DAΦNE TECHNICAL NOTE

INFN - LNF, Accelerator Division

Frascati, March 2, 1993

Note: **RF-9**

STABILITY ANALYSIS AND PULSE RESPONSE OF AN RF FEEDBACK SYSTEM

M. Migliorati, L. Verolino

INTRODUCTION

The control of RF cavities with high beam loading is a subject of great concern for the design and operation of high luminosity collider such as DAΦNE. Perturbations of the accelerating voltage can generate undesirable beam behaviour, like large oscillations of the beam itself that could cause particles loss by lack of longitudinal acceptance. To alleviate or even completely suppress these effects, a proper control of the RF power amplifier is necessary. Preliminary calculations have shown[1,2] that in DAΦNE it is possible to avoid Sands and Robinson instabilities for any beam current up to 5 Amps by properly setting the RF system. The price to pay is that the stability region around the working point reduces, especially for high currents, getting the machine operations too sensitive to longitudinal transient oscillations.

In this note we shall study the stability and the pulse response of a proposed feedback system sketched in Figure 1. The feedback loop automatically generates the correct compensating signal[3], which keeps the controlled parameter V constant. One can consider the RF feedback as a means to reduce the output impedance of the RF amplifier: its effect is equivalent to introducing a shunt resistance across the accelerating gap, the value of which is inversely proportional to the loop gain. In such a way the total shunt resistance can be reduced by a large factor, allowing a large beam current to be stored in the ring[4]. The idea is the same as the cathode follower[5] with its low output impedance that shunts the cavity.

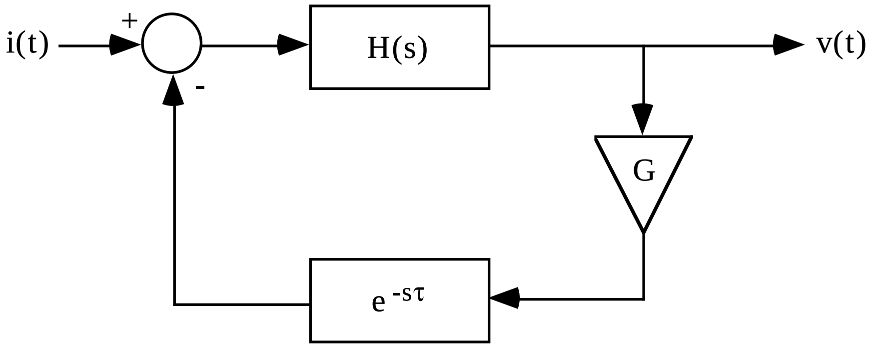

Figure 1: RF feedback scheme.

It is customary to analyze the RF cavity without feedback, in the vicinity of the main resonance, as an RLC circuit, with a transfer function, in the domain of the complex variable $s=\sigma+j\omega$, described by:

$$H(s) = \frac{\omega_r R_s}{Q} \frac{s}{s^2+2\alpha s+\omega_r^2} , \qquad (1)$$

where ω_r is the resonant angular frequency, R_s is the shunt resistance, Q is the quality factor of the cavity and $\alpha = \omega_r / 2Q$ is the damping factor.

In Section 1 we shall study the stability conditions of the feedback system, obtaining some relations between the delay τ and the cavity resonant angular frequency ω_r . In Section 2 we shall give an expansion for the pulse response in the time domain; this response can be very useful for the study of the transient phenomena in a cavity in presence of the feedback. Some considerations about inserting this feedback system in a time domain simulation code on the longitudinal dynamics of a beam will conclude, in Section 3, the paper.

1 - STABILITY CONDITIONS

Let us start by studying the stability conditions in the Laplace transformed s-domain. A feedback circuit is stable if it can be established that its closed-loop transfer function has no poles in the positive side of the s-plane. In practice it is found more convenient to work with the open-loop gain function rather than with the transfer function of the feedback system. In fact the first one is directly measurable. Moreover in our particular case, the closed-loop analysis would become very difficult because of the delay term $\exp(-s\tau)$. The open-loop gain function:

$$(s)= \frac{\omega_r R_s}{Q} \frac{G s e^{-s\tau}}{s^2+2\alpha s+\omega_r^2} \qquad (2)$$

shows two simple poles with negative real part. Putting $x=\omega/\omega_r$,the function (2) can be rewritten in the dimensionless form:

$$F(jx)= G\, R_s \frac{jx e^{-j\omega_r \tau x}}{\left(1-x^2\right)Q+jx} . \qquad (3)$$

Nyquist's criterion for stability states[6] that the feedback circuit is stable if the plot of F(jx) does not enclose the critical point (-1,0) of the F(jx)-plane as x varies from $-\infty$ to $+\infty$. The required plotting of F(jx) can be simplified by observing that the magnitude of F(jx) is an even function of x, while its phase angle is an odd function of x; therefore, it is necessary only to plot F(jx) for positive frequencies from $x=0$ to $x=\infty$.

The Figure 2 immediately suggests that in the case $\omega_r\tau=0$ (cavity without feedback ring) the system is always stable.

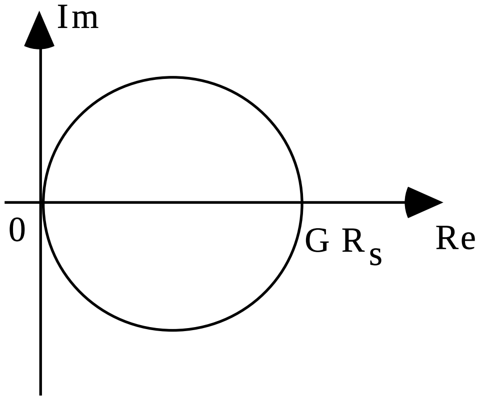

Figure 2: Nyquist diagram for $\omega_r\tau=0$.

The magnitude of the function (3):

$$|F(jx)| = GR_S \frac{|x|}{\sqrt{(1-x^2)^2 Q^2 + x^2}} \quad , \qquad (4)$$

points out that **when $GR_S<1$ the system described by equation (2) is always stable**, being $|F(jx)|<GR_S$. On the contrary, if $GR_S>1$, the problem of the stability is not so easy to solve, the stability depending on the values of GR_S, Q and $\omega_r\tau$. In Figure 3 an unstable case is shown.

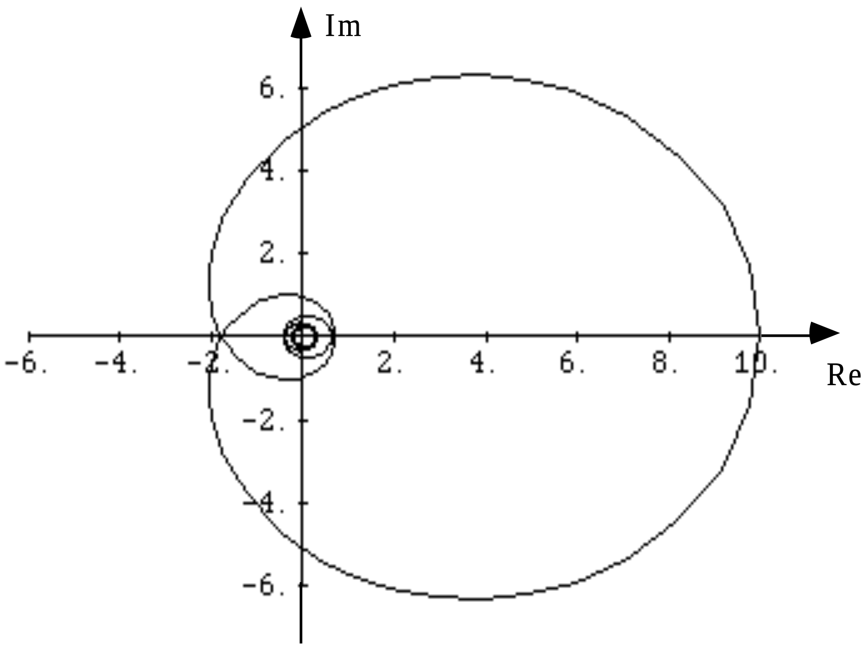

Figure 3: Nyquist diagram for $\omega_r\tau=2\pi$ 500, $GR_S=10$, Q=5000.

Thus, if $GR_s>1$, one can formulate the stability problem in this way: let us say ξ_k (k=1,2,...) the zeros of the imaginary part of F(jx); the system can be considered stable if the real part of F(jx), evaluated for $x=\xi_k$, is greater then -1. So, splitting equation (3) into its real and imaginary part:

$$\mathrm{Re}\left[F(jx)\right] = GR_s \frac{x\left(1-x^2\right)Q\sin(\omega_r\tau x)+x^2\cos(\omega_r\tau x)}{\left(1-x^2\right)^2Q^2+x^2} ;$$

$$\mathrm{Im}\left[F(jx)\right] = GR_s \frac{x\left(1-x^2\right)Q\cos(\omega_r\tau x)-x^2\sin(\omega_r\tau x)}{\left(1-x^2\right)^2Q^2+x^2} ,$$

after some algebraic manipulations, the fundamental system for the stability can be written as[1]

$$GR_s\cos(\omega_r\tau x) > -1 , \qquad (5)$$

$$\mathrm{tg}(\omega_r\tau x) = Q \frac{1-x^2}{x} . \qquad (6)$$

It is worth noting that one has to solve the transcendental equation (6) first, and then choose the parameters in order to verify the inequality (5). This problem has not an elementary solution; in the following we shall discuss possible approximate solutions and inequalities to fix the boundaries of the stability problem. The fundamental system (5) and (6) can be reformulated using the new variable $u=\omega_r\tau x$, getting:

$$GR_s\cos(u) > -1 , \qquad (7)$$

$$g(u) = \frac{Q}{\omega_r\tau} \frac{\left(\omega_r\tau\right)^2-u^2}{u} . \qquad (8)$$

---

[1]The same result can be obtained using the Pontriagin's theorem[7].The exponential polynomial

$$P(s,e^s) = e^{s\tau}\left(s^2 +2\alpha s +\omega_r^2\right) + G \frac{\omega_r R_s}{Q} s ,$$

when $s=j\omega$ can be splitten into the real part and the imaginary one. Pontriagin's stability criteria states that the time lag system is stable if, and only if, all the zeros ξ, of the real part are real, simple, and the product between the first derivative with respect to ω of the real part and the imaginary part has to be negative for every such zero.

The problem has been written in this way because the functional dependence of the solution has been divided into two parts: in the equation there are only the parameters Q and $\omega_r\tau$, while the only GR_s is in the inequality. An example of graphical solutions of equation (8) is given in Figure 4.

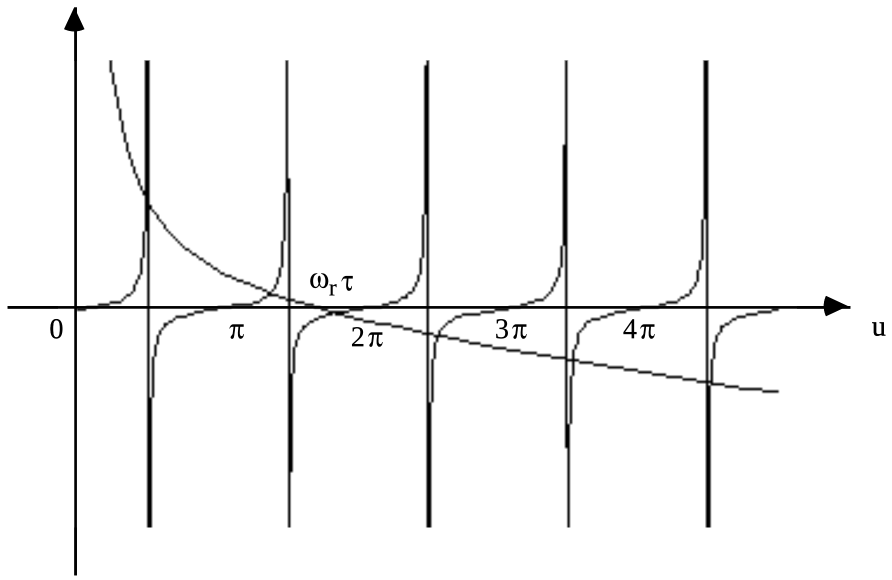

Figure 4: graphical solution of equation (8).

Another way to discuss the stability of our system, i.e. another way to read Nyquist's criterion, is the following: since the exponential function due to the delay does not change the magnitude of the open-loop gain function [equation (4)] but it displaces the phase of all the points of a angle $\Delta\varphi = \omega_r\tau x$, the range of x, potentially dangerous, is the one where F(jx) exceeds unitary circle. Formally, this range is given by $|F(jx)|>1$, or:

$$-\frac{\sqrt{(GR_s)^2-1}}{2Q}+\sqrt{1+\frac{\left[(GR_s)^2-1\right]^2}{4Q^2}} < x < \frac{\sqrt{(GR_s)^2-1}}{2Q}+\sqrt{1+\frac{\left[(GR_s)^2-1\right]^2}{4Q^2}}\ ; \qquad (9)$$

Therefore one could draw a plot in the plane of the complex function F(jx) with x varying according to the inequality (9); the system is stable if the plot passes through the real axis only one time for positive values, otherwise it is unstable (Figure 5).

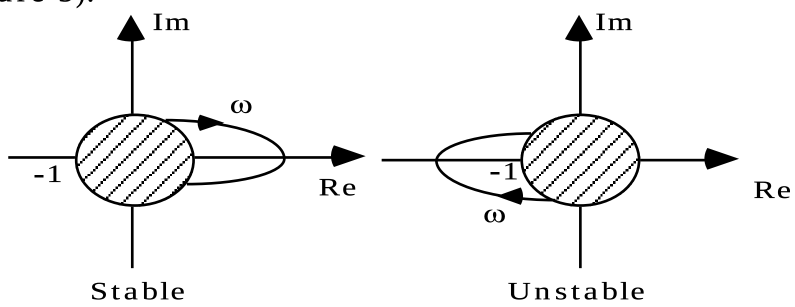

Figure 5: possible situations.

An useful approximation of equation (9) can be given if

$$GR_s \gg 1 \,, \qquad \frac{GR_s}{Q} \ll 1 \,; \tag{10}$$

under these hypothesis, that we shall use frequently in the following, one can write:

$$1 - \frac{GR_s}{2Q} < x < 1 + \frac{GR_s}{2Q} \,. \tag{11}$$

We are now ready to compute a first instability region. Let us call "a" and "b" the two roots of the equation $|F(jx)|=1$ given in (9) and $\varphi(a)$ and $\varphi(b)$ the corresponding phase angles; the system is certainly **unstable** if the absolute value of the difference of these two phase angles is greater than 2π. Because the phase angle is:

$$\varphi(x) = \frac{\pi}{2} - \omega_r\tau x - tg^{-1}\frac{x}{Q\left(1-x^2\right)} \,, \qquad (x>0),$$

making use of the approximations (10), it is not so difficult to obtain[2] from $|\varphi(a) - \varphi(b)| > 2\pi$

$$\omega_r\tau > \left(3\pi + \frac{2}{GR_s}\right)\frac{Q}{GR_s} \,. \tag{12}$$

Coming back to the fundamental system (7) and (8), we have to discuss in more details its solutions, trying to impose other inequalities for stability. Two cases have been considered, and even though they cover all the possible values of $\omega_r\tau$, they just give two regions into which we are sure to have stability or instability. Outside these regions we can not say anything. The first case studied is valid when

$$2\pi N - \frac{\pi}{2} < \omega_r\tau < 2\pi N + \frac{\pi}{2} \,, \tag{13}$$

with $N = 0,1,2\ldots$ .The (13) can be written as:

$$\omega_r\tau = 2\pi N + \alpha\frac{\pi}{2} \,, \tag{14}$$

with $|\alpha| < 1$. In this case the closest solution of the equation (8) to $\omega_r\tau$, gives a positive cosine, so that the equation (7) is certainly verified.

---

[2] Only for shortness we do not derive the complete expansion, using the exact roots given in equation (9).

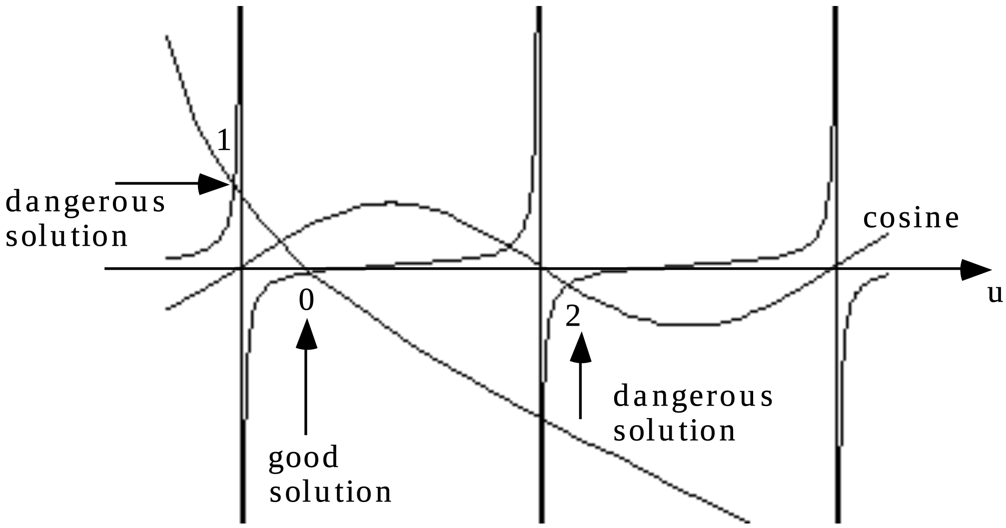

Figure 6: stability case.

The possible dangerous solutions, as shown in Figure 6, are then the (1) and (2). If the two roots of the equation |F(jx)|=1 are within the interval

$$2\pi N - \frac{\pi}{2} < u < 2\pi N + \frac{\pi}{2}\,, \tag{15}$$

this means that |F(jx)| is bigger than 1 in a region where the cosine function is positive. As a consequence we have stability for the system. Using the formulae, we can say that, if the inequality (15) is verified, we are sure to be stable, that is:

$$\frac{2N-0.5}{2N-0.5\,\alpha} \le -\frac{\sqrt{(GR_s)^2-1}}{2Q} + \sqrt{1+\frac{\left[(GR_s)^2-1\right]^2}{4Q^2}}\;;$$
$$\frac{\sqrt{(GR_s)^2-1}}{2Q} + \sqrt{1+\frac{\left[(GR_s)^2-1\right]^2}{4Q^2}} \le \frac{2N+0.5}{2N-0.5\,\alpha}\,. \tag{16}$$

It is worthy to mention that if $N \rightarrow \infty$ ($\omega_r\tau \rightarrow \infty$), the previous inequalities are gradually less verified whatever be the values of Q and GR_S. Nothing can be said if the roots of the equation |F(jx)|=1 lie outside the interval given by (15).

In the second case, i.e. when

$$(2N+1)\,\pi - \frac{\pi}{2} < \omega_r\tau < (2N+1)\,\pi + \frac{\pi}{2} \tag{17}$$

that is

$$\omega_r\tau = (2N+1)\,\pi + \alpha\frac{\pi}{2}\,, \tag{18}$$

the most dangerous solution of the equation (8) is the closest to the value $\omega_r\tau$, as shown in Figure 7, since now the cosine function is negative.

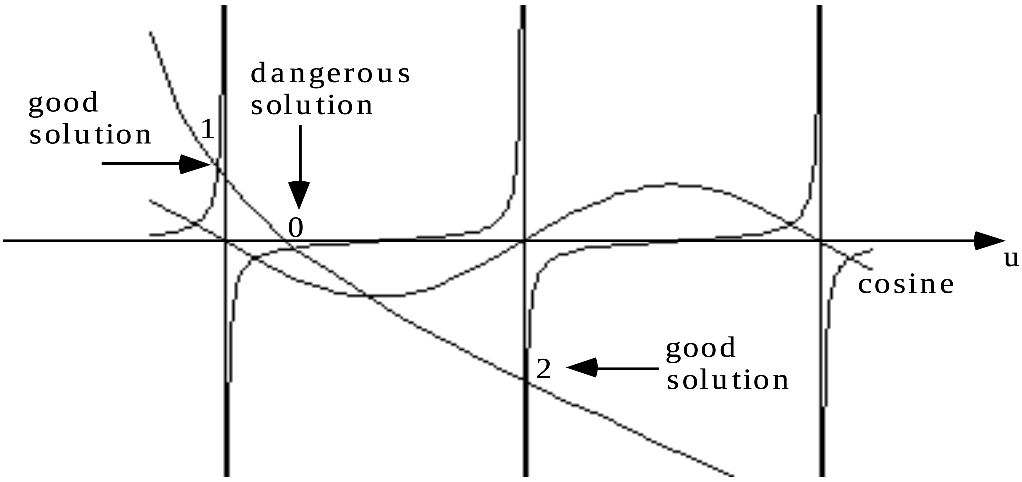

Figure 7: instability case.

Named E_1 and E_2 the two values within which $GR_s<1$ (if they exist), then, if

$$E_1 < \omega_r\tau < E_2 \ , \qquad (19)$$

that means

$$(2N+1)\pi - \cos^{-1}\frac{1}{GR_s} < \omega_r\tau < (2N+1)\pi + \cos^{-1}\frac{1}{GR_s} \ , \qquad (20)$$

we are sure to have instability. The inequality (20) can be written as

$$|\alpha| < \frac{2}{\pi}\cos^{-1}\frac{1}{GR_s} \ . \qquad (21)$$

Let us note that if $GR_s \to \infty$, then the instability occurs for every value of α , i.e. all over the interval around $(2N+1)\pi$; so it is very dangerous to work in these conditions. As in the previous case, nothing can be said if the (21) is not satisfied.

2 - PULSE RESPONSE

The closed-loop transfer function can be easily obtained in the s-plane from Figure 1. Such a function can be written as:

$$H_c(s) = \frac{V(s)}{I(s)} = \frac{H(s)}{1 + Ge^{-s\tau}H(s)} \ , \qquad (22)$$

where the subscript c means closed-loop. We shall find an expansion of the inverse Laplace transform

$$h_c(t) = L^{-1}\left[H_c(s)\right] , \tag{23}$$

i.e. the pulse response of the feedback system, making use of the well known expansion[8]

$$\frac{1}{1+z} = \sum_{n=0}^{\infty} (-1)^n z^n , \qquad (| z | < 1) . \tag{24}$$

Substituting (24) in the definition (22) and using the change $s=p\omega_r$, the closed loop function becomes:

$$_c(p\omega_r) = \sum_{n=0}^{\infty} (-1)^n G^n e^{-n\tau p\omega_r} H^{n+1}(p\omega_r) , \tag{25}$$

where[3]

$$G \left| e^{-\tau p\omega_r} H(p\omega_r) \right| < 1 .$$

Equation (25) implies we need to find the inverse Laplace transform of the infinite set of functions:

$$F_n(p) = \left(\frac{p}{p^2 + \frac{p}{Q} + 1} \right)^{n+1} , \tag{26}$$

that shows two poles for

$$p_0 = \frac{1}{2Q}\left(-1+j\sqrt{4Q^2-1}\right) = j\ e^{j\varphi} ;$$

$$\bar{p}_0 = \frac{1}{2Q}\left(-1-j\sqrt{4Q^2-1}\right) = -j\ e^{-j\varphi} ,$$

where we introduced the phase angle φ , defined as

$$\mathrm{tg}\ \varphi = \frac{1}{\sqrt{4Q^2-1}} .$$

[3]It will be shown in the Appendix that this inequality, strictly speaking, is not necessary.

It has to be noted that the phase angle φ vanishes if $Q \rightarrow \infty$. The technique of partial fractions[6] is powerful to expand the functions (26) as:

$$F_n(p) = \sum_{k=1}^{n+1} \left[\frac{A_k}{(p-p_0)^k} + \frac{\overline{A}_k}{(p-\overline{p}_0)^k} \right] , \qquad (27)$$

where the expansion coefficients A_k are given by the well known formula[10]

$$_k = \frac{1}{(n+1-k)!} \left[(p-p_0)^{n+1} \, F_n(p) \right]^{(n+1-k)}_{p=p_0} . \qquad (28)$$

Equation (28) , after some algebraic manipulations, can be written as

$$A_k = \frac{p_0^k}{(p_0-\overline{p}_0)^{n+1}} \, C_k(n) \qquad \left[k=1,2,...,(n+1) \right] , \qquad (29)$$

where $C_k(n)$ is a short form for

$$C_k(n) = \left|C_k(n)\right| \, e^{j\varphi_k(n)} = (n+1) \sum_{r=0}^{n+1-k} \frac{(-1)^r}{r!} \, \frac{(n+r)!}{(k+r)! \, (n+1-k-r)!} \left(\frac{p_0}{p_0-\overline{p}_0} \right)^r ,30)$$

Finally equations (29) and (30) , together with the causality of the inverse Laplace transform of $F_n(p)$, enables us to conclude that the pulse response of the feedback system in the range $M\tau \leq t < (M+1)\tau$ is:

$$_c(t) = \frac{2\omega_r R_s}{\sqrt{4Q^2-1}} \sum_{n=0}^{M} \left(- \frac{G R_s}{\sqrt{4Q^2-1}} \right)^n q_n(t-n\,\tau) , \qquad (31)$$

where the functions $q_n(t)$ are defined as:

$$q_n(t)=\exp\left(-\frac{\omega_r t}{2Q}\right) \sum_{k=1}^{n+1} \left|C_k(n)\right| \frac{(\omega_r t)^{k-1}}{(k-1)!} \sin\left[\frac{\omega_r t}{2Q}\sqrt{4Q^2-1} + k\varphi - (n-k)\frac{\pi}{2} + \varphi_k(n) \right] .$$

3 - CONCLUSIONS

In this paper we have analyzed a possible feedback system for a RF cavity with high beam loading. The study of the stability has been performed in order to find the regions where the feedback parameters (i.e. loop gain and time delay) should be defined.

Furthermore it has been worked out the response to a single pulse that gives us the possibility of developing the cavity voltage in the time domain taking into account both the driving and the beam loading voltage.

This last analysis has been performed with the aim of developing a time domain simulation code on the longitudinal beam dynamics.

APPENDIX

The representation in the time domain of the pulse response of a RLC circuit comes from equation (1) and it is[8]

$$(t) = \frac{2\omega_r R_s}{\sqrt{4Q^2-1}} \cos\left(\frac{\omega_r t}{2Q}\sqrt{4Q^2-1} + \varphi\right) \exp\left(-\frac{\omega_r t}{2Q}\right) u(t) \ , \qquad \text{(A1)}$$

where u(t) is the unit step function and the other symbols are the same used previously; this expansion gives also the response of the feedback circuit for $0 \le t < \tau$, because during this period the feedback ring does not work. If $Q \gg 1$, equation (A1) assumes the approximate but simple form

$$h(t) \approx \frac{\omega_r R_s}{Q} \cos(\omega_r t) \exp\left(-\frac{\omega_r t}{2Q}\right) u(t) \ .$$

The pulse response of the feedback scheme can be expressed by mean of equation (22) in the s plane, or making use of the convolution theorem[9]; putting $i(t) = \delta(t)$ to compute $v(t) = h_c(t)$ in Figure 1, the integral equation for the pulse response can be written as

$$h_c(t) = h(t) - G \int_{\tau}^{t} h(t-u) \ h_c(u-\tau) \, du \ , \qquad (t \ge \tau) \ . \qquad \text{(A2)}$$

It is interesting to note how equation (A2) takes into account the causality of the system, suggesting that we can compute the output $h_c(t)$ **without any limitation**, only knowing the solution during the initial τ seconds. Thus a recursive solution can be found by splitting the upper integration bound t into a certain number of multiple of τ. In this way we can decide to compute, for example, the solution in the interval $\tau \le t < 2\tau$; the knowledge of $h_c(t)$

$$h_c(t) = h(t) \qquad \text{when } 0 \le t < \tau \ ,$$

implies immediately that[8]

$$h_c(t) = A \exp\left[-\frac{\omega_r(t-\tau)}{2Q}\right] \cdot$$

$$\cdot \left\{\cos(\omega_0 t+\varphi) - \frac{A}{2\omega_0}\sin\left[\omega_0(t-\tau)\right] - \frac{A}{2}(t-\tau)\cos\left[\omega_0(t-\tau)+2\varphi\right]\right\} ,$$

where for shortness we called

$$\omega_0 = \frac{\omega_r}{2Q}\sqrt{4Q^2-1} \quad ,$$

$$A = \frac{2\omega_r G R_s}{\sqrt{4Q^2-1}} \exp\left(\frac{\omega_r \tau}{2Q}\right) \quad .$$

If one continues the iterations, the previous solution (31) can be again obtained.

ACKNOWLEDGMENTS

The authors want to thank A. Gallo, L. Palumbo, and V. G. Vaccaro for the helpful discussions and suggestions.

REFERENCES

[1] R. Boni, A. Gallo, *Stationary beam loading on DAΦNE cavity accelerating mode*, DAΦNE Note RF-5, 1992.

[2] A. Gallo, *An RF feedback for DAΦNE*, DAΦNE Note RF-6, 1992.

[3] S. Giordano, M. Puglisi, *A cathode follower power amplifier*, IEEE Transactions on Nuclear Science, NS-32, No. 5, 1983, p. 3408.

[4] D. Boussard, *Beam loading*, CERN 87-03, 1987, p. 626.

[5] D. Boussard, *Control of cavities with high beam loading*, IEEE Transactions on Nuclear Science, NS-32, No. 5, 1985, p. 1852.

[6] L. D. Harris, *Introduction to feedback systems*, John Wiley, New York, 1968.

[7] S. R. Koscielniak, *RF system aspects of longitudinal beam control (in the low current regime)* , TRI-DN-91-k183, August 1991.

{8] I. S. Gradshteyn, I. M. Ryzhik, *Tables of Integrals, Series, and Products*, Academic Press, 1980.

[9] M. L. Krasnov, A. I. Kiselev, G. I. Makarenko, *Integral equations*, Mir, 1976.

[10] G. N. Watson, E. T. Whittaker, *A course of modern analysis*, Cambridge University Press, 1927.