Nearly Conformal QCD and AdS/CFT

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First Workshop on Quark-Hadron Duality and the Transition to pQCD

Outline

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- ADS/CFT and QCD
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- Outlook
The Holographic Correspondence

- Original correspondence between $\mathcal{N} = 4$ SYM at large $N_C$ and the low energy supergravity approximation to Type IIB string on $AdS_5 \times S^5$:
  Maldacena, hep-th/9711200.

  \[
  \begin{array}{ccc}
  \text{Warped higher dim space} & \leftrightarrow & \text{Conformal d = 4 spacetime boundary} \\
  \text{Type IIB (AdS}_5 \times S^5) & \leftrightarrow & \mathcal{N} = 4 \text{ SYM (SO(4, 2) } \otimes \text{ SU(4))} \\
  ? & \leftrightarrow & \text{QCD}
  \end{array}
  \]

$SO(4, 2)$ is isomorphic with the isometries of $AdS^5$, and $SU(4) \sim SO(6)$ with $S^5$.

- Description of strongly coupled gauge theory using a dual gravity description!

- QCD is fundamentally different from SYM theories where all matter is in the adjoint rep of $SU(N_C)$. Introduction of quarks in the fundamental rep is dual to an open string sector:
  Gross and Ooguri, hep-th/9805129; E. Witten, hep-th/9805112.
AdS/CFT and QCD

Bottom-Up Approach

- Nonperturbative derivation of dimensional counting rules of hard exclusive glueball scattering for gauge theories with mass gap dual to string theories in warped space:
  Polchinski and Strassler, hep-th/0109174.

- Deep inelastic structure functions at small $x$:
  Polchinski and Strassler, hep-th/0209211.

- Derivation of power falloff of hadronic light-front Fock wave functions, including orbital angular momentum, matching short distance behavior with string modes at AdS boundary:
  Brodsky and de Téramond, hep-th/0310227.

- Low lying hadron spectra, chiral symmetry breaking and hadron couplings in AdS/QCD:
• Gluonium spectrum (top-bottom):
  Csaki, Ooguri, Oz and Terning, hep-th/9806021; de Mello Kock, Jevicki, Mihaiescu and Nuñez, hep-th/9806125; Csaki, Oz, Russo and Terning, hep-th/9810186; Minahan, hep-th/9811156; Brower, Mathur and Tan, hep-th/0003115.

• D3/D7 branes (top-bottom):

• Other aspects of high energy scattering in warped spaces:

• Branes in Minkowski space:
  Siopsis, hep-th/0503245.
Strongly Coupled Conformal QCD and Holography

- Conformal Theories are invariant under the Poincaré and conformal transformations with $M^\mu\nu, P^\mu, D, K^\mu$, the generators of $SO(4, 2)$.

- QCD appears as a nearly-conformal theory in the energy regimes accessible to experiment. Invariance of conformal QCD is broken by quark masses and quantum loops (running coupling). For $\beta = d\alpha_s(Q^2)/d\ln Q^2 = 0$ (fixed point theory), PQCD is a conformal theory: Parisi, Phys. Lett. B 39, 643 (1972).

- Phenomenological success of dimensional scaling laws for exclusive processes $d\sigma/dt \sim 1/s^{n-2}$ (n total number of constituents), implies QCD is a strongly coupled conformal theory at moderate but not asymptotic energies (PQCD predicts powers of $\alpha_s$ and logs).

- Theoretical and empirical evidence that $\alpha_s(Q^2)$ has an IR fixed point (constant in the IR): Alkofer, Fischer and Llanes-Estrada, hep-th/0412330; Brodsky, Menke, Merino and Rathsman, hep-ph/0212078; Deur, this conference.
Scale Transformations

- Isomorphism of $SO(4, 2)$ of conformal QCD with the isometries of AdS space

$$ds^2 = \frac{R^2}{z^2} (\eta_{\mu, \nu} dx^\mu dx^\nu - dz^2) : x^\mu \rightarrow \lambda x^\mu, \ z \rightarrow \lambda z,$$

maps scale transformations into the holographic coordinate $r = \frac{R^2}{z}$.

- String mode in $r$ is the extension of the hadron wf into the fifth dimension.

- Different values of $r$ correspond to different scales at which the hadron is examined.

- Invariant separation between quarks:

$$x_\mu x^\mu \rightarrow \lambda^2 x^2, \ r \rightarrow \frac{r}{\lambda}.$$

- The AdS boundary at $r \rightarrow \infty$ correspond to the $Q \rightarrow \infty$, UV zero separation limit.
**Confinement**

- QCD is a confining theory in the infrared with mass gap $\Lambda_{QCD}$ and a well-defined spectrum of color-singlet states.
- There is a maximum separation of quarks and a minimum value of $r$.
- AdS space should end at a finite value $r_o = \Lambda_{QCD} R^2 = 1/z_o$.
- Cutoff at $r_o$ breaks conformal invariance and allows the introduction of the QCD scale.
- Non-conformal metric dual to a confining gauge theory:

\[
d s^2 = \frac{R^2}{z^2} e^{2A(z)} (\eta_{\mu\nu} dx^\mu dx^\nu - dz^2) + ds_X^2,
\]

where $A(z) \to 0$ as $z \to 0$. Asymptotically $AdS \times X$, where $X$ is a 5-dim compact manifold if 4-dim gauge theory is dual to a critical (10 dim) string (Polchinski and Strassler).

- Simplified model: metric factor $e^{2A(z)}$ is a step function. Analog to the MIT bag model, but with boundary conditions on the holographic coordinate.

- Conformal behavior at short distances and confinement at large distances $\implies$ Truncated AdS/CFT model.
Classical Correspondence and Interpolating Operators

- Duality between string in $AdS_{d+1}$ and $N_C \to \infty$ limit of a conformal gauge theory given by the generating functionals of the string and CFT theories at the AdS $d$-dim boundary.

- $d$-dim generating functional in presence of external source $\Phi_o$:

$$Z_{QCD} = [\Phi_o(x)] = \int \mathcal{D}[\psi, \overline{\psi}, A] \exp \left\{ iS_{QCD} + i \int d^d x \Phi_o \mathcal{O} \right\}.$$

- $d + 1$-dim string partition function:

$$Z_{grav}[\Phi(x, z)] = \int \mathcal{D}[\Phi] e^{iS_{grav}[\Phi]}.$$

- Boundary condition:

$$Z_{grav}[\Phi(x, z)_{z=0} = \Phi_o(x)] = Z_{QCD}[\Phi_o].$$

Gubser, Klebanov and Polyakov, hep-th/9802109; Witten, hep-th/9802150
• Near the boundary of $AdS_{d+1}$ space $z \to 0$:

$$\Phi(x, z) \to z^\Delta \Phi_+(x) + z^{d-\Delta} \Phi_-(x).$$

• $\Phi_-(x)$ is the boundary source: $\Phi_- = \Phi_o$ (Non-normalizable mode)

• $\Phi_+(x)$ is the response function (normalizable mode): $\langle O \rangle_{\Phi_o} = (2\Delta - d) \Phi_+(x)$

Klebanov and Witten, hep-th/9905104

• Dimensions: $[\Phi] = 0$, $[\Phi_-] = [\Phi_o] = d - \Delta$, $[\Phi_+] = [O] = \Delta$

• The AdS/QCD correspondence is interpreted as a classical duality between the valence state of a hadron in the asymptotic 4-dim boundary and the lightest mass string mode in $AdS_5 \times S^5$.

• The physical string modes $\Phi(x, z) \sim e^{-iP \cdot x} f(r)$, are plane waves along the Poincaré coordinates with four-momentum $P^\mu$ and hadronic invariant mass states $P_\mu P^\mu = M^2$.

• For large-$r$ $f(r) \sim r^{-\Delta}$. The dimension $\Delta$ of the string mode, is the same dimension of the interpolating operator $O$ which creates a hadron out of the vacuum: $\langle P \mid O \mid 0 \rangle \neq 0$.

• Quarks in the fundamental representation are introduced at the AdS boundary and we compute their wavefunctions as they propagate into the bulk.
• Interpolating operators $\mathcal{O}(x)$ at the boundary: $\mathcal{O} = G^a_{\mu\nu} G^{a\mu\nu}$, $\mathcal{O} = \bar{\psi} \gamma_5 \psi$, ...

• QCD/gravity duality dictionary:

<table>
<thead>
<tr>
<th>QCD (4-dim)</th>
<th>Gravity (5-dim)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Hadron int. op. $\mathcal{O}$</td>
<td>Normalizable mode $\Phi(x, z)$</td>
</tr>
<tr>
<td>Hadron mass $\mathcal{M}$</td>
<td>Eigenvalues of 5-dim WF</td>
</tr>
<tr>
<td>Conformal dim $\Delta$</td>
<td>5-dim mass $\mu$</td>
</tr>
<tr>
<td>Large Coupling</td>
<td>Small coupling</td>
</tr>
<tr>
<td>Large $Q$</td>
<td>Small $z$</td>
</tr>
<tr>
<td>Mass gap $\Lambda_{QCD}$</td>
<td>Cutoff $z = z_o$</td>
</tr>
</tbody>
</table>
Quantum Fluctuations and Boundary Excitations

- Higher Fock components of a hadron wf and states with non-zero orbital angular momentum are manifestations of quantum fluctuations of QCD.

- Metric fluctuations of the bulk geometry about the the fixed AdS background should correspond to quantum fluctuations of Fock states above the valence state.

- For large Lorentz spin, orbital excitations in the boundary correspond to quantum fluctuations in the AdS sector. See: Gubser, Klebanov and Polyakov, hep-th/0204051.

- Higher orbital excitations are matched quanta to quanta with fluctuations around the spin $0, \frac{1}{2}, 1, \frac{3}{2}$ string solutions on $AdS_5$. Identification avoids huge string dimensions $\Delta \sim (g_s N_C)^{\frac{1}{4}}$ at large $N_C$ for spin $> 2$.

- The large-$r$ asymptotic behavior of each string mode is matched with the conformal dimension of the boundary interpolating operators for each hadron state, maintaining conformal invariance: an $L$ quantum excitation corresponds to a five dimensional mass $\mu$ in the bulk.

- Allowed values of $\mu$ determined asymptotically requiring that the dimensions are spaced by integers: spectral relation $(\mu R)^2 = \Delta(\Delta - 4)$. For large L: $\mu \simeq L/R$ (string results).
Glueball Spectrum

- AdS wave function with effective mass $\mu$:

$$[z^2 \partial_z^2 - (d - 1)z \partial_z + z^2 \mathcal{M}^2 - (\mu R)^2] f(z) = 0,$$

where $\Phi(x, z) = e^{-iP \cdot x} f(z)$ and $P_\mu P^\mu = \mathcal{M}^2$.

- Glueball interpolating operator with twist -dimension minus spin- two, and conformal dimension $\Delta = 4 + L$

$$\mathcal{O}_{4+L} = F D \{ \ell_1 \ldots D \ell_m \} F,$$

where $L = \sum_{i=1}^{m} \ell_i$ is the total internal space-time orbital momentum.

- Normalizable scalar AdS mode ($d = 4$):

$$\Phi_{\alpha,k}(x, z) = C_{\alpha,k} e^{-iP \cdot x} z^2 J_{\alpha} (z \beta_{\alpha,a} \Lambda_{QCD})$$

with $\alpha = 2 + L$ and scaling dimension $\Delta = 4 + L$. 
• 4-d mass spectrum from boundary conditions on the normalizable string mode at \( z = z_0 \),
\[ \Phi(x, z_0) = 0, \]
given by the zeros of Bessel functions \( \beta_{\alpha,k} \):
\[ M_{\alpha,k} = \beta_{\alpha,k} \Lambda_{QCD} \]

Fig: Gluonium orbital string modes for \( \Lambda_{QCD} = r_o/R^2 = 0.26 \text{ GeV} \).

• \( L = 0 \) lowest glueball state \( \Theta^{++} \):
\[ M = 1.34 \text{ GeV}, \quad \Lambda_{QCD} = 0.26 \text{ GeV}. \]

• Lattice results: \( N_C = 3 \), \( M = 1.47 - 1.64 \text{ GeV} \)
Meson Spectrum

- Wave eq. in AdS for a vector field $\Phi_\mu$ with polarization along Poincaré coordinates:

$$\left[ z^2 \partial_z^2 - (d-1)z \partial_z + z^2 M^2 - (\mu R)^2 + d-1 \right] f_\mu(z) = 0,$$

where $\Phi_\mu(x,z) = e^{-iP\cdot x} f_\mu(z)$ and $P_\mu P^\mu = M^2$ ($\Phi_z = 0$ gauge).

- Vector meson: twist-two, dimension $\Delta = 3 + L$

$$\mathcal{O}_{3+L}^\mu = \bar{\psi}\gamma^\mu \{D_{\ell_1} \cdots D_{\ell_m}\} \psi, \quad L = \sum_{i=1}^m \ell_i.$$

- Normalizable AdS vector mode:

$$\Phi_{\alpha,k}^\mu(x,z) = C_{\alpha,k} e^{-iP\cdot x} z^2 J_\alpha (z \Lambda_{QCD}) \epsilon^\mu,$$

with $\alpha = 1 + L$ and $\Delta = 3 + L$.

- 4-d mass spectrum $\Phi^\mu(x,z_o) = 0$: $M_{\nu,n} = \alpha_{\nu,n} \Lambda_{QCD}$.

- Pseudoscalar mesons: $\mathcal{O}_{3+L} = \bar{\psi} \gamma_5 D_{\{\ell_1} \cdots D_{\ell_m\}} \psi$ ($\Phi_\mu = 0$ gauge).
Fig: Light meson orbital spectrum: 4-dim states dual to vector fields in the bulk, $\Lambda_{QCD} = 0.26$ GeV
Baryon Spectrum

- Solve the full 10-dim Dirac, $\mathcal{D} \hat{\Psi} = 0$, since baryons are charged under the $SU(4) \sim SO(6)$ R-symmetry of $S^5$ (string y-junction) - baryon number conservation?

- $\hat{\Psi}$ is expanded in terms of eigenfunctions $\eta_{\kappa}(y)$ of the Dirac operator on the compact space $X$ with eigenvalues $\lambda_{\kappa}$:

$$\hat{\Psi}(x, z, y) = \sum_{\kappa} \Psi_{\kappa}(x, z) \eta_{\kappa}(y)$$

- From the 10-dim Dirac equation, $\mathcal{D} \hat{\Psi} = 0$:

$$\left[ z^2 \partial_z^2 - dz \partial_z + z^2 \mathcal{M}^2 - (\lambda_{\kappa} + \mu)^2 R^2 + \frac{d}{2} \left( \frac{d}{2} + 1 \right) + (\lambda_{\kappa} + \mu) R \hat{\Gamma} \right] f(z) = 0,$$

$$i \mathcal{D}_X \eta(y) = \lambda \eta(y),$$

where $\Psi(x, z) = e^{-iP \cdot x} f(z)$, $P_{\mu} P^\mu = \mathcal{M}^2$ and $\hat{\Gamma} u_\pm = \pm u_\pm$. For $AdS_5$, $\hat{\Gamma}$ is the four-dim chirality operator $\gamma_5$.

Henningson and Sfetsos; Muck and Viswanathan
• $\mu$ determined asymptotically by spectral comparison with orbital excitations in the boundary:

\[ \mu = L/R \] and $\lambda$ are the eigenvalues of the Dirac equation on $S^{d+1}$:

\[ \lambda \kappa R = \pm \left( \kappa + \frac{d}{2} + \frac{1}{2} \right), \quad \kappa = 0, 1, 2... \]

• Baryon: twist-three, dimension $\Delta = \frac{9}{2} + L$

\[ \mathcal{O}_{\frac{9}{2}+L} = \psi D\{\ell_1 \dots D\ell_q \psi D\ell_{q+1} \dots D\ell_m\} \psi, \quad L = \sum_{i=1}^{m} \ell_i. \]

• Normalizable AdS fermion mode (lowest KK-mode $\kappa = 0$):

\[ \Psi_{\alpha,k}(x, z) = C_{\alpha,k} e^{-iP \cdot x} z^5 [J_\alpha(z^2 \beta_{\alpha,k} \Lambda_{QCD}) \mu_+(P) + J_{\alpha+1}(z^2 \beta_{\alpha,k} \Lambda_{QCD}) \mu_-(P)]. \]

where $\mu^- = \frac{\gamma \cdot P}{P} \mu^+$, $\alpha = 2 + L$ and $\Delta = \frac{9}{2} + L$.

• 4-d mass spectrum $\Psi(x, z_0)^\pm = 0 \implies$ parallel Regge trajectories for baryons!

\[ \mathcal{M}_{\nu,n}^+ = \alpha_{\nu,n} \Lambda_{QCD}, \quad \mathcal{M}_{\nu,n}^- = \alpha_{\nu+1,n} \Lambda_{QCD} \]

• Spin-$\frac{3}{2}$ Rarita-Schwinger eq. in AdS similar to spin-$\frac{1}{2}$ in the $\Psi_z = 0$ gauge for polarization along Minkowski coordinates, $\Psi_\mu$. See: Volovich, hep-th/9809009.
- $SU(6)$ multiplet structure for $N$ and $\Delta$ orbital states, including internal spin $S$ and $L$.

<table>
<thead>
<tr>
<th>$SU(6)$</th>
<th>$S$</th>
<th>$L$</th>
<th>Baryon State</th>
</tr>
</thead>
<tbody>
<tr>
<td>56</td>
<td>$\frac{1}{2}$</td>
<td>0</td>
<td>$N_{\frac{1}{2}}^+(939)$</td>
</tr>
<tr>
<td></td>
<td>$\frac{3}{2}$</td>
<td>0</td>
<td>$\Delta_{\frac{3}{2}}^+(1232)$</td>
</tr>
<tr>
<td>70</td>
<td>$\frac{1}{2}$</td>
<td>1</td>
<td>$N_{\frac{1}{2}}^-(1535)$ $N_{\frac{3}{2}}^- (1520)$</td>
</tr>
<tr>
<td></td>
<td>$\frac{3}{2}$</td>
<td>1</td>
<td>$N_{\frac{1}{2}}^- (1650)$ $N_{\frac{3}{2}}^- (1700)$ $N_{\frac{5}{2}}^- (1675)$</td>
</tr>
<tr>
<td></td>
<td>$\frac{1}{2}$</td>
<td>1</td>
<td>$\Delta_{\frac{1}{2}}^- (1620)$ $\Delta_{\frac{3}{2}}^- (1700)$</td>
</tr>
<tr>
<td>56</td>
<td>$\frac{1}{2}$</td>
<td>2</td>
<td>$N_{\frac{3}{2}}^+ (1720)$ $N_{\frac{5}{2}}^+ (1680)$</td>
</tr>
<tr>
<td></td>
<td>$\frac{3}{2}$</td>
<td>2</td>
<td>$\Delta_{\frac{1}{2}}^+ (1910)$ $\Delta_{\frac{3}{2}}^+ (1920)$ $\Delta_{\frac{5}{2}}^+ (1905)$ $\Delta_{\frac{7}{2}}^+ (1950)$</td>
</tr>
<tr>
<td>70</td>
<td>$\frac{1}{2}$</td>
<td>3</td>
<td>$N_{\frac{5}{2}}^- N_{\frac{7}{2}}^-$</td>
</tr>
<tr>
<td></td>
<td>$\frac{3}{2}$</td>
<td>3</td>
<td>$N_{\frac{3}{2}}^- N_{\frac{5}{2}}^- N_{\frac{7}{2}}^- (2190) N_{\frac{9}{2}}^- (2250)$</td>
</tr>
<tr>
<td></td>
<td>$\frac{1}{2}$</td>
<td>3</td>
<td>$\Delta_{\frac{5}{2}}^- (1930) \Delta_{\frac{7}{2}}^-$</td>
</tr>
<tr>
<td>56</td>
<td>$\frac{1}{2}$</td>
<td>4</td>
<td>$N_{\frac{7}{2}}^+ N_{\frac{9}{2}}^+ (2220)$</td>
</tr>
<tr>
<td></td>
<td>$\frac{3}{2}$</td>
<td>4</td>
<td>$\Delta_{\frac{5}{2}}^+ \Delta_{\frac{7}{2}}^+ \Delta_{\frac{9}{2}}^+ \Delta_{\frac{11}{2}}^+ (2420)$</td>
</tr>
<tr>
<td>70</td>
<td>$\frac{1}{2}$</td>
<td>5</td>
<td>$N_{\frac{9}{2}}^- N_{\frac{11}{2}}^-$</td>
</tr>
<tr>
<td></td>
<td>$\frac{3}{2}$</td>
<td>5</td>
<td>$N_{\frac{7}{2}}^- N_{\frac{9}{2}}^- N_{\frac{11}{2}}^- (2600) N_{\frac{13}{2}}^-$</td>
</tr>
</tbody>
</table>
Fig: Predictions for the light baryon orbital spectrum for $\Lambda_{QCD} = 0.22$ GeV
Hadronic Form Factor in Space and Time-Like Regions

SJB and GdT in preparation

- The form factor in AdS/QCD is the overlap of the normalizable modes dual to the incoming and outgoing hadron $\Phi_I$ and $\Phi_F$ and the non-normalizable mode $J$, dual to the external source (hadron spin $\sigma$):

$$F(Q^2)_{I \rightarrow F} = R^{3+2\sigma} \int_0^\infty \frac{dz}{z^{3+2\sigma}} e^{(3+2\sigma)A(z)} \Phi_F(z) J(Q, z) \Phi_I(z)$$

$$\simeq R^{3+2\sigma} \int_0^{z_0} \frac{dz}{z^{3+2\sigma}} \Phi_F(z) J(Q, z) \Phi_I(z),$$

- $J(Q, z)$ has the limiting value 1 at zero momentum transfer, $F(0) = 1$, and has as boundary limit the external current, $A^\mu = \epsilon^\mu e^{iQ \cdot x} J(Q, z)$. Thus:

$$\lim_{Q \rightarrow 0} J(Q, z) = \lim_{z \rightarrow 0} J(Q, z) = 1.$$

- Solution to the AdS Wave equation with boundary conditions at $Q = 0$ and $z \rightarrow 0$:

$$J(Q, z) = z Q K_1(zQ).$$

Polchinski and Strassler, hep-th/0209211; Hong, Yong and Strassler, hep-th/0409118.
• Propagation of external perturbation suppressed inside AdS.

• At large enough \( Q \sim r/R^2 \), the interaction occurs in the large-\( r \) conformal region. Important contribution to the FF integral from the boundary near \( z \sim 1/Q \).

\[
\mathbf{J}(Q, z), \ \Phi(z)
\]

• Consider a specific AdS mode \( \Phi^{(n)} \) dual to an \( n \) partonic Fock state \( |n\rangle \). At small \( z \), \( \Phi^{(n)} \) scales as \( \Phi^{(n)} \sim z^{\Delta_n} \). Thus:

\[
F(Q^2) \rightarrow \left[ \frac{1}{Q^2} \right]^{\tau-1},
\]

where \( \tau = \Delta_n - \sigma_n, \sigma_n = \sum_{i=1}^{n} \sigma_i \). The twist is equal to the number of partons, \( \tau = n \).
Prediction for the pion form factor in the holographic model (numerical analysis):

\[ F(Q) \]

\[
\begin{array}{cc}
\text{space-like } Q & \text{time-like } Q \\
-2 & 0 \\
0 & 1 \\
1 & 2 \\

s \leftrightarrow t \text{ duality!}
\end{array}
\]
Holographic Model for Light-Front Wavefunctions

SJB and GdT in preparation

- Define the transverse center of momentum $\vec{R}_\perp$ of a hadron in terms of the energy momentum tensor $T^{\mu\nu}$

$$\vec{R}_\perp = \frac{1}{P^+} \int d\sigma^- \int d^2\vec{r}_\perp T^{++} \vec{r}_\perp.$$  

- In terms of partonic variables:

$$x_i \vec{r}_{\perp i} = \vec{R}_\perp + \vec{b}_{\perp i},$$

where $\vec{r}_{\perp i}$ are the physical coordinates and $\vec{b}_{\perp i}$ are frame-independent internal coordinates:

$$\vec{R}_\perp = \sum_i x_i \vec{r}_{\perp i}, \quad \sum_i \vec{b}_{\perp i} = 0.$$
• The normalizable string modes $\Phi_\alpha$ obey the completeness relation:

$$\sum_\alpha \Phi_\alpha(z)\Phi_\alpha(z') = \left(\frac{Re^A(z)}{z}\right)\delta(z - z').$$

• Mapping of string modes to impact space representation of LFWF, which also span a complete basis.

• Two-parton $n = 2$ LFWF including orbital angular momentum $\ell = 0, 1, 2 \ldots$ and radial modes $k = 1, 2, 3, \ldots$ is to first approximation:

$$\psi_{n,\ell,k}(x, b) = B_{n,\ell,k} x(1 - x)\frac{J_{n+\ell-1}(b\beta_{n-1,k}\Lambda_{QCD})}{b},$$  \hspace{1cm} (1)

where $b = |\vec{b}_\perp|$. 
Two-parton ground state LFWF in impact space \( \psi(x, b) \) for \( n = 2, \ell = 0, k = 1 \).
Two-parton first orbital exited state in impact space $\psi(x, b)$ for $n = 2$, $\ell = 1$, $k = 1$. 
Two-parton first radial exited state LFWF in impact space $\psi(x, b)$ for $n = 2$, $\ell = 0$, $k = 2$. 
Outlook

- Only one scale $\Lambda_{QCD}$ determines hadronic spectrum (slightly different for mesons and baryons).
- Ratio of Nucleon to Delta trajectories determined by zeroes of Bessel functions.
- String modes dual to baryons extrapolate to three fermion fields at zero separation in the AdS boundary.
- Only dimension $3, \frac{9}{2}$ and 4 states $\bar{q}q, qqq,$ and $gg$ appear in the duality at the classical level!
- Non-zero orbital angular momentum and higher Fock-states require introduction of quantum fluctuations.
- Simple description of space and time-like structure of hadronic form factors.
- Dominance of quark-interchange in hard exclusive processes emerges naturally from the classical duality of the holographic model. Modified by gluonic quantum fluctuations.
- Covariant version of the bag model with confinement and conformal symmetry.