Three-dimensional theory of quantum free-electron laser

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Hypothesis

- mean field theory → 1- particle Hamiltonian
- Pauli principle is not relevant
- no space charge effect

Preparata, PRA 38, 233 (1988) ; J.Gea-Banacloche, Phys. Rev. A 31, 3 (1985)

Variables and parameters

$$\bar z=\frac{z}{L_g};\quad z_1=\frac{z-vt}{\beta L_c};\quad p=\frac{mc(\gamma-\gamma_0)}{\hbar k_r};\quad \vartheta=(k_r+k_L)z-c(k_r-k_L)-\delta\bar z;\quad \delta=\frac{mc(\gamma_0-\gamma_r)}{\bar\rho\hbar k_r}\qquad L_g=\frac{\lambda_L}{8\pi\sqrt{\bar\rho}};\ L_c=\frac{2\lambda_r}{\lambda_L}L_g$$

$$\bar x_t=(\bar x,\bar y);\ \bar p_t=(\bar p_x,\bar p_y);\ \bar x_t=\frac{x_t}{\sigma};\ \bar p_t=\frac{\gamma\sigma}{\lambda_c}\frac{dx_t}{dz}\qquad |A(\bar x,\bar z,z_1)|^2=\frac{\langle N_{ph}\rangle}{N_e}\qquad g(\bar x_t,\bar z)$$ laser wiggler profile

3D quantum Hamiltonian

$$\hat H(\bar z)=\frac{\hat p^2}{2\bar\rho^{3/2}}+\frac{\alpha b}{2}\hat{\bar p}_t^2+\left[\frac{\xi}{2\rho\sqrt{\bar\rho}}\left(1-|g(\hat{\bar x}_t,\bar z)|^2\right)-\frac{bX}{4}\alpha^2\hat{\bar p}_t^2\right]\hat p+\frac{\xi}{\alpha\rho\sqrt{\bar\rho}X}|g(\hat{\bar x}_t,\bar z)|^2$$
$$-i\left[g^*\left(\hat{\bar x}_t,\bar z\right)A\left(\hat{\bar x}_t,\bar z,z_1\right)e^{i\hat\vartheta}-g\left(\hat{\bar x}_t,\bar z\right)A^*\left(\hat{\bar x}_t,\bar z,z_1\right)e^{-i\hat\vartheta}\right]$$

3D Wigner function

$$W_n(\vartheta,\bar x_t,\bar p_t)=w_n(\vartheta,\bar x_t,\bar p_t)+\sum_{n'\in Z}\mathrm{sinc}\left[\left(n-n'-\frac{1}{2}\right)\pi\right]w_{n+\frac12}(\vartheta,\bar x_t,\bar p_t)\qquad s=n,n+\frac{1}{2}\quad N\in Z$$

$$w_s(\vartheta,\bar x_t,\bar p_t)=\frac{1}{2\pi^3}\int d^2\bar x'\int_{-\pi}^{\pi}d\vartheta' e^{-2i(s\vartheta'+\bar p_t\cdot\bar x_t')}\langle\vartheta+\vartheta',\bar x_t+\bar x_t'|\hat\rho|\vartheta-\vartheta',\bar x_t-\bar x_t'\rangle$$

$$\frac{\partial\hat\rho}{\partial t}=-i\left[\hat H,\hat\rho\right]$$

Exact dynamic equation for w_s

$$\frac{\partial w_s(\vartheta,\bar x_t,\bar p_t)}{\partial\bar z}=-\left[\left(\frac{s}{\bar\rho^{3/2}}+\frac{\xi}{2\rho\sqrt{\bar\rho}}-\frac{bX}{2}\alpha^2\bar p_t^2\right)-\frac{bX}{8}\alpha^2\nabla_{\bar x}^2\right]\frac{\partial}{\partial\vartheta}w_s(\vartheta,\bar x_t,\bar p_t)-b\left(\alpha-\alpha^2Xs\right)\bar p_t\cdot\nabla_{\bar x_t}w_s(\vartheta,\bar x_t,\bar p_t)$$
$$+i\frac{\xi}{2\rho\sqrt{\bar\rho}\alpha X}(1-\alpha Xs)\iint\frac{d^2\bar k}{2\pi}\tilde G e^{-i\bar x\cdot\bar k}\left[w_s\left(\vartheta,\bar x_t,\bar p_t-\frac{\bar k}{2}\right)-w_s\left(\vartheta,\bar x_t,\bar p_t+\frac{\bar k}{2}\right)\right]$$
$$+\frac{\xi}{4\rho\sqrt{\bar\rho}}\iint\frac{d^2\bar k}{2\pi}\tilde G e^{-i\bar x\cdot\bar k}\frac{\partial}{\partial\vartheta}\left[w_s\left(\vartheta,\bar x_t,\bar p_t-\frac{\bar k}{2}\right)+w_s\left(\vartheta,\bar x_t,\bar p_t+\frac{\bar k}{2}\right)\right]$$
$$+\left[e^{i\vartheta}\iint\frac{d^2\bar k}{2\pi}\tilde F e^{-i\bar x\cdot\bar k}-e^{-i\vartheta}\iint\frac{d^2\bar k}{2\pi}\tilde F^* e^{i\bar x\cdot\bar k}\right]\left[w_{s-\frac12}\left(\vartheta,\bar x_t,\bar p_t-\frac{\bar k}{2}\right)-w_{s+\frac12}\left(\vartheta,\bar x_t,\bar p_t+\frac{\bar k}{2}\right)\right]$$

$$F=g^*A;\ G=|g|^2$$

3D Parameter

$$b=\frac{L_g}{\beta^*}=\frac{L_g}{\sigma^2}\varepsilon_r,$$
$$a=\frac{L_g}{Z_r}$$
$$X=k_r\varepsilon_r,$$
$$\xi=\frac{a_w^2}{1+a_w^2},$$
$$\alpha=\frac{\lambda_c}{\varepsilon_n}.$$

$$p_t=\alpha\bar p_t=\frac{\sigma\gamma}{\varepsilon_n}\frac{dx_t}{dz}$$

Classical limit for transverse variables

$$\frac{\partial w_s}{\partial\bar z}=\sum_{n=0}^{+\infty}\alpha^nF_n(\theta,\bar x_t,p_t,s)=$$
$$\alpha^0\left\{-\left[\left(\frac{s}{\bar\rho^{3/2}}+\frac{\xi(1-|g|^2)}{2\rho\sqrt{\bar\rho}}-\frac{bX}{2}p_t^2\right)\frac{\partial}{\partial\vartheta}-bp_t\cdot\nabla_{\bar x_t}+\frac{\xi}{2\rho\sqrt{\bar\rho}X}\nabla_{\bar x_t}|g|^2\cdot\nabla_{p_t}\right]w_s+\left(g^*Ae^{i\vartheta}-gA^*e^{-i\vartheta}\right)\left[w_{s+1/2}-w_{s-1/2}\right]\right\}$$
$$+\alpha^1\left\{\left[-bXsp_t\cdot\nabla_{\bar x_t}+\frac{\xi}{2\rho\sqrt{\bar\rho}}s\nabla_{\bar x_t}|g|^2\cdot\nabla_{p_t}\right]w_s-\frac{i}{2}\nabla_{\bar x_t}\left(g^*Ae^{i\vartheta}-gA^*e^{-i\vartheta}\right)\cdot\nabla_{p_t}\left[w_{s+1/2}+w_{s-1/2}\right]\right\}+\ldots\ldots$$

Procedure

$$\iint\frac{d^2q}{2\pi}f(q)e^{-ix\cdot q}\left[w_s\left(p-\frac{q}{2}\right)-w_s\left(p+\frac{q}{2}\right)\right]=-i\sum_{n=0}^{+\infty}\frac{(-1)^n}{2^{n-1}}\frac{\partial_p^n w_s(p)}{(2n+1)!}\partial_x^{2n+1}f(x)$$

$$\iint\frac{d^2q}{2\pi}f(q)e^{-ix\cdot k}\left[w_s\left(p-\frac{q}{2}\right)+w_s\left(p+\frac{q}{2}\right)\right]=\sum_{n=0}^{+\infty}\frac{(-1)^n}{2^{n-1}}\frac{\partial_p^n w_s(p)}{(2n)!}\partial_x^{2n}f(x)$$

$$\alpha=\frac{\lambda_c}{\varepsilon_n}\approx10^{-6}$$

$$\bar\rho=\rho\frac{mc\gamma_0}{\hbar k}>>1$$

Working equations

$$\frac{\partial w_s}{\partial\bar z}=-\left\{\frac{s}{\bar\rho^{3/2}}+\frac{\xi}{2\rho\sqrt{\bar\rho}}(1-|g|^2)-\frac{bX}{2}p_t^2\right\}\frac{\partial w_s}{\partial\vartheta}-bp_t\cdot\nabla_{\bar x_t}w_s+\left(g^*Ae^{i\vartheta}-gA^*e^{-i\vartheta}c.c.\right)\left[w_{s+1/2}-w_{s-1/2}\right]+\frac{\xi}{\rho\sqrt{\bar\rho}X}\nabla_{\bar x_t}|g|^2\cdot\nabla_{p_t}w_s$$

$$\frac{\partial A}{\partial\bar z}+\frac{\partial A}{\partial z_1}-ia\nabla_{\bar x}^2A=g\sum_{n=-\infty}^{+\infty}\int d^2\bar p_t\int_{-\pi}^{\pi}d\theta e^{-i\theta}w_{n+1/2}(\theta,\bar x,\bar p_t,z_1,\bar z)+i\delta A$$

Full classical limit

discrete Wigner

$$\frac{s}{\bar\rho}\to\bar p;\quad W_n(\vartheta,\bar x,\bar p_t)\to\bar\rho\,W(\vartheta,\bar p,\bar x,\bar p_t)$$

continuous Wigner

$$\left[w_{s+1/2}-w_{s-1/2}\right]\to\bar\rho\left[W\left(\bar p+\frac{1}{2\bar\rho}\right)-W\left(\bar p-\frac{1}{2\bar\rho}\right)\right]\to\frac{\partial W}{\partial\bar p}$$

Vlasov

$$\frac{\partial W_C}{\partial\bar z_C}+\left\{\bar p+\frac{\xi}{2\rho}\left(1-|g|^2\right)-\frac{b_CX}{2}\left|\bar p_t\right|^2\right\}\frac{\partial W_C}{\partial\theta}+b_C\bar p_t\cdot\nabla_{\bar x_t}W_C$$
$$-\left(g^*A_Ce^{i\theta}-gA_C^*e^{-i\theta}.\right)\frac{\partial W_C}{\partial\bar p}-\frac{\xi}{2\rho X}\left(\nabla_{\bar x}|g|^2\right)\nabla_{\bar p_t}W_C=0$$
$$\frac{\partial A}{\partial\bar z_C}+\frac{\partial A}{\partial z_{1C}}-ia\nabla_{\bar x}^2A_C=g\int d^2\bar p_t\int_{-\infty}^{+\infty}d\bar p\int_{-\pi}^{\pi}d\theta e^{-i\theta}W+i\delta_CA_C$$

$$\bar z=\sqrt{\bar\rho}\,\bar z_c;\quad z_1=\sqrt{\bar\rho}\,z_{1C};\ A=\sqrt{\bar\rho}\,A_C;\quad \delta=\delta_C/\sqrt{\bar\rho};\ b=b_C/\sqrt{\bar\rho};\ a=a_C/\sqrt{\bar\rho}$$

Conclusions

- We have derived a novel 3D model for a Quantum FEL with a laser wiggler. The model included transverse emittance effect, radiation diffraction and slippage, laser wiggler profile and propagation.
- The 3D electrons' Wigner function combines discrete longitudinal variables and continuous transversal variables.
- In the classical limit the model reduces to a continuous Vlasov model for the classical FEL with a laser wiggler.