The $a_0 - a_2$ pion scattering length from $K^+ \to \pi^+\pi^0\pi^0$ decay

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Summary

- Pion scattering lengths.
- Old and new methods for measurement of scattering lengths.
- Theoretical precision.
- Further theoretical work.

N. Cabibbo, arXiv: hep-ph 0405001

$\pi - \pi$ Scattering

(neglecting I-spin breaking)

A complete set of conserved quantum numbers:

Energy of π in CM	$E_\pi; \qquad s_{\pi\pi} = M_{\pi\pi}^2 = 4E_\pi^2$
Total 4-momentum of $\pi\pi$ pair	$P^\mu = \{E, \vec{P}\}$
Angular momentum in CM	$l,\ m$
Isotopic spin of pair	$I,\ I_3$
Velocity of π in CM	$v = k/E_\pi = \left(\frac{s_{\pi\pi} - 4m_\pi^2}{s_{\pi\pi}}\right)^{1/2}$

The S-matrix is diagonal in this basis, and since $S^\dagger S = 1$:

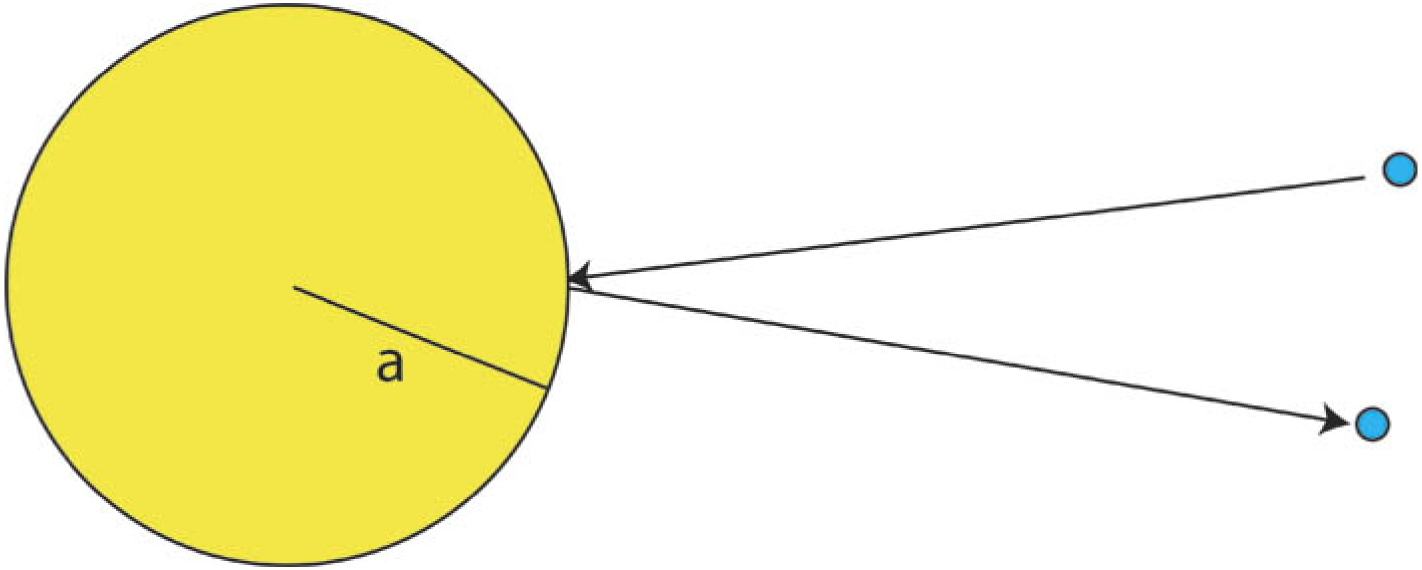
$$S|\ell, m, I, I_3 P >= \exp(2i\delta_I^\ell)|\ell, m, I, I_3 P >$$

Low energy behavior of scattering lengths:

$$\delta_{\mathbf{I}}^\ell = (\mathbf{k})^{\ell+\mathbf{1}}\, \mathbf{a}_{\mathbf{I}}^\ell$$

We will discuss ways of measuring the S-wave scattering lengths, a_0, a_2

Semi-classical view of scattering length



The particle scattered by a solid sphere of radius a is advanced in space by $2\,a$, so that:

Incoming wave:	e^{-ikr}
Outgoing wave	$e^{ik(r+2\,a)} = e^{2i\delta} e^{ikr}$
so that:	$\delta = ka$

The definition of "scattering length" is due to Enrico Fermi ($\sim$ 1925).

Pion scattering lengths and chiral dynamics

We would like to measure the scattering lengths because there is an accurate prediction of their value in the Standard Model. The first result was obtained by Steven Weinberg in 1966.

Weinberg 1966 : $$a_0 m_{\pi^+} = \frac{7m^2_{\pi^+}}{16\pi f^2_\pi} = 0.159$$

$$a_2 m_{\pi^+} = \frac{-m^2_{\pi^+}}{8\pi f^2_\pi} = -0.045$$

The prediction arises from the fact that pions act as Goldstone bosons for the nearly exact, but spontaneously broken $SU(2) \times SU(2)$ chiral symmetry. There is an excellent derivation in the chapter on *soft pions* in S. Coleman, "*Aspects of symmetry*".

These results are difficult to generalize with the original methods of Weinberg, but there is an alternative: Chiral Perturbation Theory.

Pion scattering lengths in $O(p^6)$ Chiral Perturbation Theory

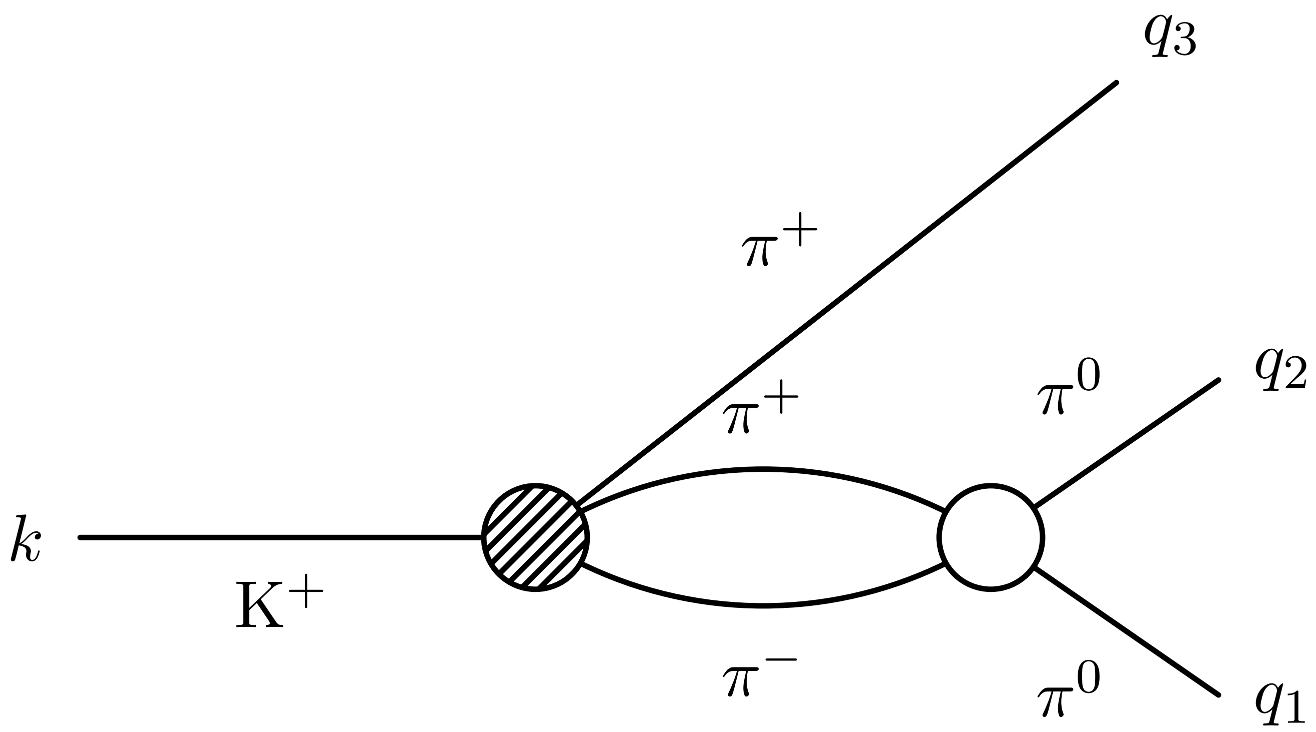
Colangelo et al. 2001 :

$$a_0 m_{\pi^+} = 0.220 \pm 0.005$$
$$a_2 m_{\pi^+} = -0.0444 \pm 0.0010$$
$$(a_0 - a_2) m_{\pi^+} = 0.265 \pm 0.004$$

These are very accurate predictions, and pose a very difficult challenge: Can this accuracy be matched by experimental results?

How to measure $a_0 - a_2$ in $K^+ \to \pi^+\pi^0\pi^0$

This is based on the effects of $\pi^+\pi^- \to \pi^0\pi^0$ re-scattering from $K^+ \to \pi^+\pi^+\pi^-$ near the threshold for that reaction.



The method is based on two fundamental properties of the **S**-matrix:

1. Unitarity: $\mathbf{S}^\dagger\mathbf{S} = 1$
2. Analyticity of the **S**-matrix elements as a function of the external momenta.

A technical clarification

I-spin is badly broken between the $\pi^0\pi^0$ and the $\pi^+\pi^-$ threshold which is just the region we are interested in!

How to define I=0 and I=2 scattering lengths?

Solution we have adopted:

Define $(a_0 - a_2)/3$ through:

$$M(\pi^+\pi^- \to \pi^0\pi^0)|_{\text{S-wave}} = imv_{\pi^+}(a_0 - a_2)/3 + O(v_{\pi^+}^2)$$

where $v_{\pi^+} = \sqrt{(s_{\pi\pi} - 4m_{\pi^+}^2)/s_{\pi\pi}}$

This is a very "interim" solution! Radiative corrections are needed to clarify the relation between this and other definitions of $(a_0 - a_2)/3$, e.g. in the decay rate of the $\pi^+\pi^-$ atom.

Unitarity of the $\mathbf{S}$-matrix

If we write: $$< f|\mathbf{S}|i >= \delta_{fi} + iT_{fi}$$

Unitarity implies that: $$(T_{fi} - T^*_{if})/i = \sum_k T^*_{kf} T_{ki}$$

Where the sum extends over possible physical states of the same energy as $|i >$ or $|f >$. T_{fi} and T_{if} are related by time reversal, so that this reduces to

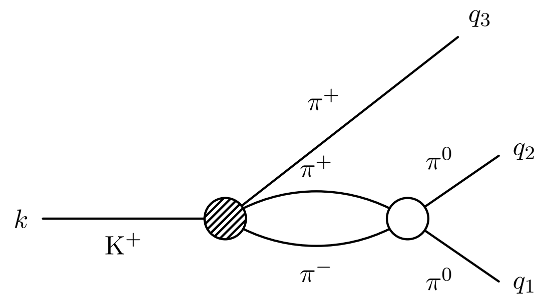
$$2\,\mathrm{Im}(T_{fi}) = \sum_k T^*_{fk} T_{ki}$$

The imaginary part of T_{fi} is determined by physical amplitudes.

$\pi^+\pi^- \to \pi^0\pi^0$ re-scattering

Let us write:

$$\mathcal{M}(K^+ \to \pi^+\pi^0\pi^0) = \mathcal{M} = \mathcal{M}_0 + \mathcal{M}_1$$



where $\mathcal{M}_1$ is the contribution of the re-scattering graph.

We must consider two cases, according to whether $M_{\pi\pi}$ is above or below the $\pi^+\pi^-$ threshold.

$$s_{\pi\pi} > 4m_{\pi^+}^2 \quad : \quad \mathcal{M}_1 = i2\frac{(a_0 - a_2)m_{\pi^+}}{3}\mathcal{M}_{+,\mathrm{thr}}\sqrt{(s_{\pi\pi} - 4m_{\pi^+}^2)/s_{\pi\pi}}$$

$$s_{\pi\pi} < 4m_{\pi^+}^2 \quad : \quad \mathcal{M}_1 = -2\frac{(a_0 - a_2)m_{\pi^+}}{3}\mathcal{M}_{+,\mathrm{thr}}\sqrt{(4m_{\pi^+}^2 - s_{\pi\pi})/s_{\pi\pi}}$$

where $\mathcal{M}_{+,\mathrm{thr}}$ is the value of $\mathcal{M}_+$ at the $\pi^+\pi^-$ threshold.

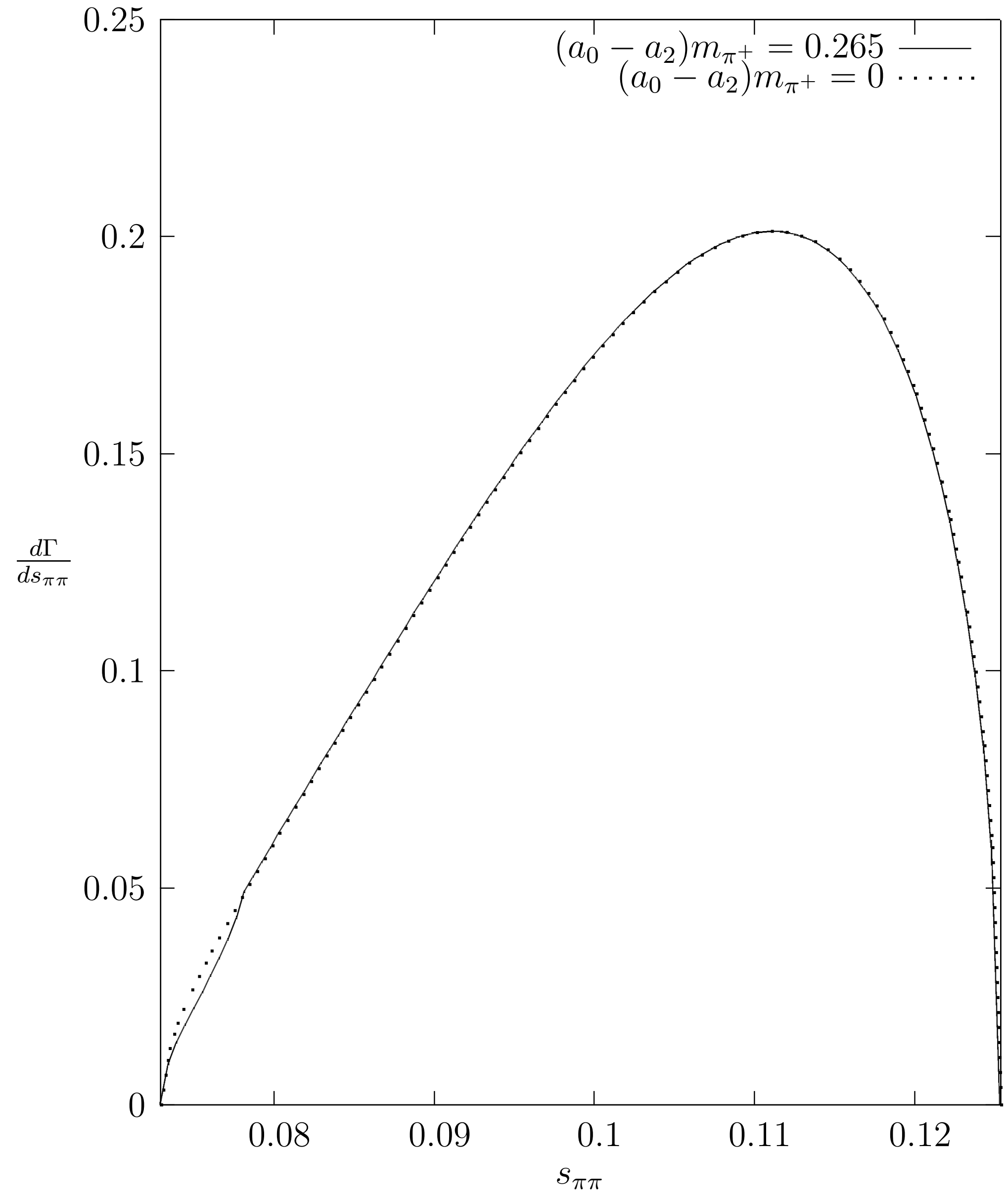
- Above threshold: $\mathcal{M}_1$ imaginary — therefore correct!
- Below threshold: $\mathcal{M}_1$ real and negative — also correct by analytic continuation!

The key to the measurement

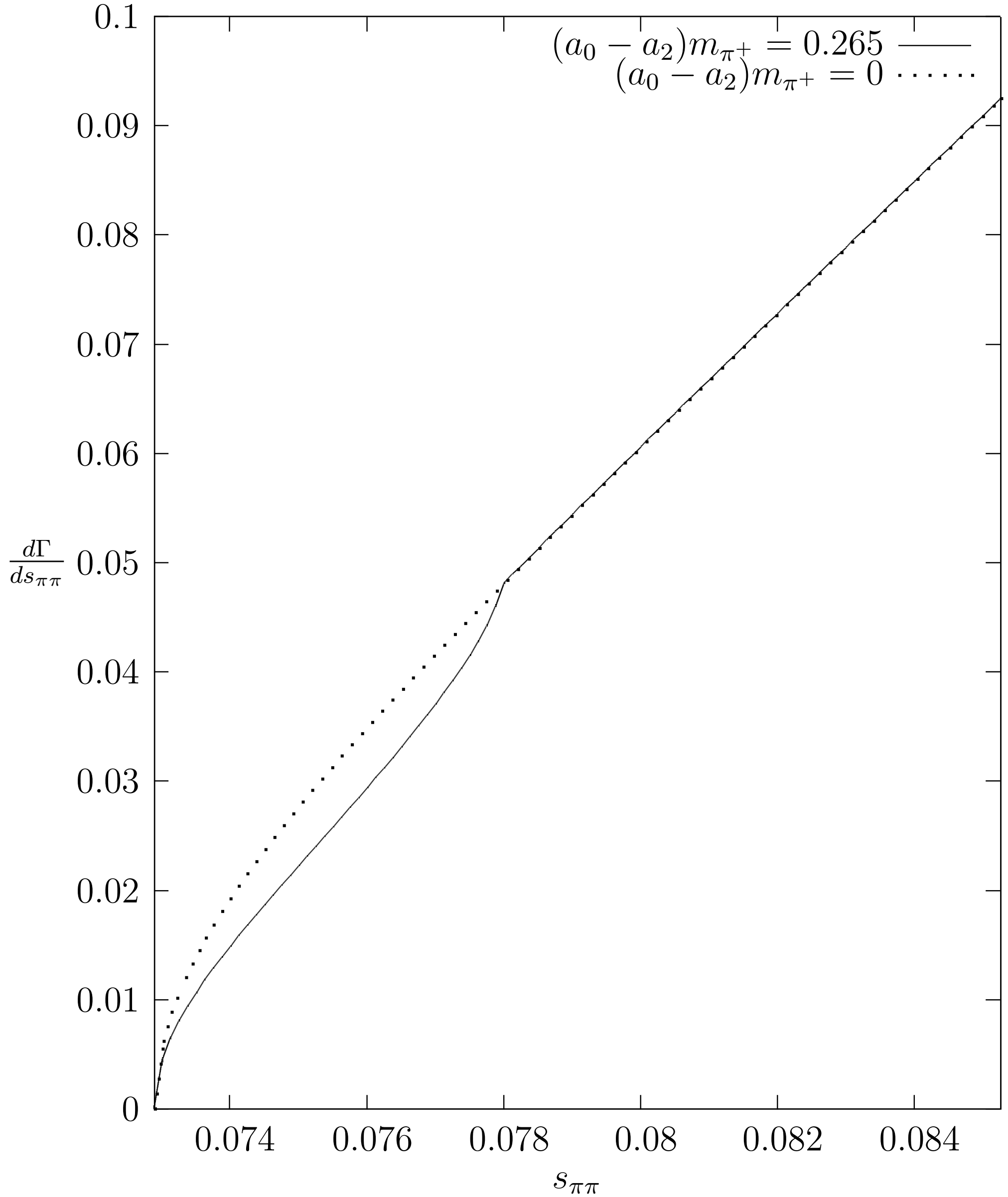
Below threshold there is an interference term
which is absent above the threshold:

$$|\mathcal{M}|^2 = \begin{cases} (\mathcal{M}_0)^2 + (\mathcal{M}_1)^2 + 2\mathcal{M}_0\mathcal{M}_1 & : \quad s_{\pi\pi} < 4m_{\pi^+}^2 \\ (\mathcal{M}_0)^2 + |\mathcal{M}_1|^2 & : \quad s_{\pi\pi} > 4m_{\pi^+}^2 \end{cases}$$

The interference term
is proportional to $a_0 - a_2$
and gives rise to a "Threshold cusp"



The $s_{\pi\pi}$ invariant mass distribution
with/without the re-scattering correction, in arbitrary units.

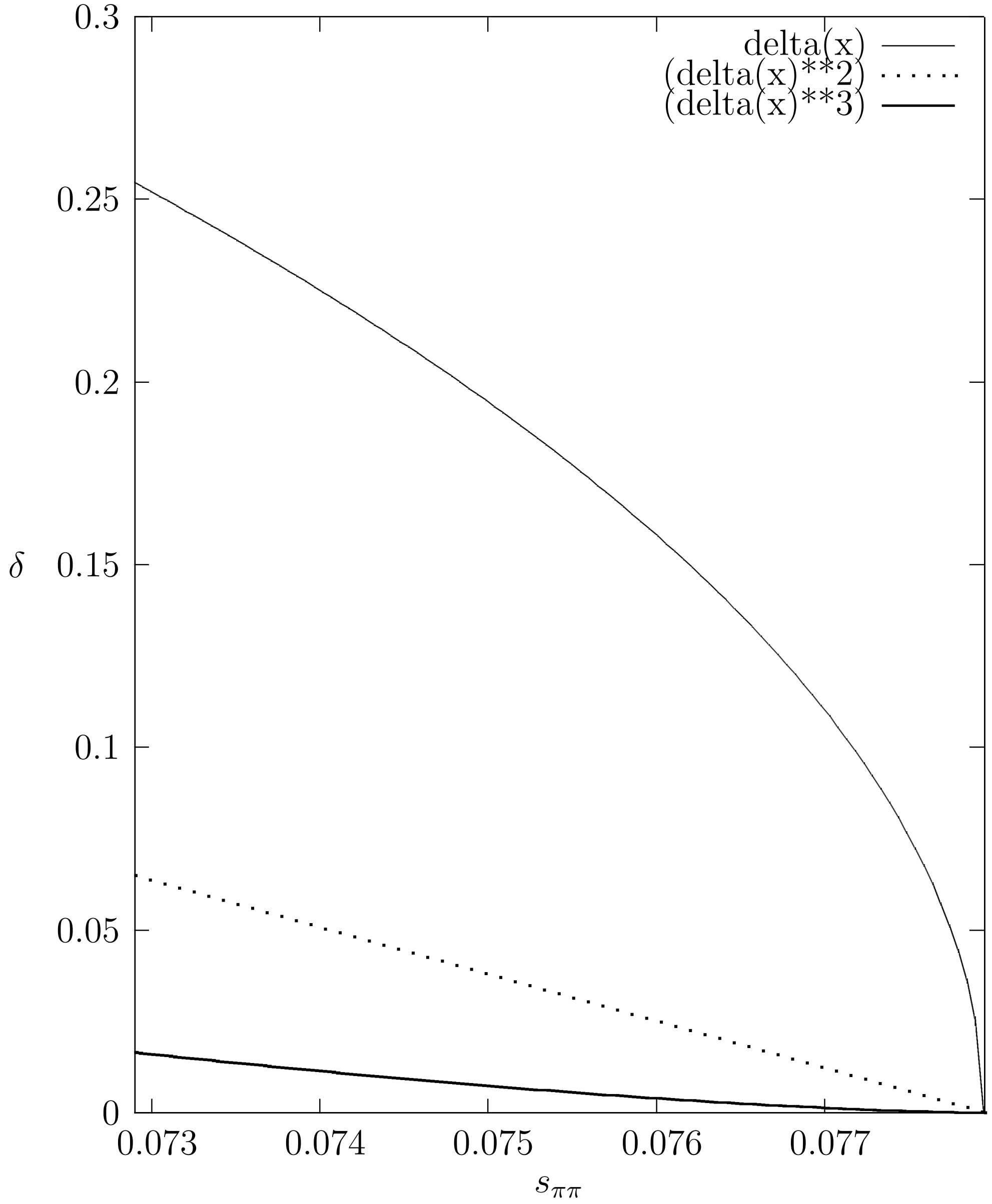


The $s_{\pi\pi}$ invariant mass distribution
with/without the re-scattering correction, in arbitrary units.

Precision

- Below threshold define $\delta = \sqrt{\frac{4m_{\pi^+}^2 - s_{\pi\pi}}{4m_{\pi^+}^2}}$
- The differential rate is predicted with errors $\sim \delta^3$
- The theoretical error on $a_0 - a_2$ is $\sim \delta_{max}^2$

One can strike a balance between theoretical uncertainty and statistical error.



Theoretical precision can be improved by restricting the range of $s_{\pi\pi}$.

Smaller effects

I have not so far discussed the contribution of other diagrams which arise from the unperturbed amplitude $\mathcal{M}_0$ with $\pi^0\pi^0 \to \pi^0\pi^0$ or $\pi^+\pi^0 \to \pi^+\pi^0$ re-scattering and similar contributions to $\mathcal{M}_+$. These contribution are generally smaller and their effect can be taken into account in the real analysis.

Prospects and space for improvement

- Experimental fit to terms $O(\delta^3)$ in differential rate.
- Compute these terms in Chiral Perturbation Theory
- Compute radiative corrections

The theoretical error can be made very small.
The experimental error can be made very small

In NA48: $>10^8$ $K^+ \to \pi^+\pi^0\pi^0$ events,

of which 3%, ($> 3 \times 10^6$) below the $\pi^+\pi^-$ threshold.

The statistical error on the effect should result $< 1\%$.