CTFF3 TECHNICAL NOTE

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The theory of beam loading in RF deflectors for CTF3

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1. Introduction

The most demanding issues in the CTF3 RF deflector design are those related to the beam dynamics, including the beam loading effect on the fundamental deflecting mode [1]. A disk-loaded waveguide working in the EH_{11} hybrid mode [2-4] already optimised for beam deflection has been considered as a possible TW solution.

In the following we report the analysis of the wake generated by the interaction between the beam and the deflecting mode in the RF deflector (beam loading) discussing the limitations of the different possible wake approximations.

2. Deflecting electromagnetic field in a disk loaded waveguide

In a disk loaded waveguide (Figure 1) the e.m. field of the first deflecting mode (EH_{11}) in the central region, in the case of small pitch approximation (λ>>D), negligible iris thickness (t/D<<1) and phase velocity equal to c, is given by the equations [2,3]:

$$\underline{E}=\begin{Bmatrix} e_r = j\dfrac{E}{8}\left(k^2a^2+k^2r^2\right)\cos(\vartheta) \\ e_\vartheta = -j\dfrac{E}{8}\left(k^2a^2-k^2r^2\right)\sin(\vartheta) \\ e_z=\dfrac{E}{2}kr\cos(\vartheta)\end{Bmatrix} \mathrm{e}^{\mathrm{j}\omega^*\left(t-\frac{z}{c}+\phi\right)} = \underline{\mathrm{e}}(r,\vartheta)\cdot \mathrm{e}^{\mathrm{j}\omega^*\left(t-\frac{z}{c}+\phi\right)}$$

$$\underline{H}=\begin{Bmatrix} Z_0h_r = j\dfrac{E}{8}\left(k^2a^2-k^2r^2-4\right)\sin(\vartheta) \\ Z_0h_\vartheta = j\dfrac{E}{8}\left(k^2a^2+k^2r^2-4\right)\cos(\vartheta) \\ Z_0h_z=-\dfrac{E}{2}kr\sin(\vartheta)\end{Bmatrix} \mathrm{e}^{\mathrm{j}\omega^*\left(t-\frac{z}{c}+\phi\right)} = \underline{\mathrm{h}}(r,\vartheta)\cdot \mathrm{e}^{\mathrm{j}\omega^*\left(t-\frac{z}{c}+\phi\right)} \quad (1)$$

where ω^* is the working frequency, $k=\omega^*/c$ and $Z_0=\sqrt{(\mu_0/\varepsilon_0)}$.

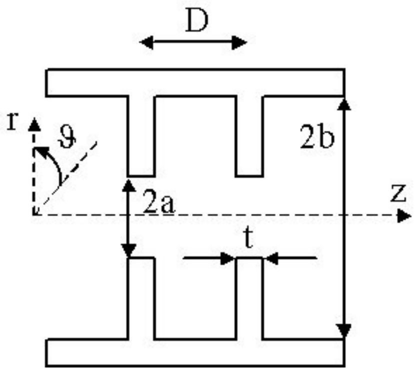

Figure 1: Sketch of a disk loaded waveguide

Using the expression of the Lorentz force acting on a particle of charge q that moves through the structure on the plane ϑ=0, with a velocity equal to c, we simply obtain:

$$F_T = \mathrm{Re}\left(E_r - Z_0 H_\vartheta\right) = -q\frac{E}{2}\sin(\phi) \qquad (2)$$

In the case of phase velocity different from c the equations of the field in the structure become more complicated [2] and, consequently, the expression of the transverse force itself.

To evaluate the beam loading in the structure one has to consider both the interaction between the travelling charges and the transverse electric field E_r (beam loading in phase) and between the travelling charges and the longitudinal electric field E_z (beam loading 90° out-of-phase)[1].

The first contribution is very similar to the beam loading of a linac accelerating section and the deflection spread along the train can be estimated obtaining a quite small value in the CTF3 case.

The second contribution is of more concern because in the combiner ring the bunch pattern is such that at a certain time the deflector will be crossed by bunch trains off axis and with a phase separation of 2π/5 generating a mutual perturbation mainly through the out-of-phase wake.

3. The general problem of mode excitation in a waveguide

In order to evaluate the beam loading out-of-phase in a disc loaded waveguide, let us consider the general problem of waveguide modes excitation by an electric current $\underline{J}$ that flows through the structure.

[1] Also in the case of phase velocity different from c the longitudinal component of the electric field is 90° out-of-phase with respect to the transverse one [2].

If we consider a set of independent modes, the general propagating field in the structure can be written, in frequency domain as ([2]):

$$\underline{E}^{\pm} = \sum_{n=0}^{N} c_n^{\pm} \underline{e}_n e^{-j(\pm\beta_n)z} = \sum_{n=0}^{N} c_n^{\pm}\left(\underline{e}_{tn} \pm e_{zn}\underline{z}_0\right) e^{-j(\pm\beta_n)z}$$
$$\underline{H}^{\pm} = \sum_{n=0}^{N} c_n^{\pm} \underline{h}_n e^{-j(\pm\beta_n)z} = \sum_{n=0}^{N} c_n^{\pm}\left(\pm\underline{h}_{tn} + h_{zn}\underline{z}_0\right) e^{-j(\pm\beta_n)z} \qquad (3)$$

where the signs "±" refer to the case of positive or negative phase velocities respectively, N is the number of excited modes and $c_n^{\pm}$ and β_n are the amplitude and the propagation constant of the n^{th} mode.

If we consider an electric density current $\underline{J}(\omega)$ at a certain section z_1-z_2 (Figure 2), it is possible to calculate the coefficients c_n^{+} by the simple formula [5]:

$$c_n^{+}(\omega) = \mp \frac{\int_V \left(\underline{e}_{tn} - e_{zn}\underline{z}_0\right) \cdot \underline{J}(\omega) e^{j\beta_n(\omega)z} dV}{2\int_{S_2} \underline{e}_{tn} \times \underline{h}_{tn} \cdot \underline{z}_0 dS_2} \qquad (4)$$

where the sign "-" refers to the case of forward waves while the sign "+" to the backward ones[3].

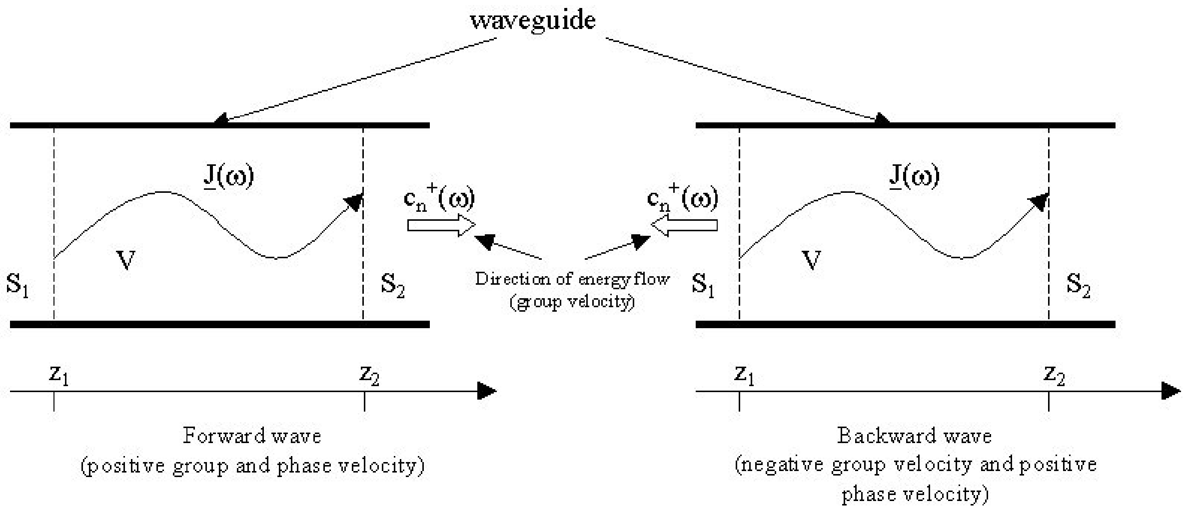

Figure 2: Sketch of a waveguide excited by an electric current

[2] With the subscripts t and z we indicate the transverse and the longitudinal component of the field respectively ($\underline{e}_n$, $\underline{e}_z$ and $\underline{e}_{tn}$ are functions of the transverse coordinates and of the frequency).

[3] For forward waves the group velocity ($v_g = d\omega/d\beta$) and the phase velocity ($v_{ph} = \omega/\beta$) have the same sign while for the backward ones they have opposite sign.

If we consider a particle of charge q that moves through the waveguide (Figure 3) at the speed of light, we can write the density current in the time domain in the simple form:

$$\underline{J}(t) = q\underline{s}_0(s)\delta\left(t - \frac{s}{c}\right)\delta(x')\delta(y') \tag{5}$$

where s is the distance along the particle trajectory, $\underline{s}_0$ is the unit vector tangent to the trajectory and (x',y') is the reference system on the plane normal to $\underline{s}_0$. In the frequency domain the equation (5) becomes:

$$\underline{J}(\omega) = q\underline{s}_0(s)e^{-j\omega\frac{s}{c}}\delta(x')\delta(y') \tag{6}$$

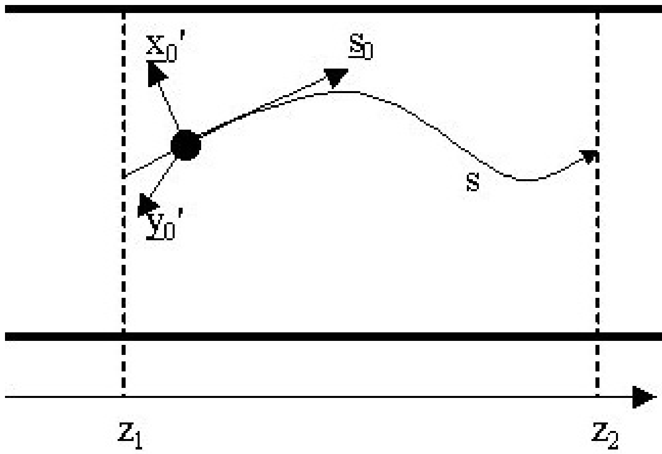

Figure 3: Sketch of a charge q moving in a waveguide

Since we are interested in the beam loading 90° out-of-phase in RF deflectors, we have to consider, in the scalar product $\underline{E}\cdot\underline{J}$ of eq. (4), only the longitudinal component of the electric field and density current. For a particle moving in the structure one has that $\underline{J} \cong J\cdot\underline{z}_0$ and we can write the coefficient $c_n^+(\omega,z_1,z_2)$ for a backward wave[4] in the form:

$$c_n^+(\omega,z_1,z_2) = -\frac{q}{4\Pi'_n(\omega)}\int_{z_1}^{z_2} e_{zn}\left(\omega,\underline{r}(z')\Big|_{\substack{particle\\trajectory}}\right)e^{-j\omega\frac{z'}{c}}e^{j\beta_n(\omega)z'}dz' \tag{7}$$

where $\underline{r}(z')|_{\text{particle trajectory}}$ is the transverse position of the particle along the structure (function of the longitudinal position z') and Π'_n is given by:

$$\Pi'_n(\omega) = \frac{1}{2}\int_{S_1}\underline{e}_{tn}\times\underline{h}_{tn}\cdot\underline{z}_0 dS_1 \tag{8}$$

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[4] We remark that the RF deflectors for CTF3 are backward structures.

It is easy to verify, from the analytical formulae [2], that[5]:

$$\Pi_n(\omega) = \frac{1}{2}\int_{S_1} \underline{e}_{tn} \times \underline{h}^*_{tn} \cdot \underline{z}_0 dS_1 = -\Pi'_n(\omega) \tag{9}$$

where Π_n is the power flow along the structure[6].

If the deflector length is L and the particle enters at z=0 we obtain, for the coefficient c_n^+ in z, the following expression:

$$c_n^+(\omega,z) = \frac{q}{4\Pi_n(\omega)}\int_z^L e_{zn}\left(\omega, \underline{r}(z')\Big|_{\substack{particle\\trajectory}}\right) e^{-j\omega\frac{z'}{c}} e^{j\beta_n(\omega)z'} dz' \tag{10}$$

The e.m. field of the n^{th} excited mode can be calculated in time domain by a Fourier integral:

$$\begin{aligned}\underline{E}_n^+(t,z,r,\vartheta) &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} c_n^+(\omega,z)\underline{e}_n(\omega,r,\vartheta)e^{-j\beta_n(\omega)z}e^{j\omega t}d\omega = \frac{1}{\pi}\mathrm{Re}\left[\int_0^{+\infty} c_n^+ \cdot \underline{e}_n \cdot e^{-j\beta_n z}e^{j\omega t}d\omega\right]\\ \underline{H}_n^+(t,z,r,\vartheta) &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} c_n^+(\omega,z)\underline{h}_n(\omega,r,\vartheta)e^{-j\beta_n(\omega)z}e^{j\omega t}d\omega = \frac{1}{\pi}\mathrm{Re}\left[\int_0^{+\infty} c_n^+ \cdot \underline{h}_n \cdot e^{-j\beta_n z}e^{j\omega t}d\omega\right]\end{aligned} \tag{11}$$

The induced transverse force in the radial direction (corresponding to the n^{th} excited mode) on a trailing particle that enter in the deflector after a time t^* from the leading one and that moves throw the structure with a velocity equal to c, can be calculated by the following Lorentz expression[7]:

$$F_{Tn}\left(t^*,z,r,\vartheta\right) = q\left(E_{nr}^+\left(\frac{z}{c}+t^*,z,r,\vartheta\right) + Z_0 H_{n\vartheta}^+\left(\frac{z}{c}+t^*,z,r,\vartheta\right)\right) \tag{12}$$

where z, r and ϑ are the cylindrical coordinates of the trailing particle itself.

4. Approximated formulae for the coefficients c_n^+ and for the wake field

Let us consider the coefficient c_1^+ corresponding to the deflecting mode in the RF deflector[8]. In order to calculate this coefficient and the corresponding e.m. field for all ω and z one has to use the equations (10, 11) and (12) respectively.

Unfortunately, this expressions are very difficult to manage numerically because all the quantities are frequency dependent and one has to compute a double integration (the first in the

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[5] This is valid, in general, for all propagating field in which the longitudinal dependence (z) and the transverse one (r,ϑ) are separated in the form $\underline{e}(r,\vartheta,z)=\underline{e}_t(r,\vartheta)\cdot\underline{e}_z(z)$ [5].

[6] In the case of backward wave Π_n is negative.

[7] We always suppose that the transverse component of the particle velocity is negligible compared with the longitudinal one.

[8] This mode is called EH_{11} mode or HEM_{11}.

z' variable and the second in ω) in order to calculate the wake field generated by a single passage of a particle.

Since this way is practically not usable to evaluate the beam loading of a multiparticle passage, we discuss some approximated formulae for the wake generated by a particle.

4.1 Dispersion curve linearization in a limited range of frequency

A typical dispersion curve for an RF deflector [2] is plotted in Figure 4. The frequency f^* is the frequency at which the phase velocity (v_{ph}) is equal to c and it corresponds to the working frequency.

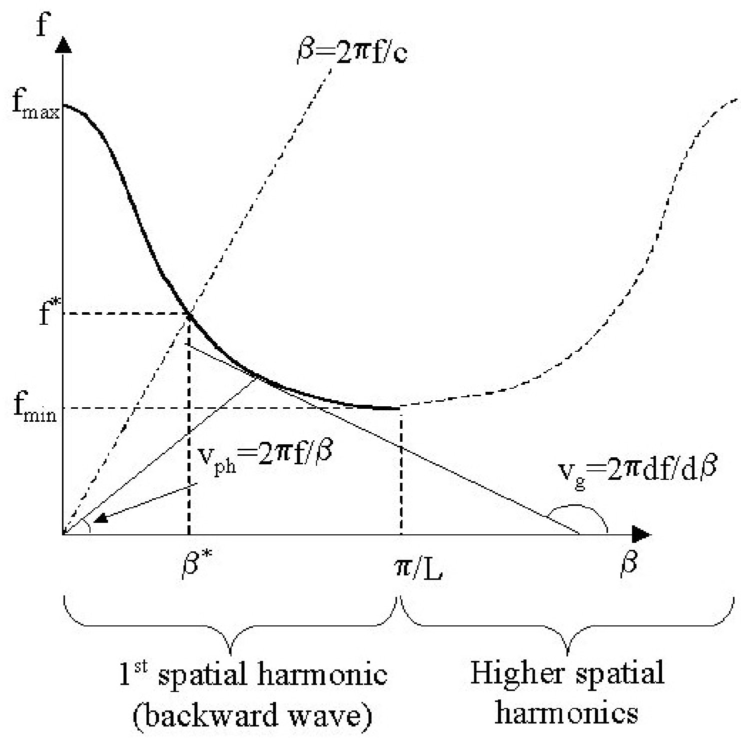

Figure 4: sketch of a typical dispersion curve for an RF deflector

If we consider the expressions (10-12), it is easy to show that the major contribution, in time domain, to the deflecting force acting on a particle 90° out-of-phase from the leading one[9], comes from a small range of frequencies near f^* ([10]).

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[9] It means that $t^*=T/4+hT$ where $T=1/f^*$ is the period of the wave in the deflector.

[10] In fact for a particle 90° out-of-phase from the leading one the real part of the coefficient $c_1^+(\omega)$ (for a fixed z) has a local maximum for $f=f^*$ (the exponential term oscillates for $f\neq f^*$) and also the deflecting force has a maximum when the particle is synchronous with the wave [2].

It is possible to linearize the expressions (10-12) near the point (β^*,f^*) obtaining the following expressions for the coefficient $c_1^+(\omega,z)$ ([11]):

$$c_1^+(\omega,z) \cong \frac{q}{4\Pi_1(\omega^*)} \begin{cases} \int_z^L e_{z1}\left(\omega^*, \underline{r}(z')\big|_{\substack{\text{particle}\\\text{trajectory}}}\right) e^{j\left(\left.\frac{d\beta}{d\omega}\right|_{\omega=\omega^*}\right)(\omega-\omega^*)z'} dz' & \omega \in \left[\omega^* - \frac{\Delta\omega}{2}, \omega^* + \frac{\Delta\omega}{2}\right] \\ 0 & \text{elsewhere} \end{cases} \quad (13)$$

where the field $e_{z1}(\omega^*,r)$ is equal to e_z of eq. (1) and $[\omega^*-\Delta\omega/2, \omega^*+\Delta\omega/2]$ is a suitable interval of the center frequency f^* ([12]).

The expression for the electric field in the time domain becomes:

$$\underline{E}_1^+(t,z) = \frac{q}{2\pi\ \ \Pi_1(\omega^*)} \mathrm{Re}\left[\underline{e}_1(\omega^*) e^{j\omega^*\left(t-\frac{z}{c}\right)} \int_z^L e_{z1}\left(\omega^*, \underline{r}(z')\big|_{\substack{\text{particle}\\\text{trajectory}}}\right) \frac{\Delta\omega}{2} \mathrm{sinc}\left[\left(t - \frac{z-z'}{v_g}\right)\frac{\Delta\omega}{2}\right] dz'\right] \quad (14)$$

where the field $\underline{e}_1(\omega^*)$ is equal to $\underline{e}$ of eq. (1) and sinc(x)=sin(x)/x.

The Lorentz force acting on a trailing particle that passes through the deflector after a time t^* from the leading one and that moves on the plane ϑ=0 is given by the formula[13]:

$$\begin{aligned} F_T(t^*,z) &= q(E_r - Z_0 H_\vartheta) = \\ &= -\frac{1}{\pi}\frac{q^2}{4\Pi_1\big|_{\omega=\omega^*}} E \sin(\omega^* t^*) \int_z^L e_{z1}\left(\omega^*, \underline{r}(z')\big|_{\substack{\text{leading}\\\text{particle}\\\text{trajectory}}}\right) \frac{\Delta\omega}{2} \mathrm{sinc}\left[\left(t^* + \frac{z}{c} - \frac{z-z'}{v_g}\right)\frac{\Delta\omega}{2}\right] dz' \cong \\ &\underset{v_g<<c}{\cong} -\frac{1}{\pi}\frac{q^2}{4\Pi_1\big|_{\omega=\omega^*}} E \sin(\omega^* t^*) \int_z^L e_{z1}\left(\omega^*, \underline{r}(z')\big|_{\substack{\text{leading}\\\text{particle}\\\text{trajectory}}}\right) \frac{\Delta\omega}{2} \mathrm{sinc}\left[\left(t^* - \frac{z-z'}{v_g}\right)\frac{\Delta\omega}{2}\right] dz' \end{aligned} \quad (15)$$

If the trajectory of the leading charge is a simple parabola on the plane ϑ=0 ([14]):

$$\underline{r}(z)\big|_{\substack{\text{leading}\\\text{particle}\\\text{trajectory}}} = r_{in} + r'_{in} z + \frac{1}{2}\frac{\Delta r'}{L} z^2 \quad (16)$$

where r_{in} and r'_{in} are the initial conditions for the leading particle, L is the deflector length and $\Delta r'$ is the angular deflection of the particle at the end of the deflector, the (16) becomes:

$$F_T(t^*,z) \cong -\frac{1}{2\pi}\frac{q^2}{4\Pi_1\big|_{\omega=\omega^*}} kE^2 \sin(\omega^* t^*) \int_z^L \left(r_{in} + r'_{in} z' + \frac{1}{2}\frac{\Delta r'}{L} z'^2\right) \frac{\Delta\omega}{2} \mathrm{sinc}\left[\left(t^* - \frac{z-z'}{v_g}\right)\frac{\Delta\omega}{2}\right] dz' \quad (17)$$

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[11] See Appendix A1.

[12] "Suitable interval" in order to have a good approximations of the exact expression for the field (11) in terms, for example of the deflecting force seen by a trailing particle.

[13] For the leading particle $t=t^*+z/c$.

[14] This is the common case for an RF deflector.

4.2 Dispersion curve linearization over an unlimited range of frequency

In this case we evaluate the wake when $\Delta\omega\rightarrow\infty$ in the equations(14-15,17). If we remember that:

$$\lim_{\Delta\omega\longrightarrow\infty}\frac{\Delta\omega}{2}\mathrm{sinc}\left[\left(t\text{-}\frac{z\text{-}z'}{v_g}\right)\frac{\Delta\omega}{2}\right]=\pi\left|v_g\right|\delta\left(z'-\left(z-tv_g\right)\right)$$

we obtain:

$$\underline{E}_1^+(t,z)=\frac{q\left|v_g\right|}{2\Pi_1\big|_{\omega=\omega^*}}\mathrm{Re}\left[\underline{e}_1\left(\omega^*\right)e^{j\omega^*\left(t-\frac{z}{c}\right)}\right]e_{z1}\left(\omega^*,\underline{r}\left(z-tv_g\right)\Big|_{\substack{leading\\ particle\\ trajectoy\\ translated\\ in\ \mathrm{z\text{-}tv_g}}}\right) \tag{14)'$$

$$F_T\left(t^*,z\right)=-\frac{q^2\left|v_g\right|}{4\Pi_1\big|_{\omega=\omega^*}}E\sin\left(\omega^*t^*\right)e_{z1}\left(\omega^*,\underline{r}\left(z-\left(t^*+\frac{z}{c}\right)v_g\right)\Big|_{\substack{leading\\ particle\\ trajectoy\\ translated\\ in\ \mathrm{z\text{-}tv_g}}}\right)\cong$$

$$\underset{\frac{v_g}{c}<<1}{\cong}-\frac{q^2\left|v_g\right|}{4\Pi_1\big|_{\omega=\omega^*}}E\sin\left(\omega^*t^*\right)e_{z1}\left(\omega^*,\underline{r}\left(z-t^*v_g\right)\Big|_{\substack{leading\\ particle\\ trajectoy\\ translated\\ in\ \mathrm{z\text{-}tv_g}}}\right) \tag{15)'$$

$$F_T\left(t^*,z\right)\cong-\frac{q^2\left|v_g\right|}{8\Pi_1\big|_{\omega=\omega^*}}kE^2\sin\left(\omega^*t^*\right)\left(r_{in}+r_{in}^{|}\left(z-t^*v_g\right)+\frac{1}{2}\frac{\Delta r^{|}}{L}\left(z-t^*v_g\right)^2\right)U\left(z-L+t^*v_g\right) \tag{17)'$$

where:

$$U(z)=\begin{cases}1 & \mathrm{z}\leq 0\\ 0 & \mathrm{z}>0\end{cases}$$

If we define [3] the R/Q of the structure as:

$$\frac{R}{Q}=\frac{\left(\frac{E}{2}\right)^2\frac{v_g}{c}}{\Pi_1\left(\omega^*\right)k} \tag{18}$$

we obtain for the field the expression:

$$\begin{aligned}\underline{E}_1^+&=-\frac{1}{2}q\omega^*\frac{R}{Q}kr\left(z-tv_g\right)\mathrm{Re}\left[\frac{\underline{e}_1\left(\omega^*\right)}{E/2}e^{j\omega^*\left(t-\frac{z}{c}\right)}\right]=\\&=-\frac{1}{2}q\omega^*\frac{R}{Q}\bigg|_{eff}\left(r\left(z-tv_g\right)\right)\mathrm{Re}\left[\frac{\underline{e}_1\left(\omega^*\right)}{E/2}e^{j\omega^*\left(t-\frac{z}{c}\right)}\right]\end{aligned} \tag{19}$$

where:

$$r\left(z-tv_g\right)=\left(r_{in}+r_{in}^{|}\left(z-tv_g\right)+\frac{1}{2}\frac{\Delta r^{|}}{L}\left(z-tv_g\right)^2\right)U\left(z-L+tv_g\right)$$

$$\left.\frac{R}{Q}\right|_{eff}(r)=\frac{R}{Q}kr$$

and for the transverse force the expression:

$$F_T\left(t^*,z\right)\cong\frac{1}{2}q^2\omega^*\frac{R}{Q}kr\left(z-t^*v_g\right)\sin\left(\omega^*t^*\right)=\frac{1}{2}q^2\omega^*\left.\frac{R}{Q}\right|_{eff}\left(r\left(z-t^*v_g\right)\right)\sin\left(\omega^*t^*\right) \qquad (20)$$

These expressions for the wake field and force correspond to what intuitively we could expect for the field generated by a passage of a particle in the RF deflector[15]: an envelope of the field (or force) that follows the profile of the leading particle trajectory ([16]) and that rigidly translates along the structure with a negative group velocity equal to v_g and with a positive phase velocity equal to ω^*/c.

5. Multiparticle passage: the steady state solution

Let us consider, now, the case of an infinite train of bunches spaced in time by T. In this case we have:

$$\underline{J}(t)=q\sum_{i=-\infty}^{\infty}\underline{s}_{0i}(s_i)\delta\left(t-\frac{s_i}{c}+iT\right)\delta\left(x'_i\right)\delta\left(y'_i\right) \qquad (21)$$

If all the bunches have the same trajectory through the RF deflector we simply obtain:

$$\underline{J}(t)=q\underline{s}_0(s)\sum_{i=-\infty}^{\infty}\delta\left(t-\frac{s}{c}+iT\right)\delta(x')\delta(y') \qquad (22)$$

that in the frequency domain becomes:

$$\underline{J}(\omega)=q\underline{s}_0(s)\delta(x')\delta(y')\omega^*e^{-j\omega\frac{s}{c}}\sum_{n=-\infty}^{\infty}\delta\left(\omega-n\omega^*\right) \qquad (23)$$

where $\omega^*=2\pi/T$.

In order to evaluate the coefficient $c_1^+(\omega,z)$ we remember that the CTF3 RF deflectors work at ~3 GHz and that the dispersion curve for the mode EH_{11} for these structures has a pass-band of the order of few hundred of MHz. The trains of bunches that we consider have, typically, a spectrum with a distance (1/T) between the δ in (23) bigger or equal to ~3 GHz [1].

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15 See Appendix A2.

16 The field E_z is proportional to the displacement r of the particle from the axis of the structure.

Substituting the term $e^{-j\omega z/c}$ in eq. (10) with the spectrum (23), we simply obtain for parabolic trajectories[17]:

$$
\begin{aligned}
c(\omega,z) &= \frac{q\omega^* Ek}{8\Pi_1\big|_{\omega=\omega^*}}\delta(\omega-\omega^*)\int_z^L\left(r_{in}+r'_{in}z'+\frac{1}{2}\frac{\Delta r'}{L}z'^2\right)dz'= \\
&= \frac{q\omega^* Ek}{8\Pi_1\big|_{\omega=\omega^*}}\delta(\omega-\omega^*)\left[r_{in}z'+\frac{1}{2}r'_{in}z'^2+\frac{1}{6}\frac{\Delta r'}{L}z'^3\right]_{z'=z}^{z'=L}
\end{aligned}
\tag{24}
$$

where $k=\omega^*/c$.

The electric field is, in this case, given by:

$$
\underline{E}_1^+(t,z)=\frac{1}{8\pi}\frac{q\omega^* Ek}{\Pi_1\big|_{\omega=\omega^*}}c(z)\mathrm{Re}\left[\underline{e}_1(\omega^*)e^{j\omega^*\left(t-\frac{z}{c}\right)}\right]
\tag{25}
$$

where:

$$
c(z)=\left[r_{in}z'+\frac{1}{2}r'_{in}z'^2+\frac{1}{6}\frac{\Delta r'}{L}z'^3\right]_{z'=z}^{z'=L}=r_{in}(L-z)+\frac{1}{2}r'_{in}\left(L^2-z^2\right)+\frac{1}{6}\frac{\Delta r'}{L}\left(L^3-z^3\right)
\tag{26}
$$

and the transverse force seen by a trailing particle of charge q that moves through the deflector after a time t^*+hT is given by:

$$
F_T(t^*,z)=-\frac{q^2kE^2}{8T\Pi_1\big|_{\omega=\omega^*}}\sin(\omega^* t^*)c(z)
\tag{27}
$$

The easiest way to calculate the wake field in the multibunch regime in the case of linearized dispersion curve over an infinite range of frequencies is to make a numerical calculation with rigid profile fields that propagate in the structure as pointed out previously.

In the next session we will compare the different results in some relevant cases.

6. Numerical computations and comparison between the different approximations

6.1 Single particle passage

Let us consider the single passage of a particle in a RF deflector with the following parameters[18]:

L = 33 cm
A ≅ 2.2 cm
B ≅ 5.7 cm

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17 In practice the spectrum (23) samples the eq. (10) at the working frequency ω^*.

18 This parameters are scaled from that of [3,4] in order to have $f^*\cong 3$GHz. In the case of small pitch approximation (λ>>D) and negligible iris thickness (t/D<<1) the values of D and t do not affect the calculation of the dispersion curve [2,3].

The dispersion curve for this structure obtained by the analytical calculations described in [2], is plotted in Figure 5[19].

By the equations (11-12), (17), (17)' is it possible to evaluate, in the correct case and in the approximated cases respectively, the transverse field excited by a leading charge and probed by a trailing particle injected with a delay t^*.

Considering the trajectory 1 of Figure 6 for the leading charge (r_{in}=0.5 mm, r'_{in}=0 and $\Delta r'$=0), we obtain for the transverse force[20] probed by a particle that enter in the structure after a time $t_1^*=T/4$ and $t_2^*=T/4+25T$ ([21]), the results plotted in Figure 7. The correct result obtained by equation (12) (solid line) is compared with those obtained in the linear approximation of the dispersion curve in the pass-band interval of the EH_{11} mode $[\omega_1,\omega_2]$ (dashed line) and with those obtained by the linear approximation of the dispersion curve in an unlimited range of frequencies (dash-dotted line). In Figures 8, 9 the same quantities for the trajectory 2 and 3 of Figure 6 are plotted.

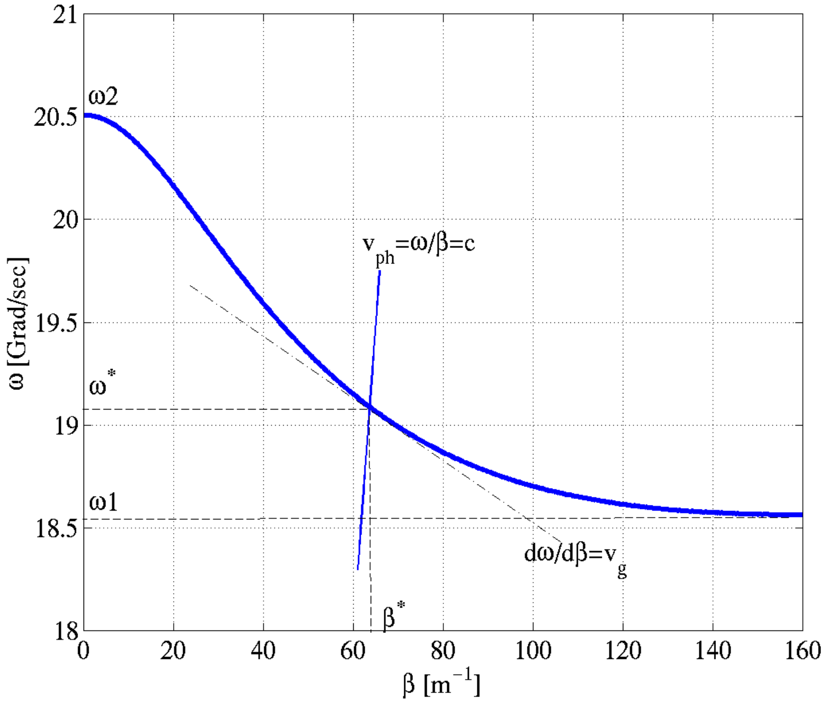

Figure 5: dispersion curve for the considered RF deflector

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19 The analytical calculation gives for this structure v_g=0,058c.

20 The force is calculated on the axis of the structure. More precisely, as shown in the eq. (17) and (17)', in the approximated cases the transverse force does not depend on the displacement of the trailing particle. Nevertheless, considering the exact field distribution in the correct calculation (11-12), there is a force dependence due to the transverse position of the trailing particle.

21 For this structure the analytical calculation gives a filling time $\tau_f=L/v_g\cong 50T$.

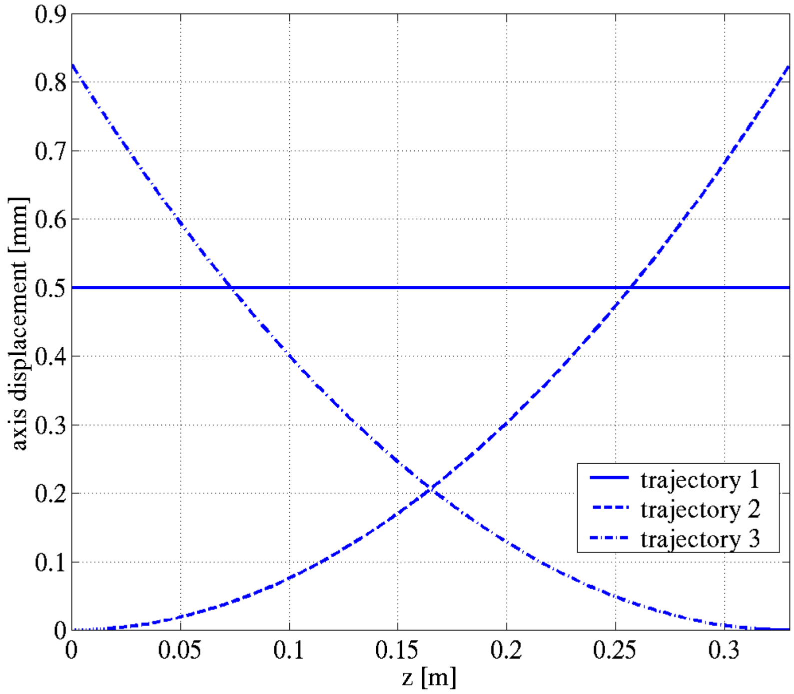

Figure 6: trajectories of the leading particle:
trajectory 1 (r_{in}=0.5 mm, r'_{in}=0 and $\Delta r'$=0)
trajectory 2 (r_{in}=0 mm, r'_{in}=5 mrad and $\Delta r'$=5 mrad)
trajectory 3 (r_{in}=0.825 mm, r'_{in}=-5 mrad and $\Delta r'$=5 mrad)

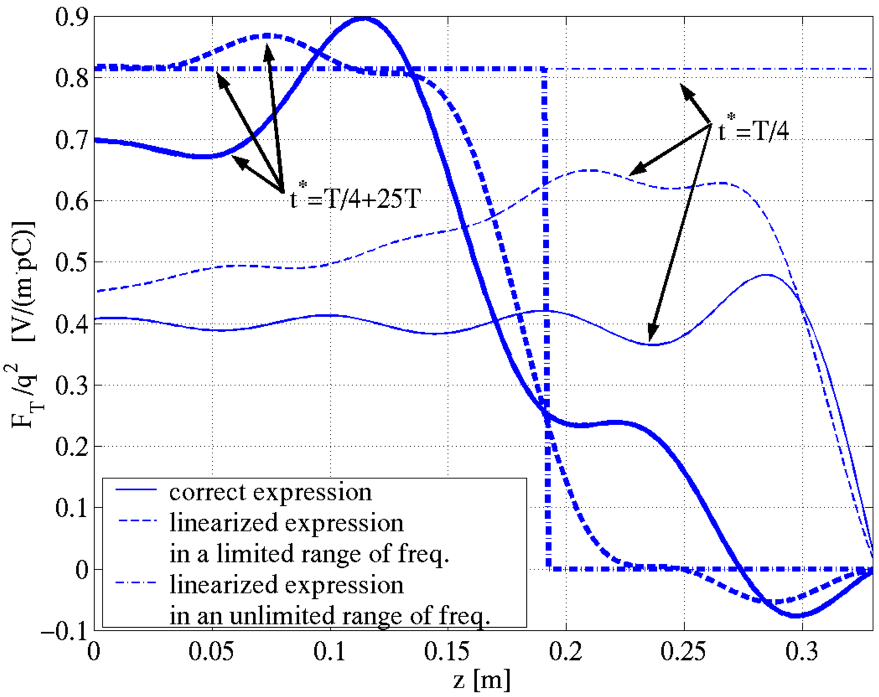

Figure 7: transverse force probed by a trailing particle
(trajectory 1 of the leading particle)

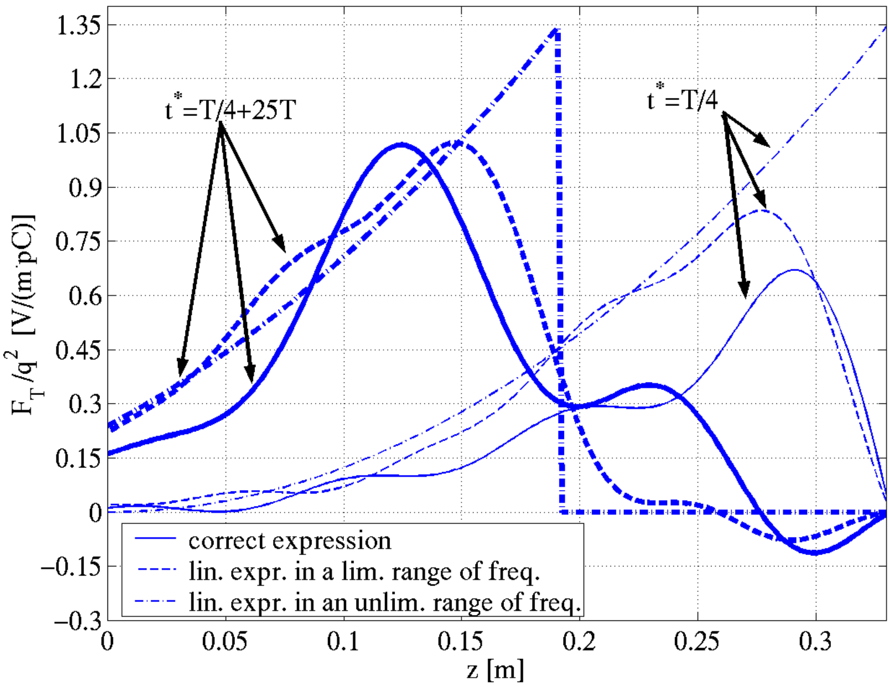

Figure 8: transverse force probed by a trailing particle (trajectory 2 of the leading particle)

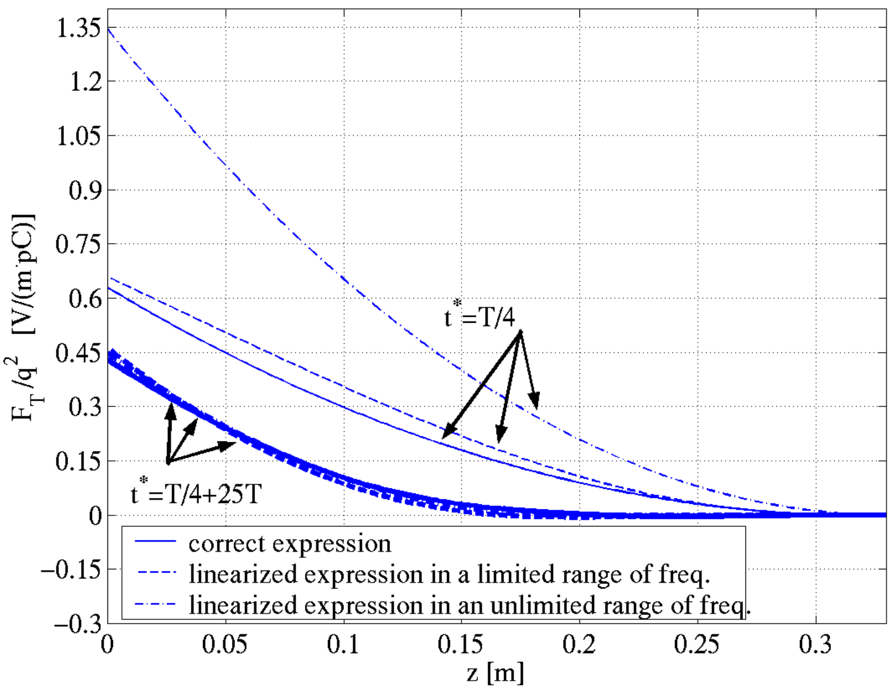

Figure 9: transverse force probed by a trailing particle (trajectory 3 of the leading particle)

If we consider now the transverse wake probed by a trailing particle that enters in the structure after a time t_n^*=T/4+nT defined as:

$$w_{\perp n} = \frac{1}{q^2}\int_0^L F_T(t_n^*, z)dz \tag{28}$$

we obtain the results plotted in Figures 9, 10 and 11 for the tree different trajectories respectively.

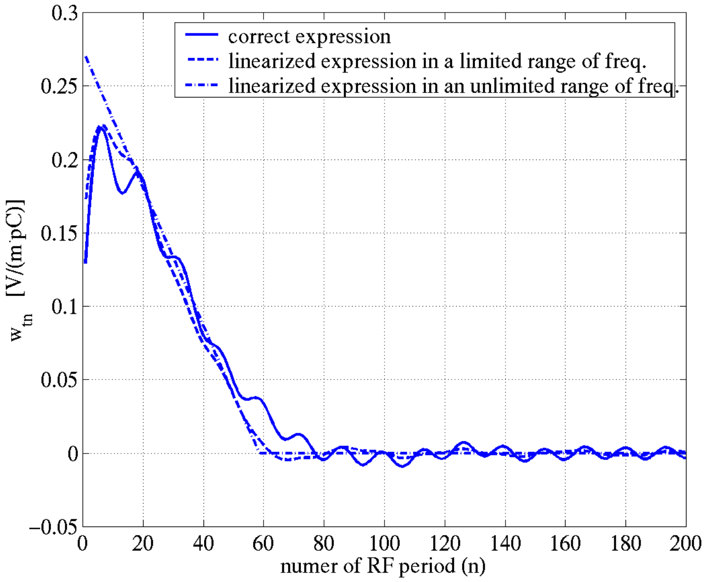

Figure 9: transverse wake probed by a trailing particle that enters in the structure after a time t_n^*=T/4+nT (trajectory 1 of the leading particle)

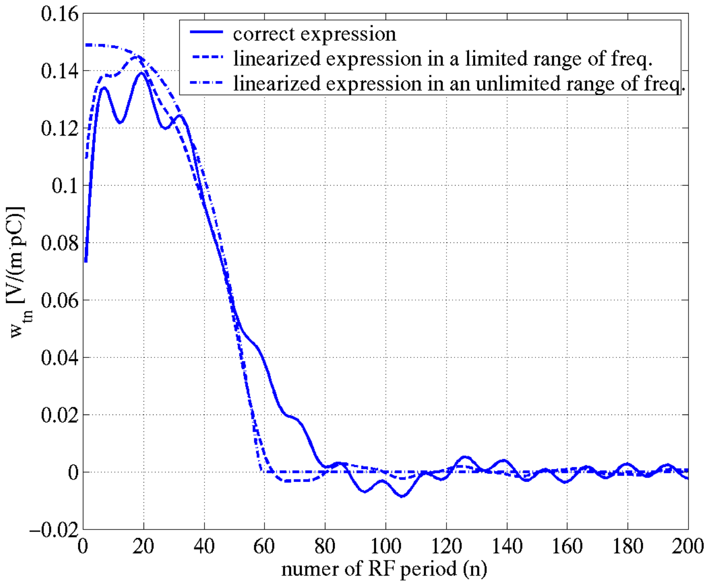

Figure 10: transverse wake probed by a trailing particle that enters in the structure after a time t_n^*=T/4+nT (trajectory 2 of the leading particle)

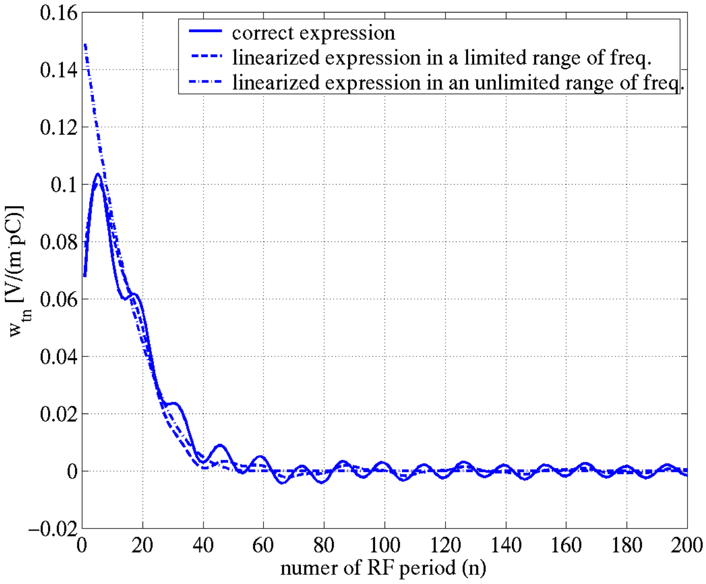

Figure 11: transverse wake probed by a trailing particle that enters in the structure after a time t_n^*=T/4+nT (trajectory 2 of the leading particle)

6.2 Multiparticle passage: the steady state solution

The results obtained in the case of multiparticle passage with t^* of eq. (27) equal to T/4 are plotted in figures 12, 13 and 1. Also in this case the correct solution (dashed line) is compared with the solution obtained by the linearized approximation over an unlimited range of frequencies (solid line) ([22]).

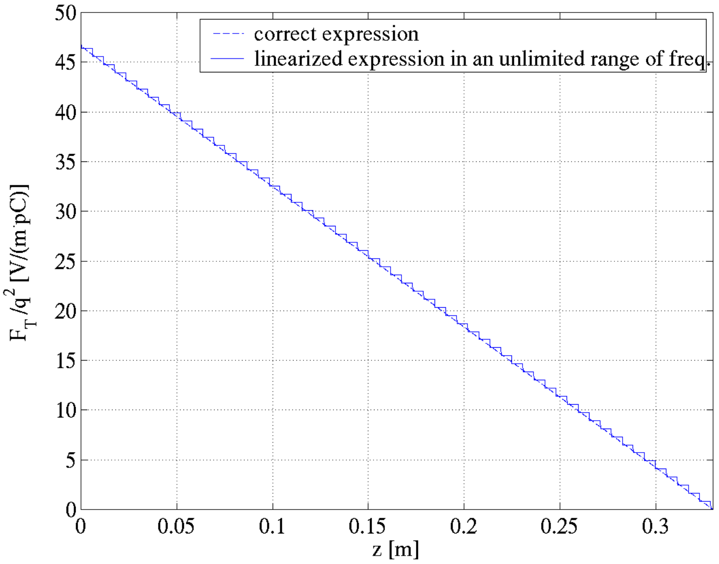

Figure 12: transverse force seen by a 90° out of phase particle in the case of multibunch regime (trajectory 1 of the train of bunches)

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[22] In the case of multibunch regime there are not differencies between the correct solution and the case of linearaized dispersion curve in a limited range of frequencies.

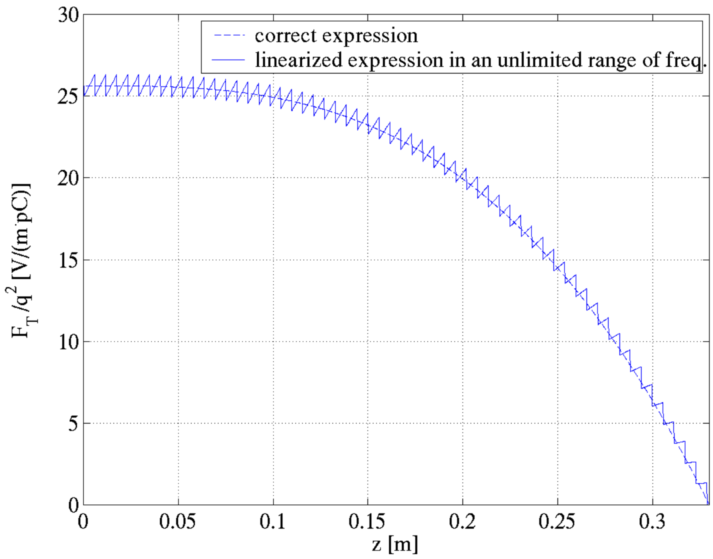

Figure 13: transverse force seen by a 90° out of phase particle in the case of multibunch regime (trajectory 2 of the train of bunches)

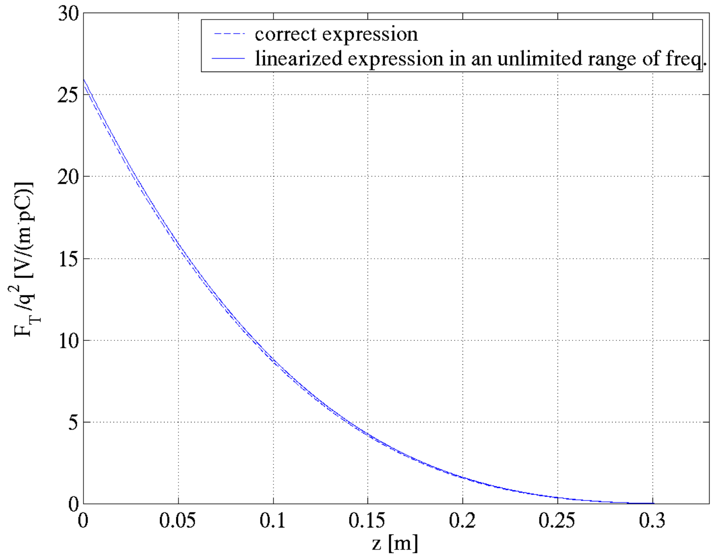

Figure 14: transverse force seen by a 90° out of phase particle in the case of multibunch regime (trajectory 3 of the train of bunches)

Conclusions

In this paper we have compared different ways of calculating the single and multi-passage wake of the fundamental EH_{11} deflecting mode of the CTF3 RF deflectors. In particular we have considered the possibility to approximate the wake with a local excitation proportional to the leading charge displacement and a rigid profile for the excited field that translates backward with the group velocity. The numerical results have shown that the amplitude of the wake obtained in this simple way is not too different with respect to the more complicated approaches. In particular, in the case of multibunch regime (steady state solution) the approximated model for the wake produces almost the same results that the correct one. The explanation is that the multibunch regime solution is the response to an almost monochromatic excitation, and therefore the details of the dispersion curve out of resonance are not relevant in this case.

The simplest model of the single passage wake can therefore be assumed to study the multibunch beam loading in the CTF3 combiner ring [1].

Appendix

A.1 Derivation of the equation (13) from eq. (10)

If we consider the expression (10) for the coefficient c_1^+ we can develop to the first order in ω the esponential term as in the following:

$$\begin{aligned} -\omega\frac{z'}{c} &\cong -\omega^*\frac{z'}{c}+\left(\omega^*-\omega\right)\frac{z'}{c} \\ \beta(\omega)z' &\cong \beta\left(\omega^*\right)z'+\left.\frac{d\beta}{d\omega}\right|_{\omega=\omega^*}\left(\omega-\omega^*\right)z' \end{aligned} \tag{A1.1}$$

Substituting in the equation (10) we obtain ([23]):

$$c_1^+(\omega,z)=\frac{q}{4\Pi_1\left(\omega^*\right)}\begin{cases} \displaystyle\int_z^L e_{z1}\left(\omega^*\right)\Big|_{\substack{particle\\trajectory}} e^{j\left(\left.\frac{d\beta}{d\omega}\right|_{\omega=\omega^*}-\frac{1}{c}\right)\left(\omega-\omega^*\right)z'}dz' \cong & \omega\in\left[\omega^*-\dfrac{\Delta\omega}{2},\omega^*+\dfrac{\Delta\omega}{2}\right] \\ \displaystyle\cong\int_z^L e_{z1}\left(\omega^*\right)\Big|_{\substack{particle\\trajectory}} e^{j\left(\left.\frac{d\beta}{d\omega}\right|_{\omega=\omega^*}\right)\left(\omega-\omega^*\right)z'}dz' & \\ 0 & \text{elsewhere} \end{cases} \tag{A1.2}$$

The second approximation in the interval $[\omega^*-\Delta\omega/2,\ \omega^*+\Delta\omega/2]$ comes out from the fact that the group velocity v_g for this kind of structures is few percent of the velocity of light.

The field is simply given by the expression (11) where we can also develop to the first order the esponential term $-j\beta(\omega)z$ obtaining:

$$\begin{aligned} &\underline{E}_n^+(t,z)\cong\frac{q}{4\pi\Pi_1\left(\omega^*\right)} \\ &\mathrm{Re}\left[\underline{e}_1^+\left(\omega^*\right)\int_z^L e_{z1}\left(\omega^*,\underline{r}(z')\Big|_{\substack{particle\\trajectory}}\right)\int_0^\infty e^{j\left.\frac{d\beta}{d\omega}\right|_{\omega=\omega^*}\left(\omega-\omega^*\right)z'}e^{-j\beta\left(\omega^*\right)z-j\left.\frac{d\beta}{d\omega}\right|_{\omega=\omega^*}\left(\omega-\omega^*\right)z}e^{j\omega t}d\omega\, dz'\right]= \\ &=\frac{q}{4\pi\Pi_1\left(\omega^*\right)} \\ &\mathrm{Re}\left[\underline{e}_1^+\left(\omega^*\right)\int_z^L e_{z1}\left(\omega^*,\underline{r}(z')\Big|_{\substack{particle\\trajectory}}\right)e^{-j\left.\frac{d\beta}{d\omega}\right|_{\omega=\omega^*}\omega^*(z'-z)-j\beta\left(\omega^*\right)z}\int_0^\infty e^{j\omega\left(t-\left.\frac{d\beta}{d\omega}\right|_{\omega=\omega^*}(z-z')\right)}d\omega\, dz'\right] \end{aligned} \tag{A1.3}$$

If we consider the integral in dω limited between $\omega^*-\Delta\omega/2$ and $\omega^*+\Delta\omega/2$ it is easy to show that:

$$\int_{\omega^*-\frac{\Delta\omega}{2}}^{\omega^*+\frac{\Delta\omega}{2}} e^{j\omega\left(t-\left.\frac{d\beta}{d\omega}\right|_{\omega=\omega^*}(z-z')\right)}d\omega=e^{j\omega^*\left(t-\left.\frac{d\beta}{d\omega}\right|_{\omega=\omega^*}(z-z')\right)}\Delta\omega\,\mathrm{sinc}\left[\left(t-\frac{z-z'}{v_g}\right)\frac{\Delta\omega}{2}\right] \tag{A1.4}$$

where sinc(x)=sin(x)/x.

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[23] $\omega^*/c=\beta(\omega^*)$.

A.2 Derivation of the eq. (19) by intuitive considerations

Let us consider only the resonant field configuration of eq. (1) with a local excitation proportional to the leading charge displacement. The energy per unit length stored in the section corresponding to the abscissa z after the charge passage is given by:

$$U = \frac{1}{2}V(z)q = \frac{1}{2}\frac{E}{2}kr(z)q \tag{A2.1}$$

where -_V(z) is the voltage seen by the charge q ([24]), r(z) is the q dispacement with respect to the axis and E/2 is the field amplitude (see formalism of eq. (1)).

If we remember the definition (18) of the R/Q and that:

$$U = \frac{\Pi\left(\omega^*\right)}{v_g} \tag{A2.2}$$

we simply obtain that the amplitude of the excited field at the abscissa z is given by:

$$\frac{E}{2} = \frac{1}{2}k\omega^*\frac{R}{Q}r(z)q \tag{A2.3}$$

Since $v_g << v_{ph}$ it is reasonable to suppose that the E field generated by the particle passage has the rigid amplitude profile given by (A2.3), a phase velocity equal to c and a negative group velocity v_g obtaining the equation (19).

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[24] The factor _ comes out from the beam loading theorem.