla}_\perp) w_s \\ & - \bar{\rho} (g^* A e^{i\theta} + \text{c.c.}) \{w_{s+1/2} - w_{s-1/2}\} - \frac{\xi}{\rho X} (\vec{\nabla}_\perp |g|^2) \cdot (\vec{\nabla}_{\mathbf{p}_\perp} w_s) = 0 \\ & \frac{\partial A}{\partial \bar{z}} + \frac{\partial A}{\partial z_1} - i a \nabla_\perp^2 A = g \sum_{m=-\infty}^{+\infty} \int d^2 p_\perp \int_{-\pi}^{\pi} d\theta e^{-i\theta} w_{m+1/2}(\theta, \vec{x}, \vec{p}_\perp, \bar{z}, z_1) \end{aligned}$$

$$s=0, \pm 1/2, \pm 1, \dots$$

transverse variables: $\vec{X}$ in units of σ (spatial spread Δx)
 $\vec{p}_\perp$ in units of $\varepsilon_n/\gamma\sigma$ ('momentum' spread $\Delta x'$)

3D parameters:

$$b = \frac{L_g}{\beta^*}, \quad \beta^* = \frac{\gamma\sigma^2}{\varepsilon_n} \text{ ('beta function')}$$

$$a = \frac{L_g}{Z_r}, \quad Z_r = \frac{4\pi\sigma^2}{\lambda_r} \text{ (radiation Rayleigh range)}$$

$$\xi = \frac{a_{w0}^2}{1 + a_{w0}^2}$$

(only for **laser wiggler**)

$$X = \frac{b}{a} = \frac{Z_r}{\beta^*} = \frac{4\pi\varepsilon_n}{\gamma\lambda_r}$$

interpretation:

$$\left\{ \dots \frac{\xi}{2\rho} (1 - |g|^2) \dots \right\} p \frac{\partial w_s}{\partial \theta} \rightarrow \text{detuning from laser wiggler } (\Delta a_w / a_w)$$

$$\left\{ \dots \frac{b^2}{4a} |\vec{p}_\perp|^2 \dots \right\} p \frac{\partial w_s}{\partial \theta} \rightarrow \text{resonance spread from emittance } (\gamma^2 \Delta \theta_{x,y}^2)$$

$$\dots b(\vec{p}_\perp \cdot \vec{\nabla}_\perp w_s) \dots \rightarrow \text{beam expansion due to emittance}$$

$$\dots - \frac{\xi}{2\rho X} (\vec{\nabla}_\perp |g|^2) \cdot (\vec{\nabla}_{p_\perp} w_s) \dots \rightarrow \text{laser wiggler gradient force}$$

3D CLASSICAL LIMIT: from the Wigner to the Vlasov equation..

$$\bar{\rho} \gg 1$$

$w_s(\theta, x, y, p_x, p_y, z, z_1) \rightarrow W(\theta, \mathbf{p}, x, y, p_x, p_y, z, z_1)$ with $\mathbf{p} = \mathbf{s} / \bar{\rho}$ continuous

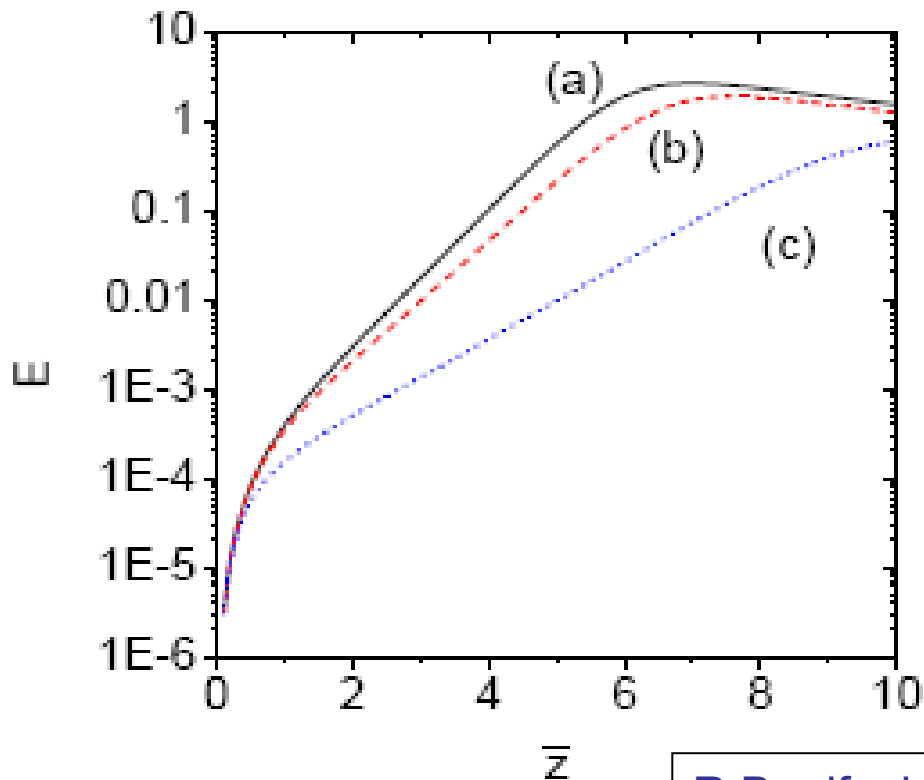
$$\begin{aligned} & \frac{\partial W}{\partial \bar{\mathbf{z}}} + \left\{ \mathbf{p} + \delta + \frac{\xi}{2\rho} (1 - |g|^2) - \frac{b^2}{4a} |\vec{\mathbf{p}}_{\perp}|^2 \right\} \frac{\partial W}{\partial \theta} + b \vec{\mathbf{p}}_{\perp} \cdot \vec{\nabla}_{\perp} W \\ & - (A e^{i\theta} + \text{c.c.}) \frac{\partial W}{\partial \mathbf{p}} - \frac{\xi}{2\rho X} (\nabla_{\bar{\mathbf{x}}} |g|^2) \cdot \vec{\nabla}_{\mathbf{p}_{\perp}} W = 0 \\ & \frac{\partial A}{\partial \bar{\mathbf{z}}} + \frac{\partial A}{\partial \mathbf{z}_1} - i a \nabla_{\perp}^2 A = g \int d^2 \mathbf{p}_{\perp} \int_{-\infty}^{+\infty} d\mathbf{p} \int_{-\pi}^{\pi} d\theta e^{-i\theta} W(\theta, \mathbf{p}, \bar{\mathbf{x}}, \vec{\mathbf{p}}_{\perp}, \mathbf{z}_1, \bar{\mathbf{z}}) \end{aligned}$$

NUMERICAL RESULTS: (see Angelo Schiavi's next talk..)

example: **inhomogenous emittance**

[(a) $\varepsilon_n=0$, (b) $\varepsilon_n=0.05$ mm mrad, (c) $\varepsilon_n=0.1$ mm mrad]

[for instance $\lambda_r=2$ Å, $\lambda_L=1\mu\text{m}$, $\gamma=36$, $I=800$ A, $\sigma=10\mu\text{m}$, $a_{w0}=0.15$, $L_g=1.3$ mm]



$$\frac{b}{2\sqrt{a}} = 0.5 \text{ (b) , } 1 \text{ (c)}$$

$$Z_r \approx 6250 L_g \quad (a = 1.6 \cdot 10^{-6})$$

$$\frac{b}{2\sqrt{a}} < 1 \Rightarrow \varepsilon_n < \frac{\gamma \lambda_r}{2\pi} \sqrt{\frac{Z_r}{L_g}}$$

CONCLUSIONS

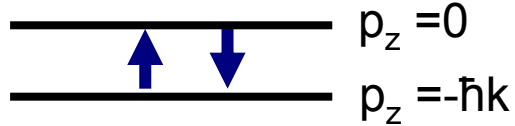
we have presented a **3D Quantum FEL** model, based on a **Wigner function** for the electron beam.

The model include:

- Optical diffraction (guiding)
- Transverse effects in the electron beam (defocusing, emittance ..)
- Laser wiggler
- The equations couple the **longitudinal** and the **transverse** dynamics.
- The longitudinal recoil is quantized in units of $\hbar k$.
- The electron beam behaves classically in its transverse motion.

QUANTUM REGIME (TWO-LEVEL APPROXIMATION)

$$\bar{\rho} < 1$$



only two momentum states $p_z = 0$ and $p_z = -\hbar k$

$$D = \int_{-\pi}^{+\pi} d\theta [w_0(\theta) - w_{-1}(\theta)] \quad \text{population difference}$$

$$B = \int_{-\pi}^{+\pi} d\theta w_{-1/2}(\theta) e^{-i\theta} \quad \text{bunching ('polarization')}$$

'quasi-Bloch' equations
(similar as for **lasers**)

$$\frac{\partial D}{\partial \bar{z}} + b \vec{p}_\perp \cdot \nabla_\perp D = -2(g^* A B^* + c.c.) + v(\nabla_\perp |g|^2) \cdot \nabla_{p_\perp} D$$

$$\frac{\partial B}{\partial \bar{z}} + b \vec{p}_\perp \cdot \nabla_\perp B = i \left\{ \Delta - \bar{\xi}(1 - |g|^2) - \frac{b^2}{4a} |\vec{p}_\perp|^2 \right\} B + g^* D A + v(\nabla_\perp |g|^2) \cdot \nabla_{p_\perp} D$$

$$\frac{\partial A}{\partial \bar{z}} + \frac{\partial A}{\partial z_1} - i a \nabla_\perp^2 A = g \int d^2 p_\perp B$$

$$L_g \rightarrow L_g / \sqrt{\bar{\rho}} \quad , \quad A \rightarrow \sqrt{\bar{\rho}} A \quad \bar{\xi} = \frac{\xi}{2\rho\sqrt{\bar{\rho}}} \quad , \quad v = \frac{\xi}{\rho\sqrt{\bar{\rho}}X} \quad , \quad \Delta = \frac{mc(\gamma_0 - \gamma_R) - \hbar k/2}{\hbar k \bar{\rho}^{3/2}}$$

How to model the INITIAL ENERGY SPREAD?

$$\frac{\partial c_n}{\partial \bar{z}} = -in \left(\frac{n}{2\bar{\rho}} + \delta \right) c_n - \bar{\rho} (A c_{n-1} - A^* c_{n+1})$$

$$\frac{\partial A}{\partial \bar{z}} + \frac{\partial A}{\partial z_1} = \sum_{n=-\infty}^{\infty} \int d\delta G(\delta) c_n c_{n-1}^*$$

$G(\delta)$; initial distribution of width σ

$$A \rightarrow A e^{i\delta \bar{z}}$$

$$c_n \rightarrow c_n e^{in\delta \bar{z}}$$

linear theory $\rightarrow$ GENERALIZED DISPERSION RELATION : $(A \propto e^{i\lambda \bar{z}})$

$$\lambda - \Delta + \bar{\rho} \int_{-\infty}^{\infty} \frac{d\delta}{\lambda + \delta} \left\{ G\left(\delta + \frac{1}{2\bar{\rho}}\right) - G\left(\delta - \frac{1}{2\bar{\rho}}\right) \right\} = 0$$

$$\left(\Delta = \delta + \frac{n}{\bar{\rho}} - \bar{\omega} \right)$$

N. Piovella, R. Bonifacio, NIMA (2006)

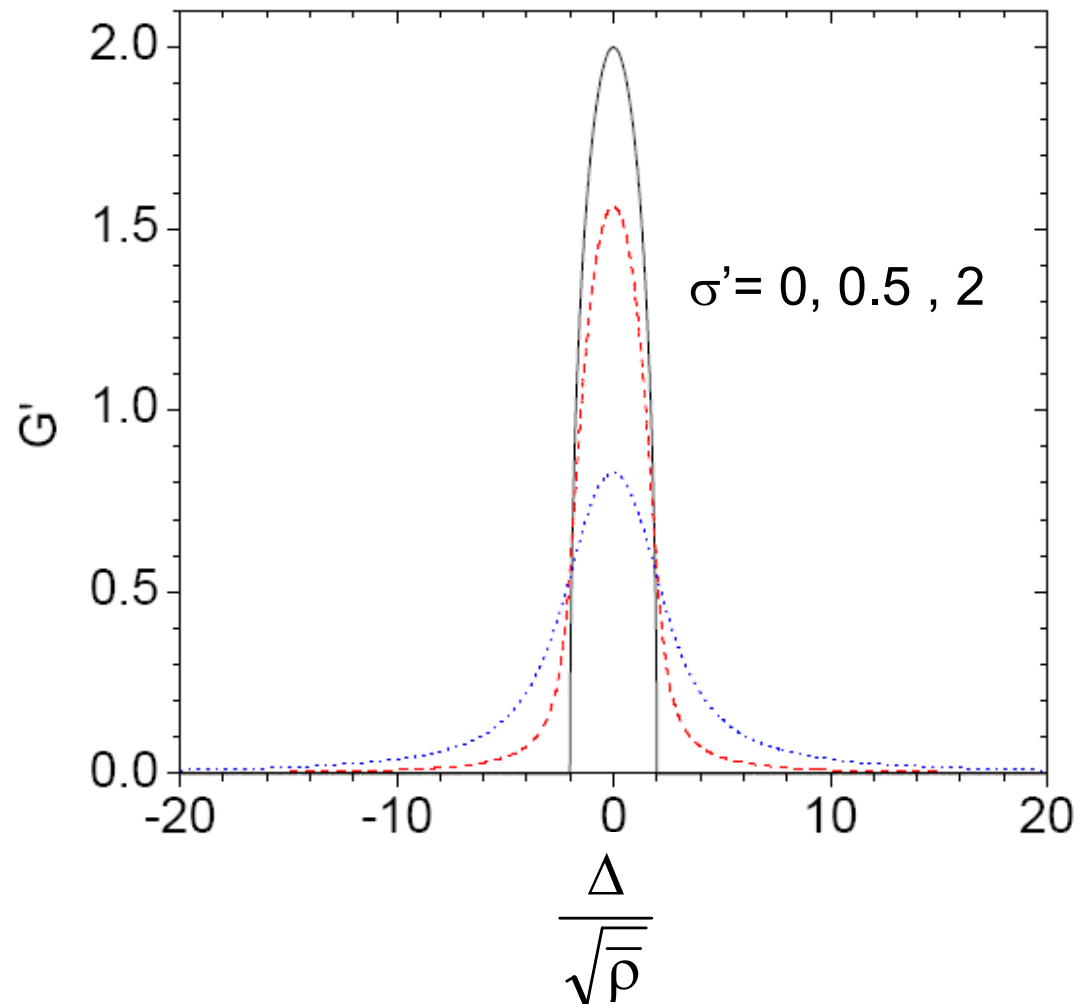
when $G(\delta)$ is Lorentzian $\rightarrow$

$$(\lambda - \Delta) \left[(\lambda - i\sigma)^2 - \frac{1}{4\bar{\rho}^2} \right] + 1 = 0$$

for $\bar{\rho} < 1$

$$\text{max gain} = \frac{1}{L_g} \left[\sqrt{4 + \sigma'^2} - \sigma' \right]$$

$$\sigma' = \frac{1}{\rho\sqrt{\bar{\rho}}} \left(\frac{\delta\gamma}{\gamma} \right)_0$$



$$\bar{\rho} = 0.1$$

$$\left(\frac{\delta\gamma}{\gamma} \right)_0 = 0.1 \rho$$

population of $n=0$
($p=-\hbar k$)

population of $n=0$
($p=0$)

$$\bar{\rho} = 0.1$$

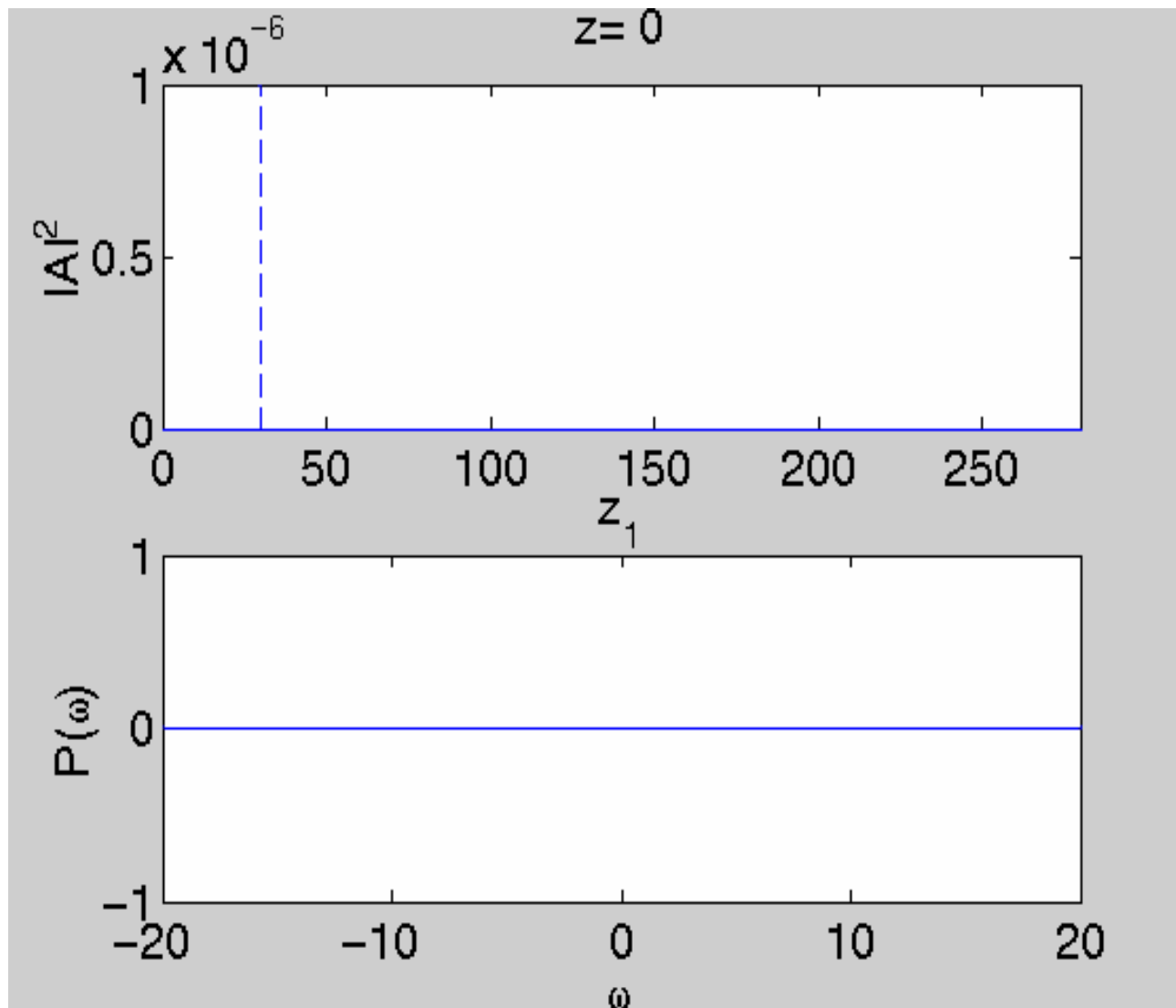
$$\left(\frac{\delta\gamma}{\gamma} \right)_0 = \rho$$

population of $n=0$
($p=-\hbar k$)

population of $n=0$
($p=0$)

$\bar{\rho} = 0.1$ **SASE** with energy spread

$$\left(\frac{\delta\gamma}{\gamma}\right)_0 = 0.5\rho$$



coherence is
preserved if

$$mc\delta\gamma_0 < \hbar k$$

but gain decreases for

$$\left(\frac{\delta\gamma}{\gamma}\right)_0 > \rho\sqrt{\bar{\rho}}$$

simulation by G.R.M Robb