Analysis of High-Gain FEL Higher-Order Paraxial Mode Coupling:

Virtual Dielectric Waveguide Eigenmode Expansion

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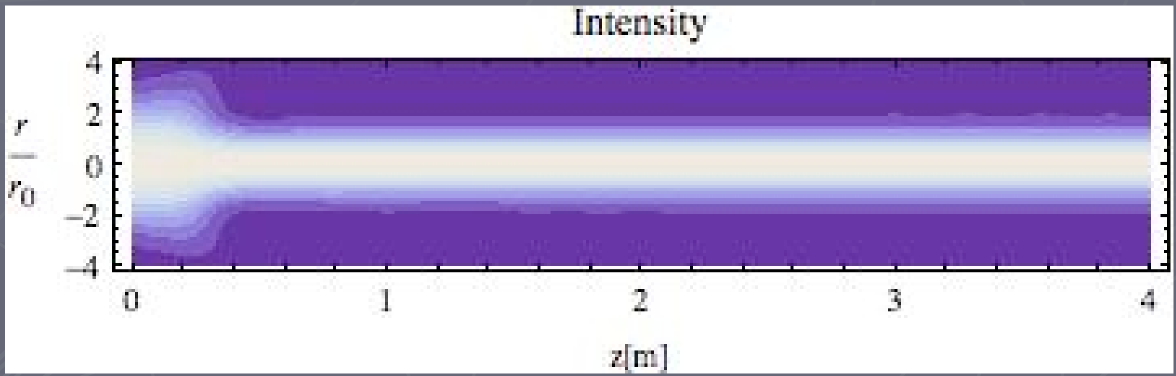
Talk Outline

- Concept and Motivation
  - virtual dielectric waveguide mode expansion
- Character of, and general applications for, higher-order paraxial modes
- Overview of FEL expansion equations
- Preliminary modeling results
- Future directions

Concept

Describe the gain-guided FEL radiation field with a basis of *guided* Hermite-Gaussian or Laguerre Gaussian modes



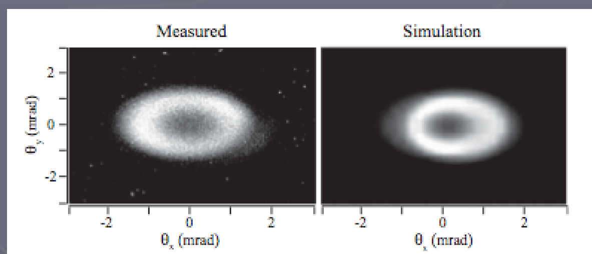
Why?

To explore the propagation and coupling characteristics of free-space paraxial light modes that are gain-guided in an FEL over many Rayleigh lengths

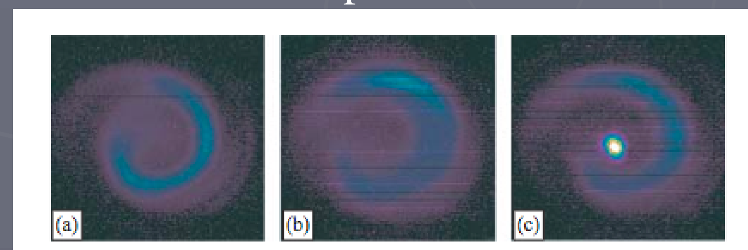
Direct Motivation

- To understand, and determine origin of, "exotic" mode structures observed in transverse intensity pattern of VISA FEL.

hollow



spiral



- To describe the guided radiation in a general, efficient way through a natural gaussian mode basis of paraxial optics
- To investigate the generation of higher-order modes in FELs for future experiments that may utilize specific spatial/phase structures (Orbital Angular Momentum) present in such modes

Example modes in paraxial optics

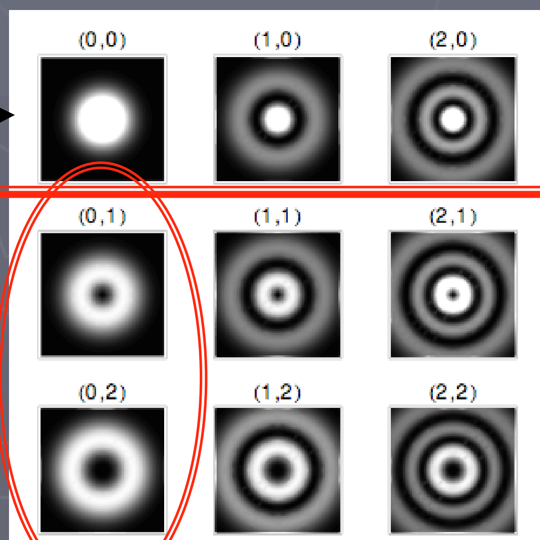
Laguerre-Gaussian Modes from Paraxial Wave Equation

$$\tilde{\mathcal{E}}_{\perp;p,l}(r_\perp, z) \propto e^{i\Phi_{p,l}(\underline{r}) - \frac{r^2}{w(z)^2}} \left(\frac{\sqrt{2}r}{w(z)}\right)^{|l|} L_p^{|l|}\left(\frac{2r^2}{w(z)^2}\right)$$

(p,l)

Fundamental (Gaussian)

OAM Modes



particularly useful for system with cylindrical symmetry

helical phase structure

l=-1 *l=1* *l=2*

The addition of multiple modes can result in spiral intensity patterns like those observed at VISA

hollow modes l>0

Why are azimuthal LG modes interesting?

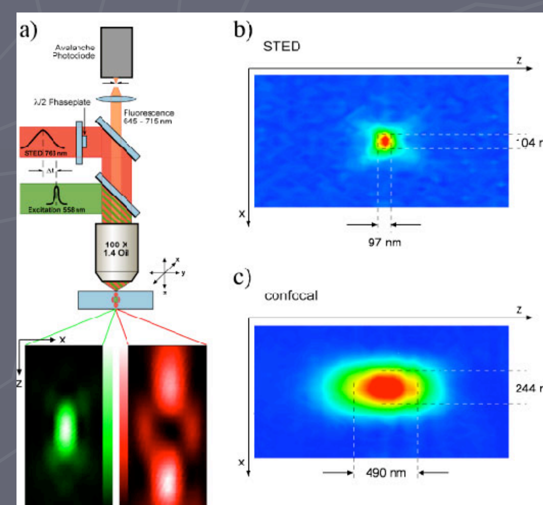
• Transfer of rotational motion, or torque, to samples. Proteins, molecules, dielectric beads, etc →

http://www.physics.gla.ac.uk/Optics/projects/spinningAndOrbiting/

• Quantum Entanglement and Encryption - store information in azimuthal modes rather than binary photon spin state. N-dimensional storage

• Knotted, braided phase topologies, singularity on axis (phase recovery tricky)

• Sub-diffraction imaging through Stimulated Emission Depletion (STED) microscopy →



http://www.mpibpc.gwdg.de/groups/hell/STED.htm

• many others...

Many of these concepts can have applications to present and/or future FEL light sources!

Can we generate LG modes like this in an FEL?

To find out what is involved, we pursue an expansion that will retain the interesting and useful aspects of LG paraxial modes, but will also efficiently (fewest # of modes) describe during gain-guided propagation

General Eigenmode Field Expansions

In the presence of a current, the transverse E-field is expanded into complete basis set with slowly-growing amplitude coefficients

Paraxial (free-space) Modes

$$\underline{\tilde{E}}_\perp(\underline{r}) = \sum_q C_q(z)\underline{\tilde{\mathcal{E}}}_{\perp q}(r_\perp, z)e^{ik_z z}$$

$$\frac{d}{dz}C_q(z) = -\frac{1}{4\mathcal{P}_q}e^{-ik_z z}\int\int \underline{\tilde{J}}_\perp(\mathbf{r})\cdot\underline{\tilde{\mathcal{E}}}^*_{\perp q}(r_\perp, z)d^2\mathbf{r}_\perp$$

Hollow waveguide Modes

$$\underline{\tilde{E}}_\perp(\underline{r}) = \sum_q C_q(z)\underline{\tilde{\mathcal{E}}}_{\perp q}(r_\perp)e^{ik_{qz} z}$$

$$\frac{d}{dz}C_q(z) = -\frac{1}{4\mathcal{P}_q}e^{-ik_{zq} z}\int\int \underline{\tilde{J}}_\perp(\mathbf{r})\cdot\underline{\tilde{\mathcal{E}}}^*_{\perp q}(r_\perp)d^2\mathbf{r}_\perp$$

Virtual Dielectric Waveguide Mode Expansion

FEL guiding process is imagined to be analogous to that of a guiding fiber, with refractive index that varies transversely

Field Expansion

$$\underline{\tilde{E}}_\perp(\underline{r}) = \sum_q C_q(z)\underline{\tilde{\mathcal{E}}}_{\perp q}(r_\perp)e^{ik_{qz}z}$$

where the modes satisfy:

$$\nabla n^2 \ll k$$

$$\nabla^2_\perp \underline{\tilde{\mathcal{E}}}_{\perp q}(r_\perp) + [n(r_\perp)^2 k^2 - k_{zq}^2]\underline{\tilde{\mathcal{E}}}_{\perp q}(r_\perp) = 0$$

Amplitude evolution equation

$$\frac{d}{dz}C_q(z) = -\frac{1}{4\mathcal{P}_q}e^{-ik_{zq}z}\int\int \underline{\tilde{J}}_\perp(\mathbf{r})\cdot \underline{\tilde{\mathcal{E}}}^*_{\perp q}(r_\perp)d^2\mathbf{r}_\perp - i\sum_{q'} C_{q'}(z)e^{-i\Delta k_{zqq'}z}\kappa^d_{q,q'}$$

where...

$$\kappa^d_{q,q'} = \frac{\omega\epsilon_0}{4\mathcal{P}_q}\int\int [n(r_\perp)^2 - 1]\underline{\tilde{\mathcal{E}}}_{\perp,q'}(r_\perp)\cdot\underline{\tilde{\mathcal{E}}}^*_{\perp q}(r_\perp)d^2\mathbf{r}_\perp$$

..results from the virtual polarization currents, and preserves diffracting characteristics of field

Expansion Mode Approach: Regimes of Applicability

During high-gain, e-beam acts like a guiding light source for otherwise free-space paraxial light propagation

Relevant quantities: z_R, L_G, L_w

High-Gain Regime: $L_G << L_w$

- Guided Mode Expansion: $L_G \leq z_R << L_w$
- Paraxial or Guided Mode Expansion: $L_G << L_w < z_R$

Paraxial+Guided (Hybrid) or Source Dependent or Co-moving Expansion: $z_R << L_G < L_w$ (strongly diffracting, may not have high gain in single pass)

Return to Paraxial Waves: Useful Solutions to guiding dielectric medium

Laguerre-Gaussian (and Hermite-Gaussian) functions are ubiquitous solutions in laser optics to the paraxial wave equation...

$$(\nabla_\perp^2 + 2ik\frac{\partial}{\partial z})\tilde{\mathcal{E}}_\perp(r_\perp, z) = 0 \quad \Rightarrow \quad \tilde{\mathcal{E}}_{\perp;p,l}(r_\perp, z) \propto e^{i\Phi_{p,l}(\underline{r}) - \frac{r^2}{w(z)^2}} \left(\frac{\sqrt{2}r}{w(z)}\right)^{|l|} L_p^{|l|}\left(\frac{2r^2}{w(z)^2}\right)$$

BUT, they also show up in the context of guided modes, as solutions to a dielectric medium with a quadratic transverse dependence

$$\nabla_\perp^2 \underline{\tilde{\mathcal{E}}}_{\perp;p,l}(r_\perp) + [n(r_\perp)^2 k^2 - k_{z;p,l}^2]\underline{\tilde{\mathcal{E}}}_{\perp;p,l}(r_\perp) = 0$$

with $n^2(r_\perp) = n_0^2[1 - 2\Delta\left(\frac{r}{a}\right)^2] \qquad \Delta \ll 1$

$$\tilde{\mathcal{E}}_{\perp;p,l}(r, \phi) \propto \sqrt{\frac{2}{\pi w_0^2}\frac{p!}{(p+|l|)!}} e^{-il\phi} e^{-\frac{r^2}{w_0^2}} (-1)^p \left(\frac{r\sqrt{2}}{w_0}\right)^{|l|} L_p^{|l|}\left(\frac{2r^2}{w_0^2}\right)$$

where $w_0^2 \equiv \frac{2a}{kn_0\sqrt{2\Delta}}$ and $k_{z;p,l}^2 = k^2 n_0^2 - \frac{4}{w_0^2}(2p + l + 1)$ ← mode dispersion

Example e-beam description: Cold-beam Excitation Equations

Linear plasma fluid (streaming e-beam) with negligible energy spread

Electron velocity & density expansions

$$v_z(\underline{r},t) = v_{z0} + \mathrm{Re}[\tilde{v}_{z1}(\underline{r})e^{-i\omega t}]$$

$$\mathsf{n}(\underline{r},t) = \mathsf{n}_0(r_\perp) + \mathrm{Re}[\tilde{\mathsf{n}}_1(\underline{r})e^{-i\omega t}]$$

Current density expansions

$$\frac{d}{dz}\tilde{J}_{z1} = -i\omega e\tilde{\mathsf{n}}_1(\underline{r})$$

$$\nabla_\perp \cdot \tilde{J}_{\perp_1} \ll \partial \tilde{J}_{z_1}/\partial z$$

$$\underline{\tilde{J}}_\perp(\underline{r}) = -\frac{1}{2}e\tilde{\mathsf{n}}_1(\underline{r})\underline{\tilde{v}}_{\perp w}e^{-ik_w z}.$$

Density Modulation equation

$$\left[\frac{d^2}{dz^2} - 2i\frac{\omega}{v_{z0}}\frac{d}{dz} + \frac{\omega_p^2(r_\perp) - \omega^2}{v_{z0}^2}\right]\tilde{\mathsf{n}}_1(\mathbf{r}) = i\frac{\omega_p^2(r_\perp)}{v_{z0}^2}\frac{\epsilon_0}{e}\sum_q (k_{zq}+k_w)C_q(z)\tilde{\mathcal{E}}_{pm,q}(r_\perp)e^{i(k_{zq}+k_w)z}$$

Coupled Excitation Equations

$$\frac{d}{dz}C_q(z) = \tilde{i}_q e^{i\theta_q z} - i\sum_{q'} C_{q'}(z)e^{-i\Delta k_{z;qq'}z}\kappa^d_{q,q'}$$

$$\left[\frac{d}{dz} - i\theta_{pr}\right]\left[\frac{d}{dz} + i\theta_{pr}\right]\tilde{i}_q(z) = i\sum_{q'} Q_{q,q'}C_{q'}(z)e^{-i\theta_{q'}z},$$

space-charge parameter

Detuning parameter

$$\theta_q = \omega/v_{z0} - (k_q + k_w)$$

Current bunching parameter

$$\tilde{i}_q(z) = -\frac{e}{8\mathcal{P}_q}e^{-i\frac{\omega}{v_{z0}}z}\int\int \tilde{\mathsf{n}}_1(\underline{r})\underline{\tilde{v}}_{\perp w}\cdot\underline{\tilde{\mathcal{E}}}^*_{\perp q}(r_\perp)d^2\mathbf{r}_\perp.$$

Mode/e-beam coupling parameter

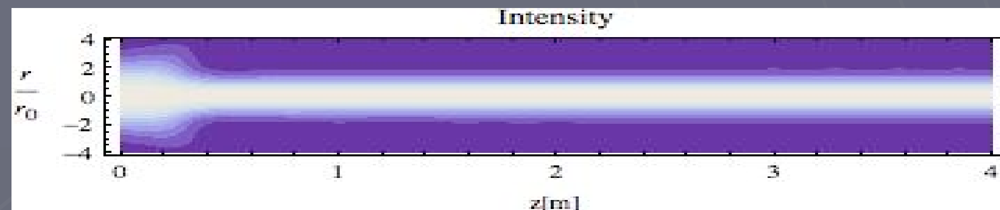
$$Q_{q,q'} = \theta_p^2 JJ\frac{\epsilon_0}{8\mathcal{P}_q}(k_{zq} + k_w)\int\int f(r_\perp)\tilde{\mathcal{E}}_{pm,q'}(r_\perp)\underline{\tilde{v}}^w_\perp\cdot\underline{\tilde{\mathcal{E}}}^*_{\perp q}(r_\perp)d^2\mathbf{r}_\perp$$

$$\tilde{\mathcal{E}}_{pm,q}(r_\perp) = \frac{1}{2}[\underline{\tilde{v}}_{\perp q}\times\underline{\tilde{\mathcal{B}}}^*_{\perp w} + \underline{\tilde{v}}^*_{\perp w}\times\underline{\tilde{\mathcal{B}}}_{\perp q}]\cdot\hat{e}_z$$

Supermode Matrix Solutions

The self-similar FEL supermode propagates as a fixed superposition of expansion modes...

$$\underline{\tilde{E}}^{sm}(\underline{r}) = \left[\sum_q b_q^{sm}\underline{\tilde{\mathcal{E}}}_q(r_\perp)\right]e^{ik^{sm}z}$$



with a characteristic wavenumber $k^{sm} = k + \delta k$

Transforming the coefficients via $C_q(z) = b_q e^{-i(\Delta k_q - \delta k)z}$ decouples the excitation equations. The supermode vectors $\underline{b}$ are then found from non-trivial solutions to the matrix eqn

$$\left[\left[(\delta k-\theta)^2-\theta_{pr}^2\right]\left[\underline{\underline{\mathbf{I}}}\delta k+\underline{\underline{\kappa}}^d-\underline{\underline{\Delta k}}\right]+\underline{\underline{Q}}\right]\underline{b}=\underline{0}$$

The dominant supermode vector $\underline{b}^{sm}$ and wavenumber are found from the determinant equation

$$\left\|\left[(\delta k-\theta)^2-\theta_{pr}^2\right]\left[\underline{\underline{\mathbf{I}}}\delta k+\underline{\underline{\kappa}}^d-\underline{\underline{\Delta k}}\right]+\underline{\underline{Q}}\right\|=0$$

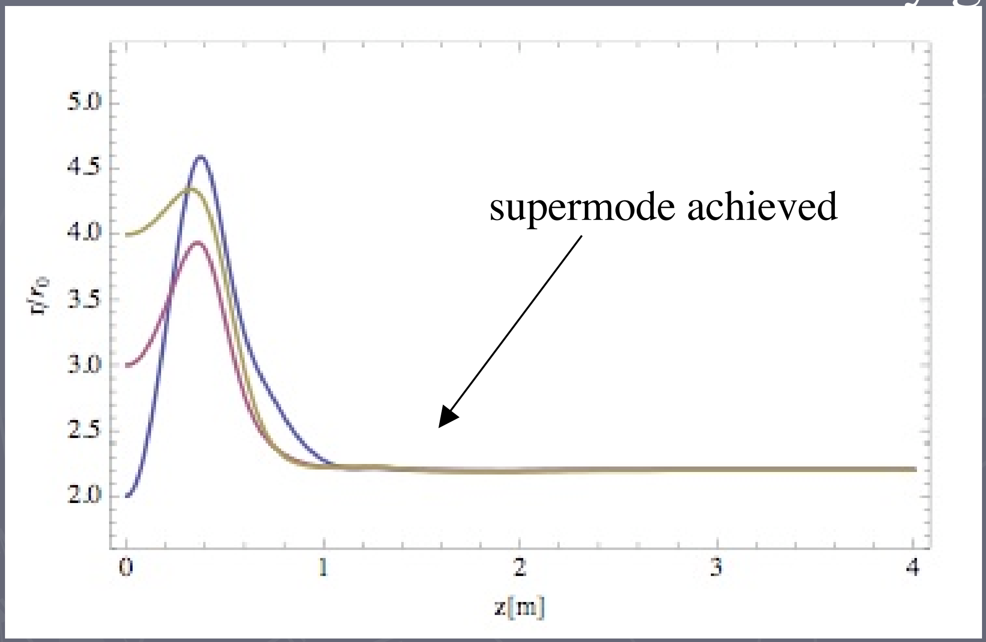
Quadratic Index Media Expansion

Expanding the FEL light field into guided LG or HG modes allows a connection with physical paraxial modes of the same form, while preserving the efficient guided description over many Rayleigh lengths.

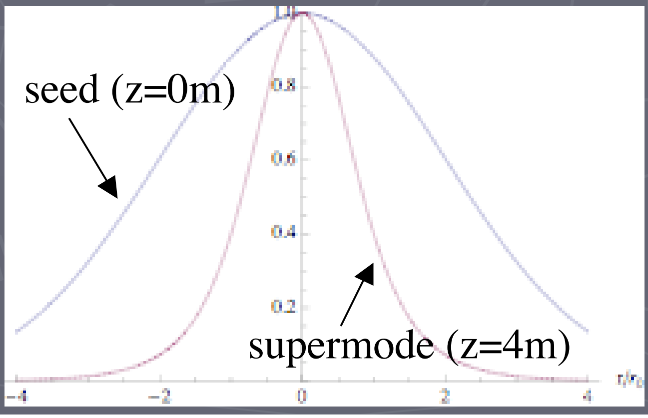
- natural description of coupling to realistic Gaussian light fields (seed laser coupling, in situ fields)
- easy transition from guided modes in FEL to free-space modes exiting (or entering) undulator
- straightforward exploration of coupling to higher-order spatial modes
    - LG light with helical phase fronts - optical orbital angular momentum studies in FEL light
- analytic solutions available for coupling to simple transverse e-beam distributions

Testing the formulation (VISA FEL)

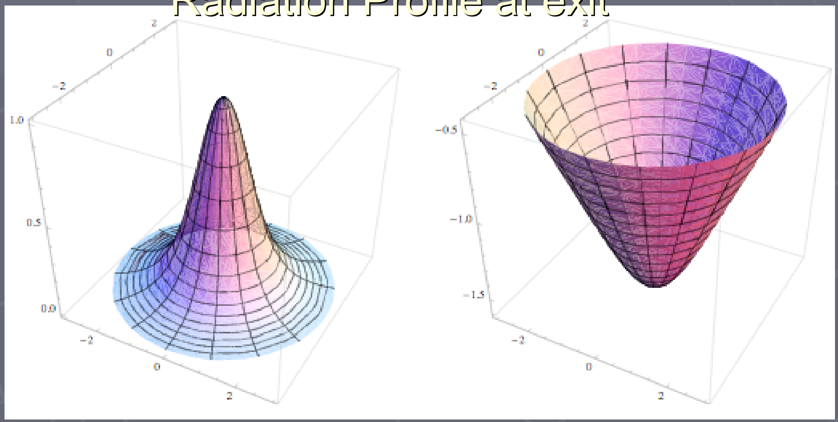
Radiation Profile evolution for various gaussian seed spot sizes, with transversely gaussian e-beam



Normalized Intensity



Radiation Profile at exit



Intensity

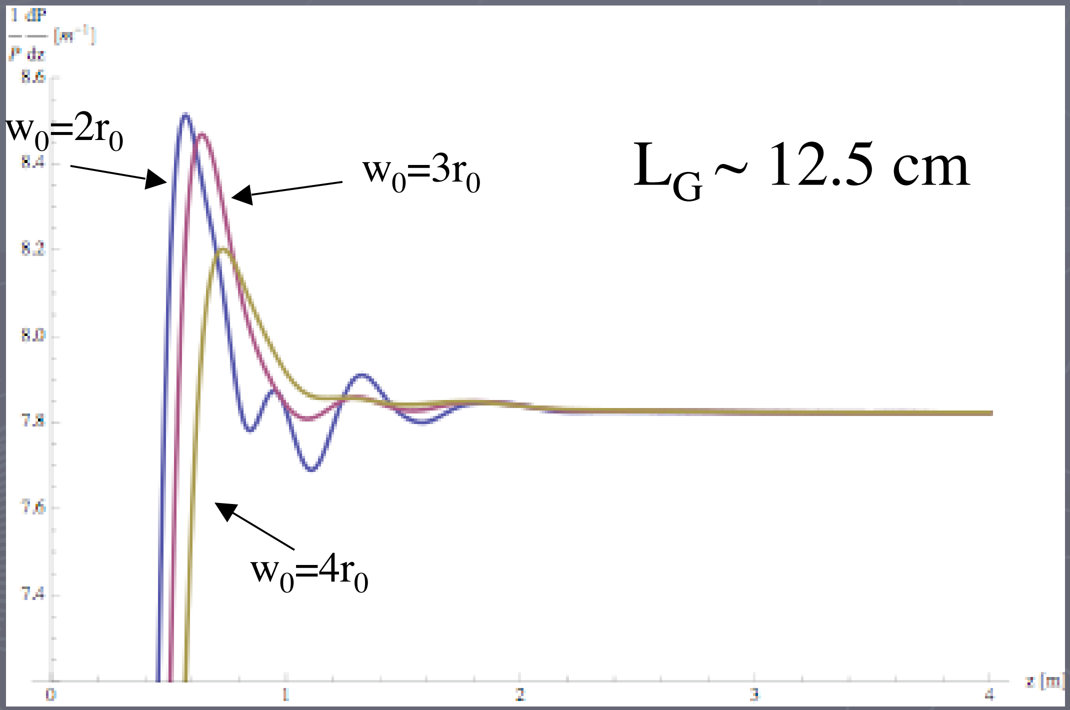
Phase

Modeling (cont)

Normalized Differential Power for Various Seed Spot Sizes

Gaussian Seed, Gaussian e-beam

$$\frac{1}{P}\frac{dP}{dz}$$



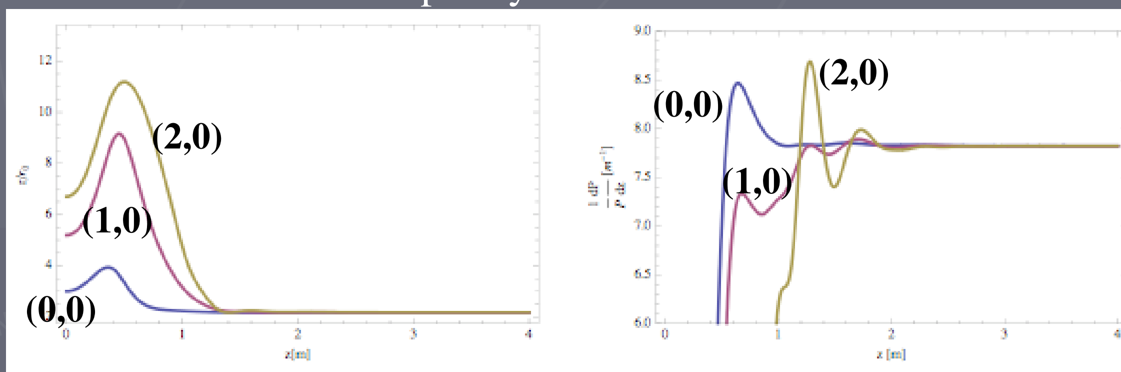
Higher-Order Mode SEEDING

Seeding with LG modes

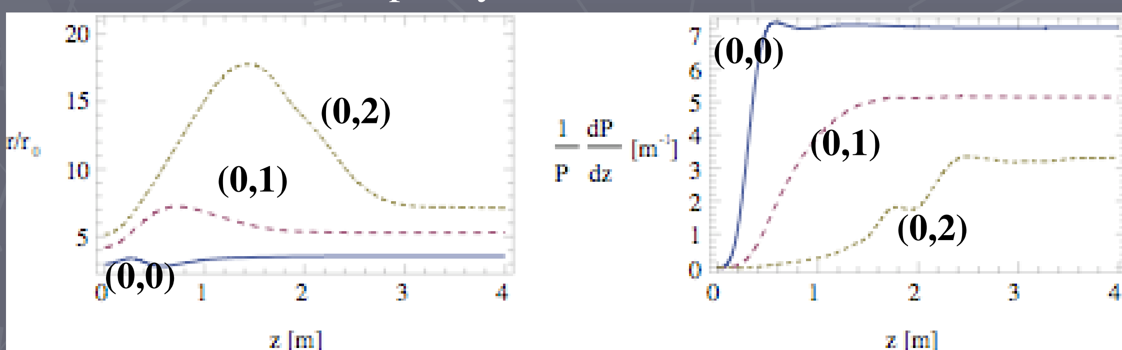
Profile evolution

Differential Power

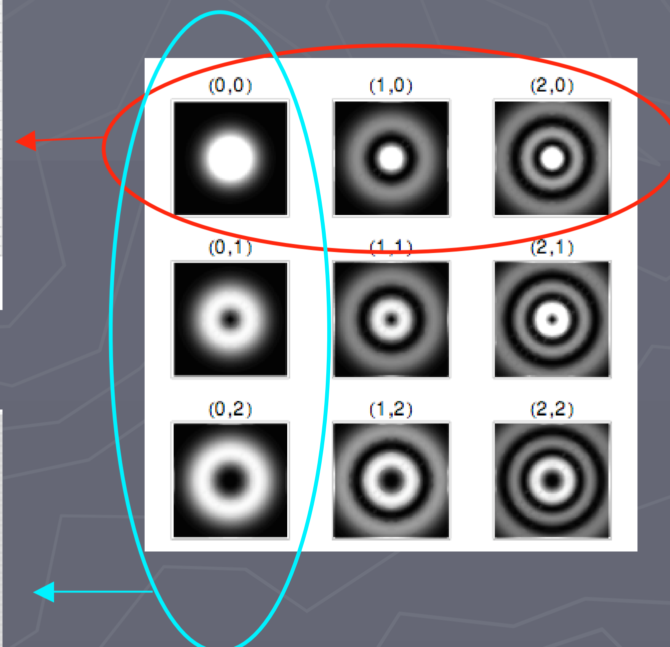
purely radial modes



purely azimuthal modes



Intensity Profiles
of LG modes



In cold beam, different azimuthal modes do not couple to each other!

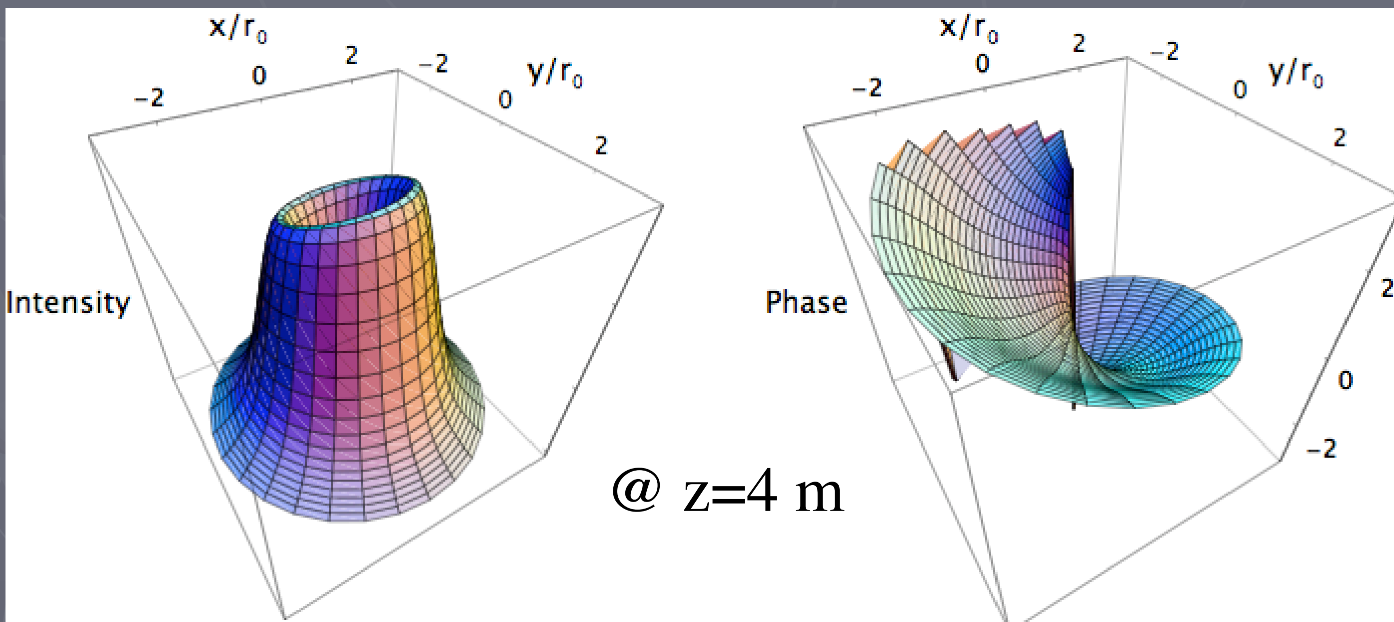
Higher-Order Mode SASE

Helical pre-Bunching with LG modes

An initial density perturbation of the form...

$$\tilde{n}_1(r,0) = n_0 \sum_{\eta} \tilde{\epsilon}_\eta e^{-i\eta\phi}$$

...couples to azimuthal LG modes, producing hollow intensity and helical phase profiles at the undulator exit:



l~1 helical phase

$$\tilde{\epsilon}_0 = 10^{-5}$$
$$\tilde{\epsilon}_1 = 8 \times 10^{-3}$$

Generate in situ Optical Angular Momentum with e-beam!

Summary: higher-order paraxial modes in FELs

- One might be able to amplify specific higher-order modes through seeding, or by selectively preparing the e-beam to generate transverse phase structure(s) in situ
- Many potential applications in chemistry, biology, etc, for both current and future FEL light sources
- This provides additional knobs for FEL research goals, and capabilities.

Conclusions

- A general expansion of the FEL signal field into eigenmodes of a guiding dielectric medium has been derived as a description of FEL light in the exponential gain regime.
- Coupled evolution excitation equations can be transformed into a single matrix equation for solutions to the FEL supermodes
- Eigenmodes of a QIM provide a natural descriptive connection to free-space modes for propagation and coupling investigations
- Radiation/e-beam coupling can be readily studied for seeding and/or pre-bunching

Future plans

- Compare higher-order seeding predictions to VISA experiment
- Explore the generation & coupling of OAM modes
- Extend eqns to include energy spread, emittance...