Theory of Nonlinear Harmonic Generation In Free-Electron Lasers with Helical Wigglers

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Motivation

Is on-axis NHG in FELs with helical wiggler suppressed ?

Answer important for both main FEL development paths

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Our answer: on-axis NHG is suppressed

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Contents

In this talk I will

1. **Describe our theory of NHG in FELs with helical wigglers**

2. **Treat a particular example-case**

3. **Explain why, in our view, literature is incorrect**

NHG as electrodynamical problem

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FEL self-consistent code:
Interaction I harmonic & beam

NHG as electrodynamical problem

Bunching at the h^{th} harmonic $\leftarrow$ FEL self-consistent code: Interaction I harmonic & beam

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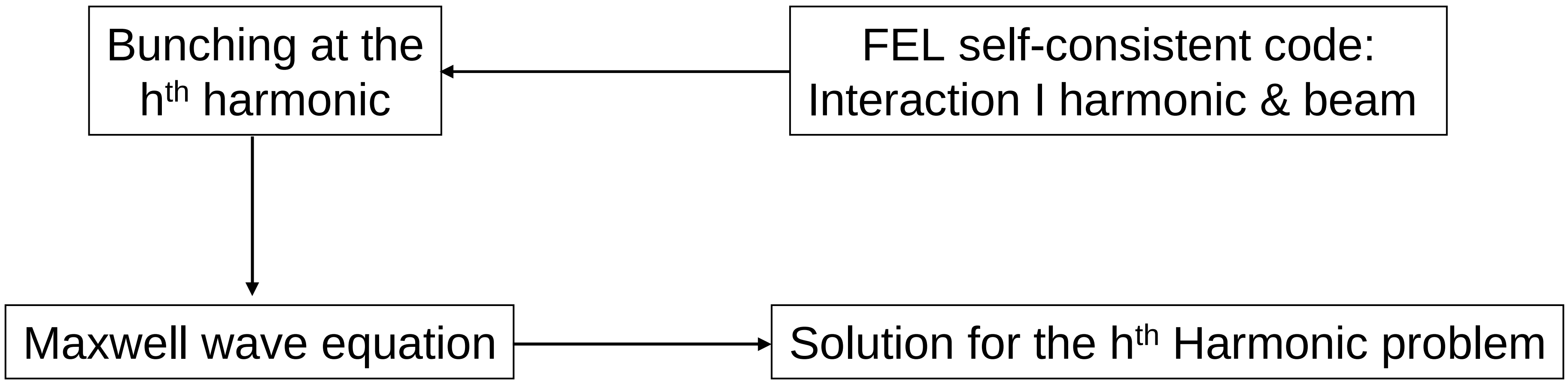
Maxwell wave equation

NHG as electrodynamical problem



NHG as electrodynamical problem

Bunching at the h^{th} harmonic ← FEL self-consistent code: Interaction I harmonic & beam

↓

Maxwell wave equation → Solution for the h^{th} Harmonic problem

Solution of the NHG problem means solution of <u>Maxwell equations</u> with "God-given" sources

NHG as electrodynamical problem

Bunching at the h^{th} harmonic

FEL self-consistent code: Interaction I harmonic & beam

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Solution for the h^{th} Harmonic problem

Solution of the NHG problem means solution of <u>Maxwell equations</u> with "code-given" sources

Maxwell equations

Space-frequency domain

We are interested in solving Maxwell Equations in paraxial approximation with respect to the F.T. of the field $\bar{E}(\omega)$

Maxwell equations

Space-frequency domain

$$\left(\nabla_\perp{}^2 + \frac{2i\omega}{c}\frac{\partial}{\partial z}\right)\left[\vec{\tilde{E}}_\perp(z,\vec{r}_\perp,\omega)\right] = -4\pi\exp\left[-\frac{i\omega z}{c}\right]\left(\frac{i\omega}{c^2}\vec{\bar{j}}_\perp - \vec{\nabla}_\perp\bar{\rho}\right)$$

$$\vec{\tilde{E}}_\perp = \vec{\bar{E}}_\perp \exp\left[-i\omega z/c\right]$$

Maxwell equations

Space-frequency domain

$$\left(\nabla_\perp{}^2 + \frac{2i\omega}{c}\frac{\partial}{\partial z}\right)\left[\vec{\tilde{E}}_\perp(z,\vec{r}_\perp,\omega)\right] = -4\pi\exp\left[-\frac{i\omega z}{c}\right]\left(\frac{i\omega}{c^2}\vec{\bar{j}}_\perp - \vec{\nabla}_\perp\bar{\rho}\right)$$

current term

Maxwell equations

Space-frequency domain

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gradient term

Maxwell equations

Space-frequency domain

$$\left(\nabla_\perp{}^2 + \frac{2i\omega}{c}\frac{\partial}{\partial z}\right)\left[\vec{\tilde{E}}_\perp(z,\vec{r}_\perp,\omega)\right] = \underbrace{-4\pi \exp\left[-\frac{i\omega z}{c}\right]\left(\frac{i\omega}{c^2}\vec{\tilde{j}}_\perp - \vec{\nabla}_\perp\bar{\rho}\right)}_{\vec{f}}$$

Maxwell equations

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$\vec{f} = \vec{f}(z;\vec{r}_\perp - \vec{r}_\perp{}^{(c)})$ from code; $\vec{r}_\perp - \vec{r}_\perp{}^{(c)}$ accounts for helical motion

Maxwell equations

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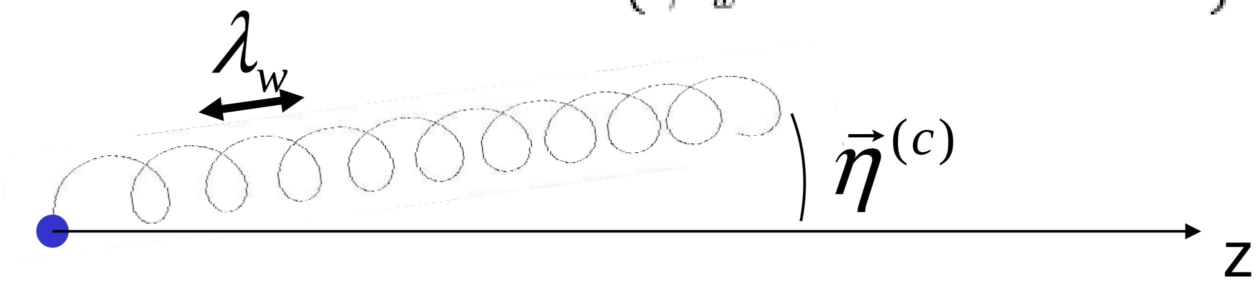
$$\vec{r}_\perp^{(c)}(z,\vec{\eta}^{(c)}) = \vec{r'}_{o\perp}(z) + \vec{\eta}^{(c)} z = \left\{\frac{K}{\gamma k_w}\left[\cos(k_w z) - 1\right] + \eta_x^{(c)} z\right\}\vec{e}_x + \left\{\frac{K}{\gamma k_w}\sin(k_w z) + \eta_y^{(c)} z\right\}\vec{e}_y$$



$k_w = 1/\bar{\lambda}_w$

Maxwell equations

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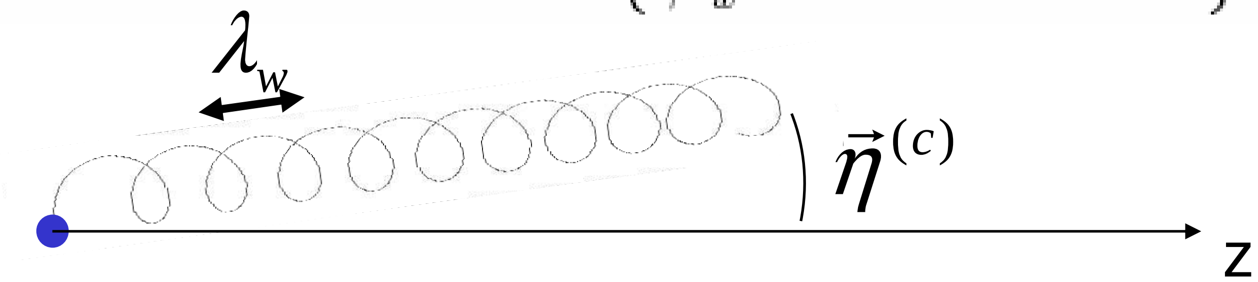

$$k_w = 1/\lambda\!\!\!^{-}_w$$

$$\vec{f} = 4\pi \exp\left[i\int_0^z d\bar{z}\frac{\omega}{2c\gamma_z^2(\bar{z},\vec{\eta}^{(c)})}\right]\left[\frac{i\omega}{c^2}\vec{v}_\perp(z,\vec{\eta}^{(c)}) - \vec{\nabla}_\perp\right]\tilde{\rho}[z;\vec{r}_\perp - \vec{r}_\perp^{(c)}(z,\vec{\eta}^{(c)})]$$

Solution of paraxial equation on-axis

Resonance approximation

$$N_w >> 1$$

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Define detuning parameter as

$$C_h = \frac{\omega}{2\bar{\gamma}_z^2} - hk_w = \frac{\Delta\omega}{\omega_r} k_w \qquad (\Delta\omega = h\omega_r - \omega)$$

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$$|C_h| << k_w \;\; i.e. \; \frac{|\Delta\omega|}{\omega_r} << 1$$

Solution of paraxial equation on-axis

Far zone solution of paraxial Maxwell equation on-axis:

$$\vec{\widetilde{E}}_\perp = \frac{i\omega}{cz_o}\int d\vec{l}' \int_{-L_w/2}^{L_w/2} dz' \widetilde{\rho}\left(z', \vec{l}'\right) \exp\left[iC_h z'\right]$$
$$\times \left\{ \left[\frac{K}{2i\gamma}\left(\exp[i(h+1)k_w z'] - \exp[i(h-1)k_w z'] \right) \right] \vec{e}_x \right.$$
$$\left. + \left[-\frac{K}{2\gamma}\left(\exp[i(h+1)k_w z'] + \exp[i(h-1)k_w z'] \right) \right] \vec{e}_y \right\}$$

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Slowly varying function of z' over λ_w

$|C_h| << k_w$

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Only h=1 survives, on axis.

Solution of paraxial equation: second harmonic

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$$\vec{\widetilde{E}}_\perp = \frac{i\omega^2\left(\vec{e}_x + i\vec{e}_y\right)}{2cz_0\omega_r}\frac{K^2}{1+K^2}\left[\left(\theta_x - \eta_x^{(c)}\right) + i\left(\theta_y - \eta_y^{(c)}\right)\right]$$
$$\times\int_{-\infty}^{\infty} dl'_x \int_{-\infty}^{\infty} dl'_y \int_{-L_w/2}^{L_w/2} dz' \exp\left\{i\frac{\omega}{c}\left[\frac{z_o\left(\theta_x^2+\theta_y^2\right)}{2} - \left(\theta_x l'_x + \theta_y l'_y\right)\right]\right\}$$
$$\times\widetilde{\rho}\left(z', l'_x, l'_y\right)\exp\left[iC_2 z'\right]\exp\left\{i\frac{\omega}{2c}\left[\left(\theta_x - \eta_x^{(c)}\right)^2 + \left(\theta_y - \eta_y^{(c)}\right)^2\right]z'\right\}$$

Circularly polarized field, vanishing on-axis

Simple model

To get further results, we need to treat a particular case

C_2=0;

$$\widetilde{\rho} = \frac{I_o a_2}{2\pi c \sigma_\perp^2} \exp\left(-\frac{l'^2_x + l'^2_y}{2\sigma_\perp^2}\right) \exp\left[i \frac{2\omega_r}{c}\left(\eta_x^{(c)} l'_x + \eta_y^{(c)} l'_y\right)\right]$$

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Gaussian transverse profile

Simple model

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Modulation wave front orthogonal to z

Simple model

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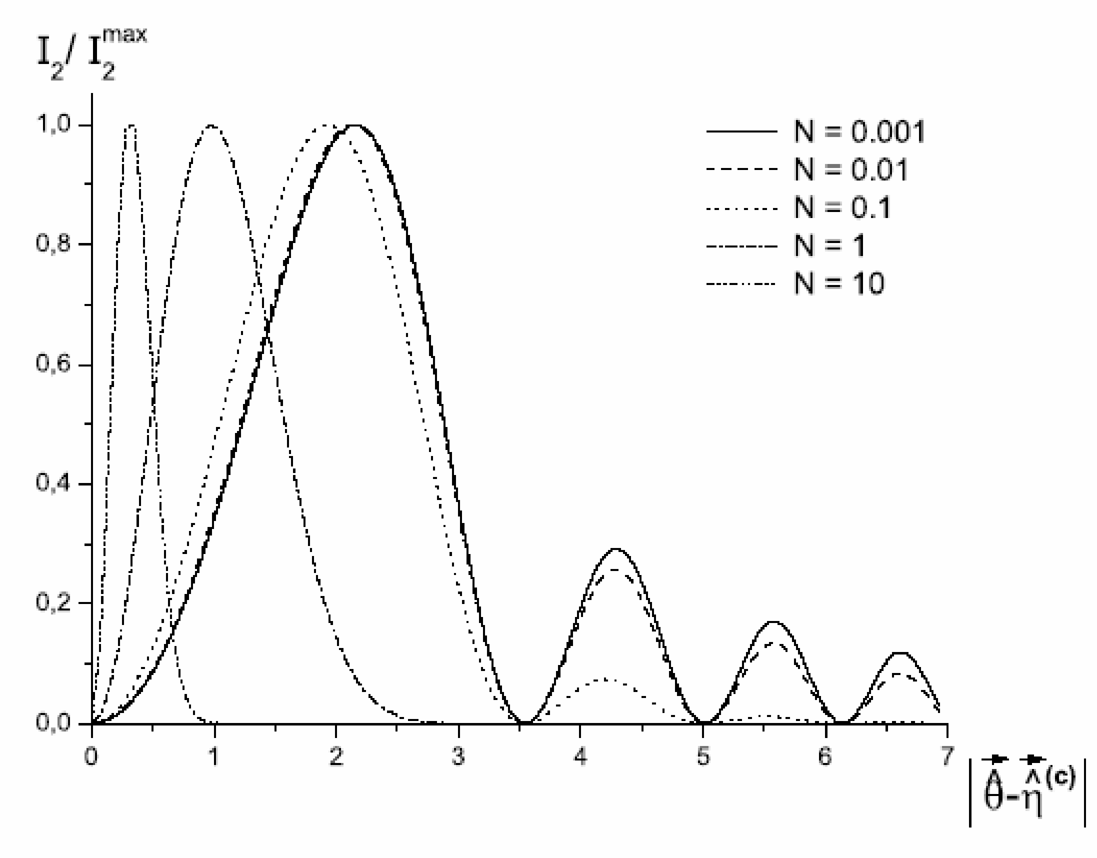
$$\begin{aligned}\vec{\tilde{E}}_\perp = {} & \frac{2iI_o a_2 L_w \omega_r \left(\vec{e}_x + i\vec{e}_y\right)}{c^2 z_o} \left(\frac{K^2}{1+K^2}\right) \left[\left(\theta_x - \eta_x^{(c)}\right) + i\left(\theta_y - \eta_y^{(c)}\right)\right] \\ & \times \exp\left[i\frac{\omega_r}{c} z_o (\theta_x^2 + \theta_y^2)\right] \exp\left\{-\frac{2\sigma_\perp^2 \omega_r^2}{c^2}\left[\left(\theta_x - \eta_x^{(c)}\right)^2 + \left(\theta_y - \eta_y^{(c)}\right)^2\right]\right\} \\ & \times \mathrm{sinc}\left\{\frac{L_w \omega_r}{2c}\left[\left(\theta_x - \eta_x^{(c)}\right)^2 + \left(\theta_y - \eta_y^{(c)}\right)^2\right]\right\} \ .\end{aligned}$$

Simple model: II harmonic directivity diagram

$$I_2\left(\left|\vec{\hat{\theta}}-\vec{\hat{\eta}}^{(c)}\right|\right) = \mathrm{const}\times\left|\vec{\hat{\theta}}-\vec{\hat{\eta}}^{(c)}\right|^2 \exp\left\{-N\left|\vec{\hat{\theta}}-\vec{\hat{\eta}}^{(c)}\right|^2\right\}\mathrm{sinc}^2\left\{\frac{1}{4}\left|\vec{\hat{\theta}}-\vec{\hat{\eta}}^{(c)}\right|^2\right\}$$

$$\hat{\theta}=\frac{\theta}{\sqrt{\lambda\!\!\!^{-}_2/L_w}} \qquad \hat{\eta}=\frac{\eta}{\sqrt{\lambda\!\!\!^{-}_2/L_w}}$$

$$N=\frac{\sigma_\perp^2}{\lambda\!\!\!^{-}_2 L_w}$$



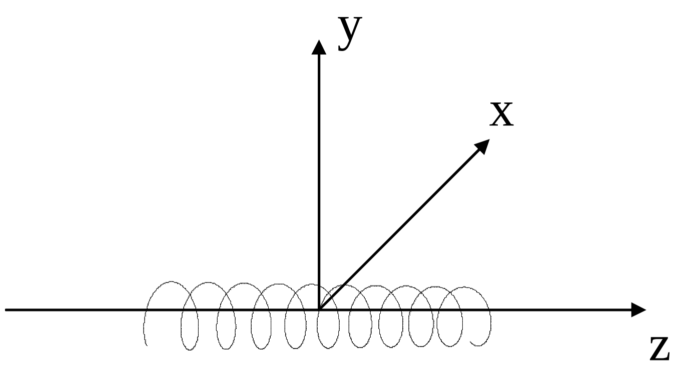
Simple model: II harmonic power

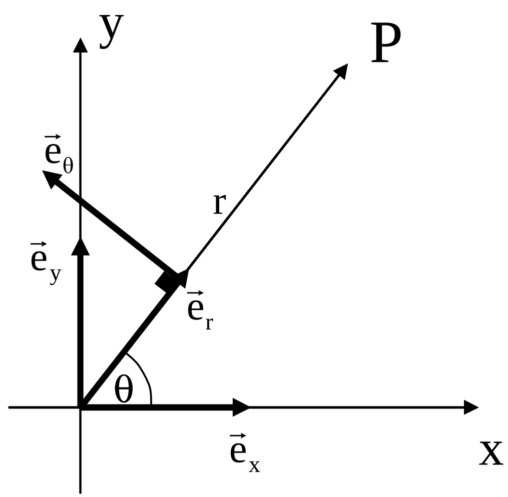
$$\hat{W}_2 = W_2 / W_o$$

$$\hat{W}_2 = F_2(N) = \ln\left(1 + \frac{1}{4N^2}\right)$$

$$W_o^{(2)} = \left(\frac{2K^2}{1+K^2}\right)^2 \frac{I_o^2}{c} a_2^2 \ .$$

Criticism to literature





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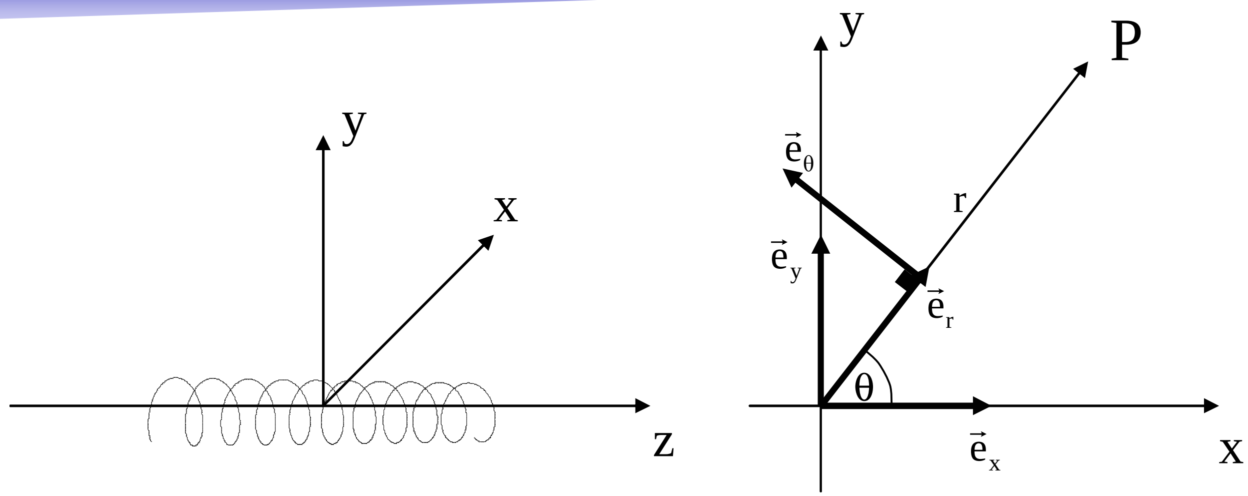

$$\vec{E}(\vec{r},t) = E_o \cdot (\vec{e}_r + i\,\vec{e}_\theta)\exp[i\phi_h] \qquad \phi_h = kz + h\theta - \omega t$$ wave phase

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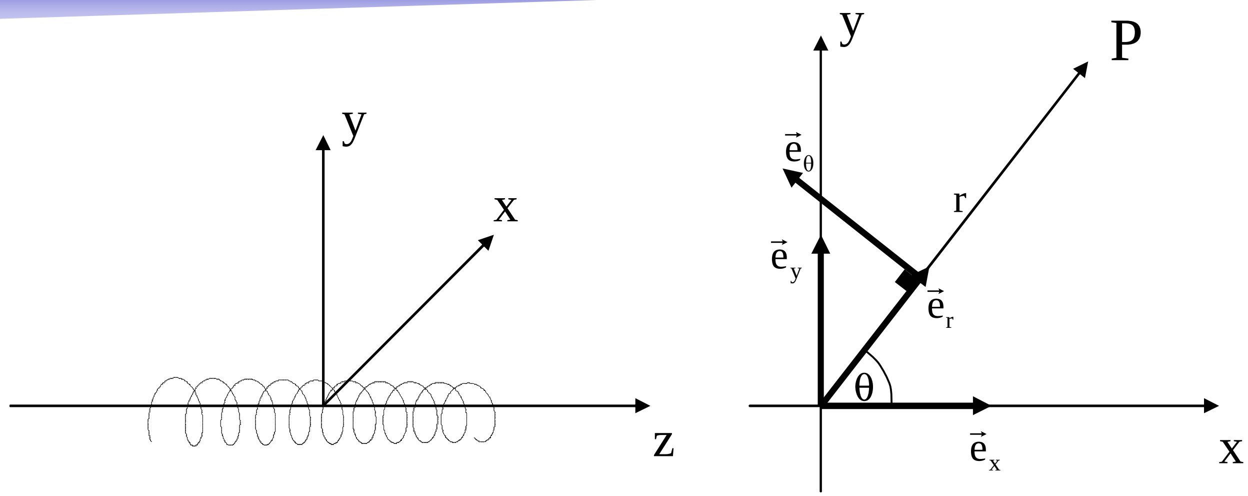

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$$\theta = k_w z$$ Azimuthal electron motion in helical wiggler

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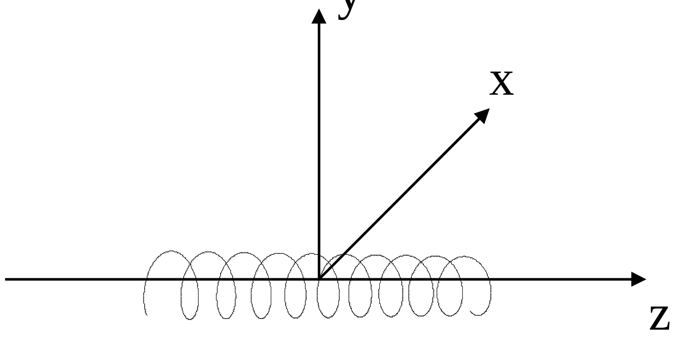

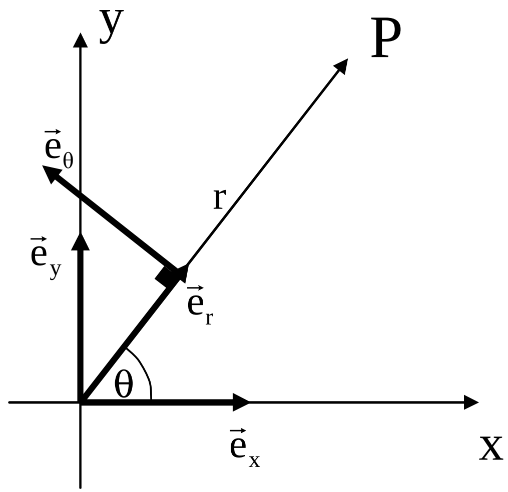

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→Phase along ptc trajectory: $(k + hk_w)z - \omega t$

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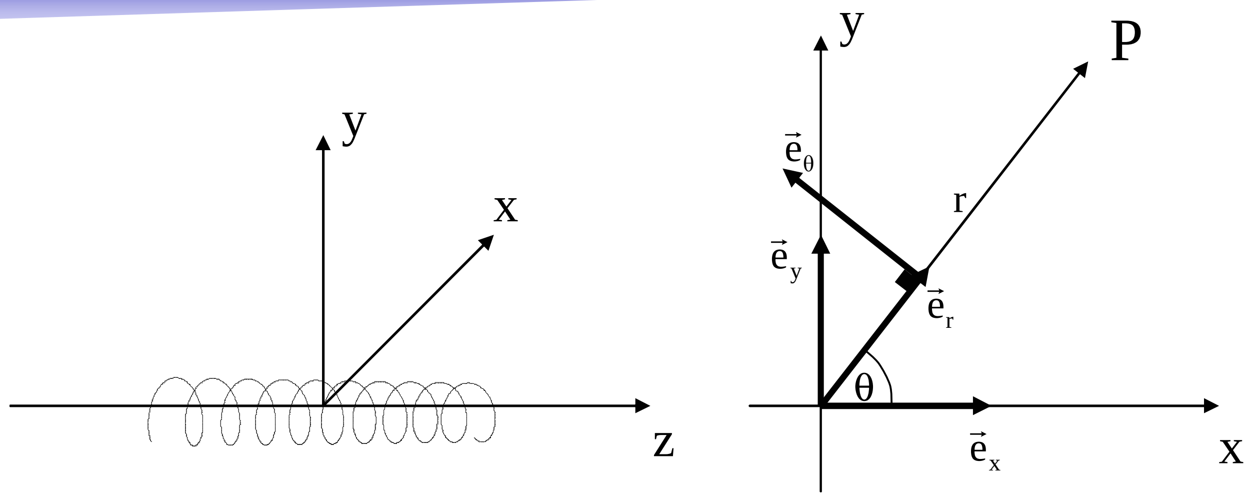

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Azimuthal resonant condition (literature)

$$k + hk_w - \omega / v_z = 0$$

Criticism to literature

"The azimuthal electron motion in helical wigglers is $\theta = k_w z$ (k_w is the wave number for the wiggler period λ_w), which couples to circularly polarized waves that vary as $\exp(i\phi_h)$, where $\phi_h = kz + h\theta - \omega t$ is the wave phase. Hence, the phase along the particle trajectories varies as $\phi_h = (k + hk_w)z - \omega t$, and the hth order azimuthal mode corresponds to the hth harmonic resonance [i.e., $\omega \approx (k + hk_w)v_z$]" [1]

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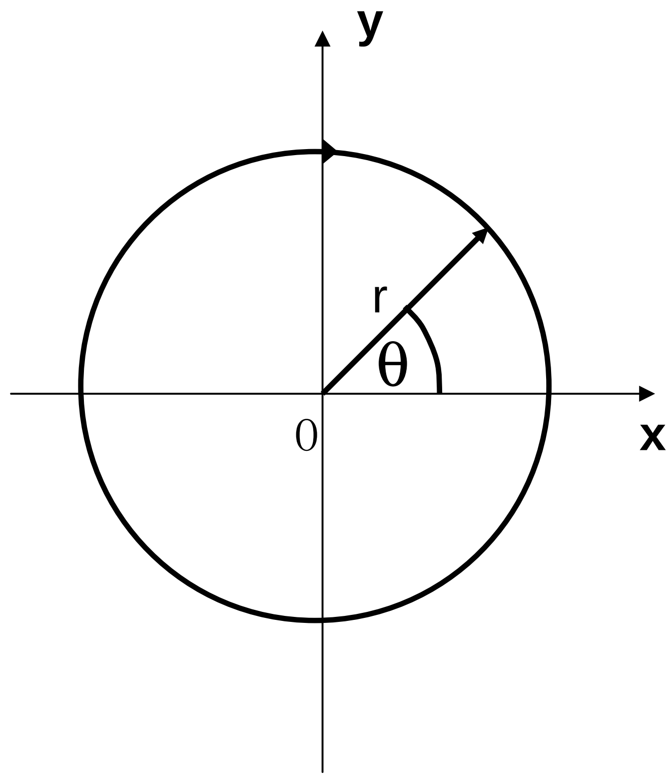
Criticism to literature

$$k + hk_w - \omega / v_z = 0$$ consequence of

$$\theta = k_w z$$

Incorrect kinematical picture

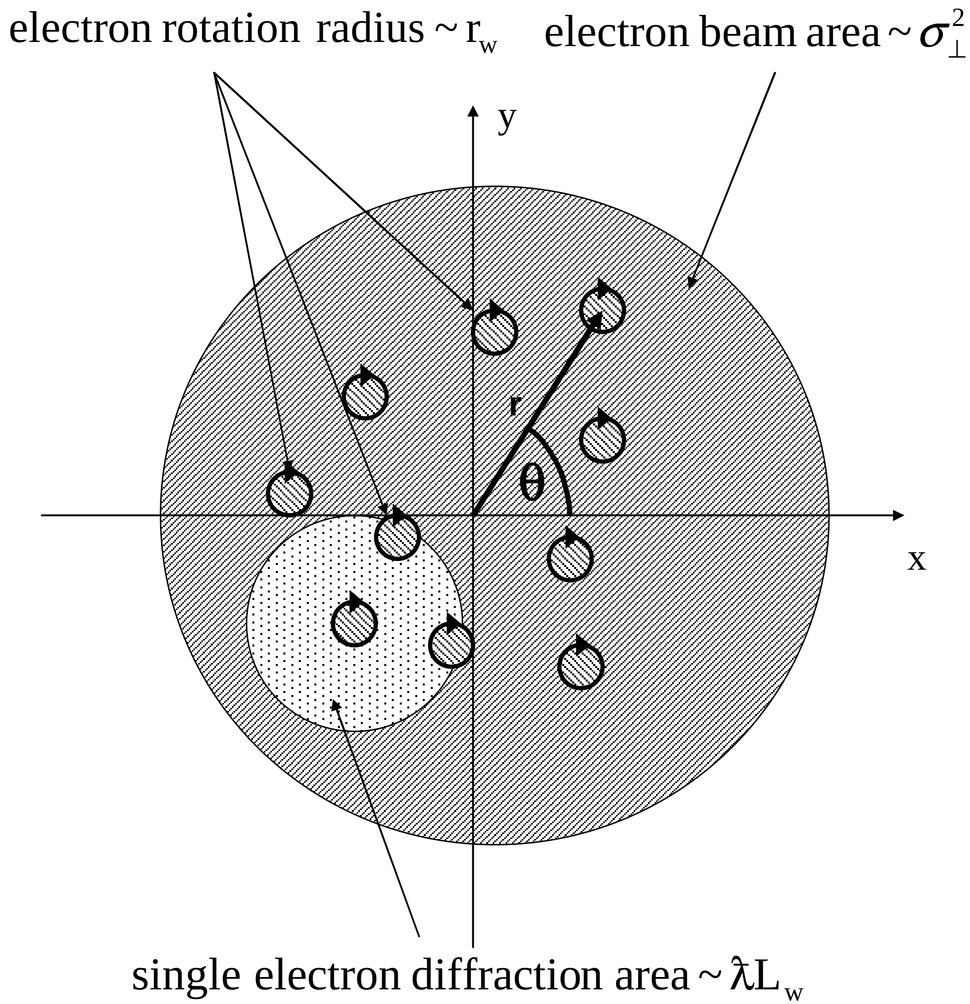


Criticism to literature

$$\frac{r_w^{\;2}}{\lambda\!\!\!^{-} L_w} << 1$$

Particles have nearly constant azimuthal position θ. There is no special resonance condition in helical undulators.

Simply, NHG emission on-axis vanishes at h>1.



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$$\theta = \text{constant}$$

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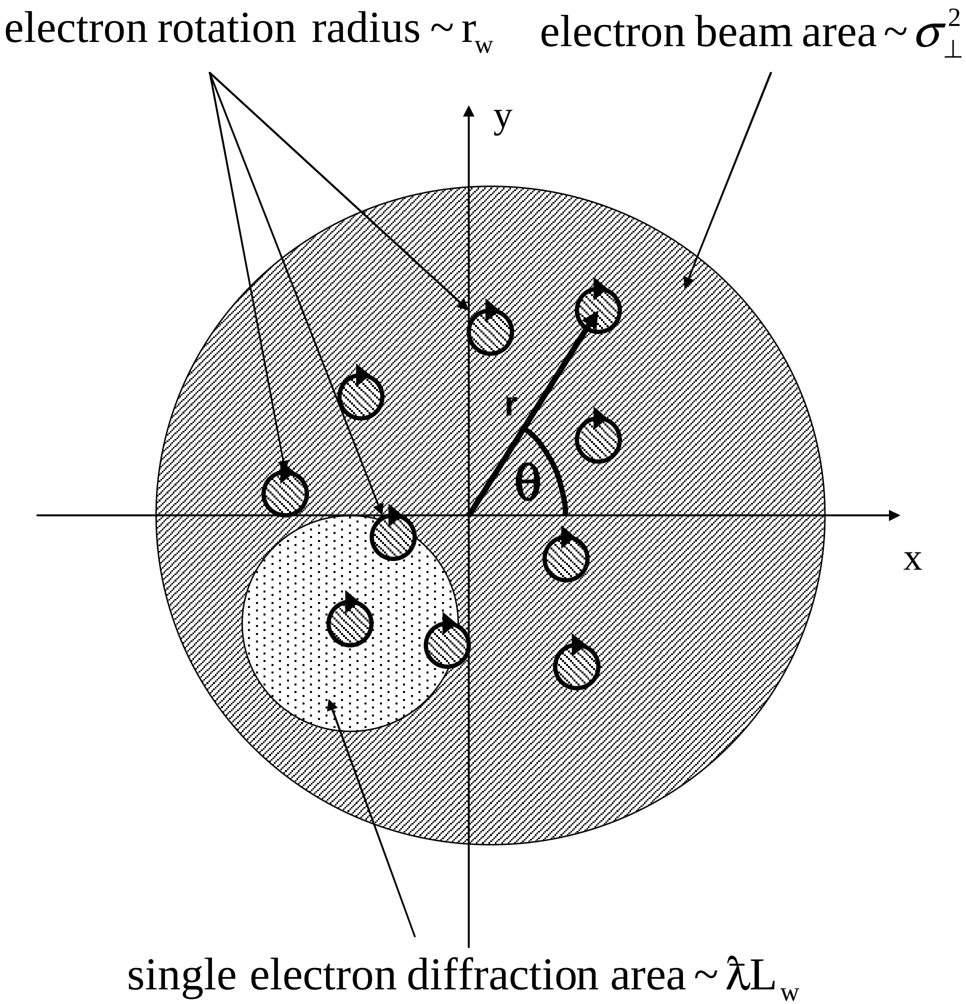

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