DAΦNE TECHNICAL NOTE

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Tune Shift in Beam-Beam Collisions with a Crossing Angle

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Abstract

Several schemes for upgrading DAFNE require a large crossing angle. In this Note we investigate the beam-beam tune shift and beam deflection dependence on the crossing angle, for non-crabbing collisions.

Beam-beam tune shift

Let us consider two ultrarelativistic bunches colliding at a horizontal angle θ as shown in Fig. 1. The strong beam moves along z-axis of the right laboratory coordinate system. The coordinate system connected with the test particles of the weak beam and denoted with the index 'p' in Fig. 1 is rotated by the angle θ with respect to the strong beam system. The vertical y-axes coincide for both systems and are directed towards us.

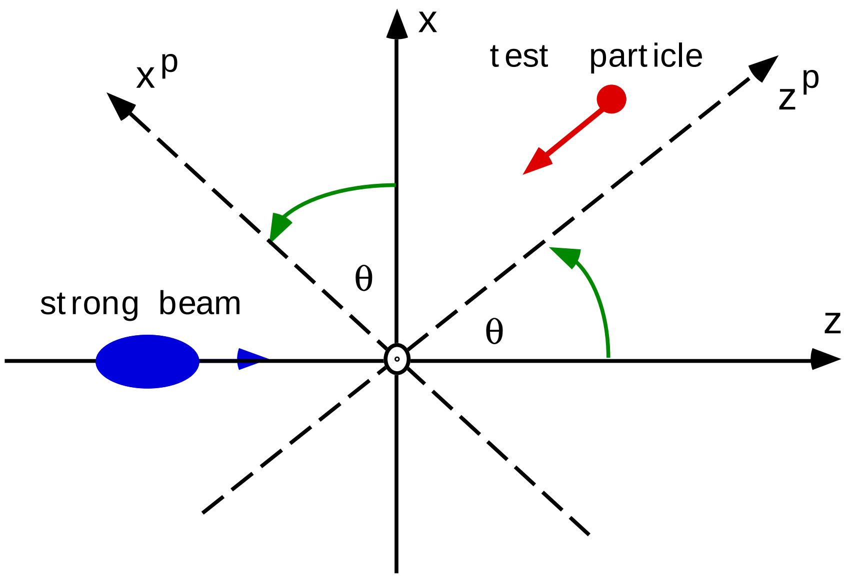

Figure 1: Scheme of beam-beam collision under a crossing angle.

The coordinate transformations from one system to the other are as follows:

$$
\begin{aligned}
&x = x^p \cos(\theta) + z^p \sin(\theta)\\
&z = z^p \cos(\theta) - x^p \sin(\theta)\\
&\textit{and}\\
&x^p = x\cos(\theta) - z\sin(\theta)\\
&z^p = z\cos(\theta) + x\sin(\theta)
\end{aligned}
$$

In the laboratory system components of the electromagnetic field, created by a 3D Gaussian bunch (strong bunch) moving with a velocity ~c is given by [1]:

$$
\begin{aligned}
E_x &= \frac{eN\gamma}{2\pi^{3/2}\varepsilon_0} x\int_0^\infty dw \frac{\exp\left\{-\frac{x^2}{\left(2\sigma_x^2+w\right)} - \frac{y^2}{\left(2\sigma_y^2+w\right)} - \frac{\gamma^2(z-ct)^2}{\left(2\gamma^2\sigma_z^2+w\right)}\right\}}{\left(2\sigma_x^2+w\right)^{3/2}\sqrt{\left(2\sigma_y^2+w\right)\left(2\gamma^2\sigma_z^2+w\right)}}\\
E_y &= \frac{eN\gamma}{2\pi^{3/2}\varepsilon_0} y\int_0^\infty dw \frac{\exp\left\{-\frac{x^2}{\left(2\sigma_x^2+w\right)} - \frac{y^2}{\left(2\sigma_y^2+w\right)} - \frac{\gamma^2(z-ct)^2}{\left(2\gamma^2\sigma_z^2+w\right)}\right\}}{\left(2\sigma_y^2+w\right)^{3/2}\sqrt{\left(2\sigma_x^2+w\right)\left(2\gamma^2\sigma_z^2+w\right)}} \qquad (1)\\
B_x &= -\frac{E_y}{c}\\
B_y &= \frac{E_x}{c}
\end{aligned}
$$

Equations of motion of a test particle belonging to the weak beam in this system are:

$$
\begin{aligned}
&x(t) = -c\sin(\theta)t &\quad& v_x = -c\sin(\theta)\\
&z(t) = -c\cos(\theta)t &\quad& v_z = -c\cos(\theta) \qquad (2)\\
&y(t) = y_o &\quad& v_y = 0
\end{aligned}
$$

The Lorentz force acting on the test particle due to the electromagnetic fields produced by the strong beam:

$$
\vec{F} = e\left(\vec{E} + \vec{v}\times\vec{B}\right) \quad \textit{with} \quad \vec{v}\times\vec{B} = -v_z B_y\vec{i} + v_z B_x\vec{j} + v_x B_y\vec{k} \qquad (3)
$$

has the following components:

$$
\begin{aligned}
F_x &= e\left(E_x - v_z B_y\right) = e\left(E_x + c\cos(\theta)B_y\right) = eE_x\left(1+\cos(\theta)\right)\\
F_y &= e\left(E_y + v_z B_x\right) = e\left(E_x - c\cos(\theta)B_x\right) = eE_y\left(1+\cos(\theta)\right) \qquad (4)\\
F_z &= ev_x B_y = -ec\sin(\theta)B_y = -eE_x\sin(\theta)
\end{aligned}
$$

Note that:

$$F_z = -\frac{F_x}{\left(1+\cos(\theta)\right)}\sin(\theta) \tag{5}$$

The force projected onto the axes of the test particle coordinate system has:

$$\begin{aligned} F_x^p &= F_x\cos(\theta) - F_z\sin(\theta) = F_x\cos(\theta) + F_x\frac{\sin^2(\theta)}{\left(1+\cos(\theta)\right)} = F_x \\ F_z^p &= F_x\sin(\theta) + F_z\cos(\theta) = F_x\sin(\theta) - F_x\frac{\sin(\theta)\cos(\theta)}{\left(1+\cos(\theta)\right)} = F_x / ctg\left(\frac{\theta}{2}\right) \\ F_y^p &= F_y \end{aligned} \tag{6}$$

According to the tune shift definitions:

$$\begin{aligned} \xi_{x^p} &= \Delta Q_{x^p} = \frac{1}{4\pi}\int_{-\infty}^{+\infty} dz^p \beta_x \left.\frac{\partial F_x^p\left(x\left(x^p,y^p,z^p\right),y\left(x^p,y^p,z^p\right),z\left(x^p,y^p,z^p\right)\right)}{\partial x^p}\right|_{x^p=y^p=0} \\ \xi_{y^p} &= \Delta Q_{y^p} = \frac{1}{4\pi}\int_{-\infty}^{+\infty} dz^p \beta_y \left.\frac{\partial F_y^p\left(x\left(x^p,y^p,z^p\right),y\left(x^p,y^p,z^p\right),z\left(x^p,y^p,z^p\right)\right)}{\partial y^p}\right|_{x^p=y^p=0} \end{aligned} \tag{7}$$

Combining eqs. (1), (4) and (6) one gets after differentiation:

$$\begin{aligned} \left.\frac{\partial F_x^p}{\partial x^p}\right|_{x^p=y^p=0} &= \frac{e^2 N}{2\pi^{3/2}\varepsilon_0 m_0 c^2}\int_0^{\infty} dw[2\gamma^2 z^2\sin^2(\theta)\left(1+\cos(\theta)\right)\left(2\sigma_x^2+w\right) \\ &+\cos(\theta)\left(\left(2\sigma_x^2+w\right)\left(2\gamma^2\sigma_z^2+w\right)+2z^2\sin^2(\theta)\right)\left(\gamma^2\left(2\sigma_x^2-2\sigma_z^2+w\right)-w\right)] \\ &\times\frac{Exp\left\{-\frac{\left(z^p\right)^2\gamma^2\left(1+\cos(\theta)\right)^2}{\left(2\gamma^2\sigma_z^2+w\right)}-\frac{\left(z^p\right)^2\sin^2(\theta)}{\left(2\sigma_x^2+w\right)}\right\}}{\left(2\sigma_x^2+w\right)^{5/2}\left(2\sigma_y^2+w\right)^{1/2}\left(2\gamma^2\sigma_z^2+w\right)^{3/2}} \end{aligned} \tag{8}$$

$$\begin{aligned} \left.\frac{\partial F_y^p}{\partial y^p}\right|_{x^p=y^p=0} &= \frac{e^2 N}{2\pi^{3/2}\varepsilon_0 m_0 c^2}\int_0^{\infty} dw\frac{\left(1+\cos(\theta)\right)}{\left(2\sigma_x^2+w\right)^{1/2}\left(2\sigma_y^2+w\right)^{3/2}\left(2\gamma^2\sigma_z^2+w\right)^{1/2}} \\ &\times Exp\left\{-\frac{\left(z^p\right)^2\gamma^2\left(1+\cos(\theta)\right)^2}{\left(2\gamma^2\sigma_z^2+w\right)}-\frac{\left(z^p\right)^2\sin^2(\theta)}{\left(2\sigma_x^2+w\right)}\right\} \end{aligned}$$

By integrating the above expressions along z^p we get formulae for the tune shifts:

$$\xi_{x^p} = \frac{r_e N \beta_x}{2\pi\gamma} \int_0^\infty \frac{dw}{\left(2\sigma_y^2 + w\right)^{1/2} \left(2\sigma_z^2 tg^2\left(\frac{\theta}{2}\right) + 2\sigma_x^2 + w + \frac{w}{\gamma^2 ctg^2\left(\frac{\theta}{2}\right)}\right)^{3/2}}$$

$$\xi_{y^p} = \frac{r_e N \beta_y}{2\pi\gamma} \int_0^\infty \frac{dw}{\left(2\sigma_y^2 + w\right)^{3/2} \left(2\sigma_z^2 tg^2\left(\frac{\theta}{2}\right) + 2\sigma_x^2 + w + \frac{w}{\gamma^2 ctg^2\left(\frac{\theta}{2}\right)}\right)^{1/2}} \qquad (9)$$

In case when $\gamma >> tg(\theta/2)$ we can neglect by the term $w/(\gamma^2 ctg^2(\theta))$ and the formulae are greatly simplified:

$$\xi_{x^p} = \frac{r_e N}{2\pi\gamma} \frac{\beta_x}{\sqrt{\left(\sigma_z^2 tg^2(\theta/2) + \sigma_x^2\right)}\left(\sqrt{\left(\sigma_z^2 tg^2(\theta/2) + \sigma_x^2\right)} + \sigma_y\right)}$$

$$\xi_{y^p} = \frac{r_e N}{2\pi\gamma} \frac{\beta_y}{\sigma_y\left(\sqrt{\left(\sigma_z^2 tg^2(\theta/2) + \sigma_x^2\right)} + \sigma_y\right)} \qquad (10)$$

Considering the last expressions and the luminosity formula obtained in [2]:

$$L = \frac{N^2}{4\pi\sigma_y \sqrt{\left(\sigma_z^2 tg^2(\theta/2) + \sigma_x^2\right)}} \qquad (11)$$

we can see that both eqs. (10) and (11) can be obtained from similar formulae for the head-on collision by simply substituting the horizontal beam size with:

$$\sigma_x \, {-}{-}{>} \sqrt{\left(\sigma_z^2 tg^2(\theta/2) + \sigma_x^2\right)} \qquad (12)$$

The kick that the test particle receives while passing through the strong beam is obtained by integrating eqs. (4). The final result is:

$$\left(x^p\right)' = \frac{2r_eN}{\gamma}(x^p - z^p tg(\theta/2)\int_0^\infty dw \frac{\exp\left\{-\frac{\left(x^p - z^p tg(\theta/2)\right)^2}{\left(2\left(\sigma_x^2+\sigma_z^2 tg^2(\theta/2)\right)+w\right)} - \frac{\left(y^p\right)^2}{\left(2\sigma_y^2+w\right)}\right\}}{\left(2\left(\sigma_x^2+\sigma_z^2 tg^2(\theta/2)\right)+w\right)^{3/2}\left(2\sigma_y^2+w\right)^{1/2}}$$

$$\left(y^p\right)' = \frac{2r_eN}{\gamma} y^p \int_0^\infty dw \frac{\exp\left\{-\frac{\left(x^p - z^p tg(\theta/2)\right)^2}{\left(2\left(\sigma_x^2+\sigma_z^2 tg^2(\theta/2)\right)+w\right)} - \frac{\left(y^p\right)^2}{\left(2\sigma_y^2+w\right)}\right\}}{\left(2\left(\sigma_x^2+\sigma_z^2 tg^2(\theta/2)\right)+w\right)^{1/2}\left(2\sigma_y^2+w\right)^{3/2}} \qquad (13)$$

$$\left(z^p\right)' = \left(x^p\right)' tg(\theta/2)$$

As we can see, a large crossing angle introduces strong coupling between the horizontal and longitudinal planes, provided that σz > σx (this is almost always true).

Conclusion

- We have obtained the formulae for the beam-beam tune shifts in collisions with a horizontal crossing angle. In particular, it has been shown that these formulae can be transformed from similar formulae for head-on collisions by substituting the horizontal beam size by $\left(\sigma_x^2+\sigma_z^2 tg^2(\theta/2)\right)^{1/2}$.
- Analyzing eqs. (10) and (11), we see that the luminosity and the tune shifts are reduced with the crossing angle. However, since
  $L \sim \left(\sigma_x^2+\sigma_z^2 tg^2(\theta/2)\right)^{-1/2}; \quad \xi_x \sim \left(\sigma_x^2+\sigma_z^2 tg^2(\theta/2)\right)^{-1}; \quad \xi_y \sim \left(\sigma_x^2+\sigma_z^2 tg^2(\theta/2)\right)^{-1/2}$
  the horizontal tune shift drops faster than the luminosity does.
- On the other hand, large crossing angles introduce strong coupling between horizontal and longitudinal planes of motion.

References

1. A. A. Zholents, Preprint 91-18, Novosibirsk, 1991.
2. O. Napoly, Particle Accelerators, 1993, Vol. 40, pp. 181-203.